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Time-Ordered Products and Schwinger Functions

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Abstract. It is shown that every system of time-ordered products for a local field theory determines a related system of Schwinger functions possessing an extended form of Osterwalder-Schrader positivity and that the converse is true provided certain growth conditions are satisfied. This is applied to the φ_3^4 theory and it is shown that the time-ordered functions and S-matrix elements admit the standard perturbation series as asymptotic expansions.

I. Introduction

The present paper is a sequel to [EEF], in which some of the existing models of field theories in 2 space-time dimensions were considered. In order to study the dependence of the S-matrix on the coupling constant, time-ordered functions were constructed in a natural way, by taking essential advantage of the local integrability of the Schwinger functions. (This very property served to define the Schwinger functions as distributions defined everywhere, including coinciding points.) In other models, more singular Schwinger functions occur, and the method of [EEF] cannot be applied. In this paper, a general discussion of the connection between Schwinger functions and time-ordered products is given and applied to the φ_3^4 theory. We show that any Wightman theory equipped with time-ordered products possesses Schwinger functions (considered as distributions over the whole Euclidean world, including coinciding points) which exhibit "extended Osterwalder-Schrader positivity". Conversely, given a set of Schwinger functions possessing this extended positivity together with growth properties similar to those of [OS2], it is possible to supplement the constructions of [OS1, OS2, G1] with a construction of time-ordered products, in a canonical manner. Finally we consider the model φ_3^4 and, starting from results accumulated in the literature [G2, GJ, Fe, FO, MS1, MS2, B, FR, C], we extend to this model the analysis of [EEF], showing in particular that the time-ordered functions and the S-matrix

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elements are $\mathscr{C}^\infty$ in the coupling constant near 0 and that their Taylor series at 0 are given by standard perturbation theory.

In the remainder of this introduction we shall state the "axioms" which respectively characterize time-ordered products and Schwinger functions and the theorems relating these two notions. Sections II and III give the proofs of these theorems and in Section IV we discuss the application to φ_3^4.

I.1. Axioms for Time-Ordered Products

(These "axioms" simply restate the standard postulates but, for reasons of convenience we take the anti-time-ordered products as the basic objects.) Here, $\mathbb{R}^\nu$ denotes the ν-dimensional Minkowski space.

T1) Hilbert Space. $\mathscr{H}$ is a Hilbert space in which a continuous unitary representation $a^0 \to U(a^0, \mathbf{0})$ of the time translation group $\mathbb{R}$ operates. There is a normalized vector Ω (vacuum) such that $U(a^0, \mathbf{0})\Omega = \Omega$ for all $a^0 \in \mathbb{R}$.

T2) Spectrum. $U(a^0, \mathbf{0}) = \exp ia^0 P^0$, $P^0 = H \geqq 0$.

T3) (Anti)-Time Ordered Products. There is a dense subspace D_0 of $\mathscr{H}$ containing Ω and invariant under $U(a^0, \mathbf{0})$. For every $n > 0$ and every $f \in \mathscr{S}((\mathbb{R}^\nu)^n)$ a linear operator $\bar{T}(f)$ is defined on D_0 and $\bar{T}(f)D_0 \subset D_0$. [We also write

$$\bar{T}(f) = \int \bar{T}(x_1, \ldots, x_n) f(x_1, \ldots, x_n) d^\nu x_1 \ldots d^\nu x_n.]$$

Furthermore, for every $f \in \mathscr{S}(\mathbb{R}^{\nu n_r})$,

$$\int \bar{T}(x_1, \ldots, x_{n_1}) \bar{T}(x_{n_1+1}, \ldots, x_{n_2}) \ldots \bar{T}(x_{n_{r-1}+1}, \ldots, x_{n_r}) \\ f(x_1, \ldots, x_{n_r}) dx_1 \ldots dx_{n_r} \psi \tag{1}$$

is defined for each $\psi \in D_0$, belongs to D_0, and depends continuously on f in the topology of $\mathscr{S}$. It is assumed that D_0 is the subspace generated by the vectors of the form (1) with $\psi = \Omega$.

T4) Symmetry. For each $n > 1$, $\bar{T}(x_1, \ldots, x_n)$ is symmetric, i.e. for each permutation π of $(1, \ldots, n)$,

$$\bar{T}(x_1, \ldots, x_n) = \bar{T}(x_{\pi 1}, \ldots, x_{\pi n}).$$

[This will allow us to denote $\bar{T}(X)$ the distribution $\bar{T}(x_{j_1}, \ldots, x_{j_r})$ where $X = \{j_1, \ldots, j_r\}$. We denote $\bar{T}(\emptyset) = 1$.]

T5) (Anti-)Causal Factorization. Let $X = \{1, \ldots, n\}$, $X = P \cup Q$, $P \cap Q = \emptyset$. Then (on D_0) the two distributions $\bar{T}(X)$ and $\bar{T}(P)\bar{T}(Q)$ coincide in the open set of $\mathbb{R}^{\nu n}$ given by

$$\{(x_1, \ldots, x_n) | \forall j \in P, \forall k \in Q, x_j^0 < x_k^0\}. \tag{2}$$

[We define $T(X)$ (the "time-ordered operators") through the polynomial expression

$$T(X) = \sum_{1 \leqq p \leqq |X|} (-1)^{|X|+p} \sum_{\{I_j\}}{}^{*} \bar{T}(I_1) \ldots \bar{T}(I_p), \tag{3}$$

where $\sum^*$ runs over the set $\{I_1, \ldots, I_p \neq \emptyset,\ I_1 \cup \ldots \cup I_p = X,\ I_j \cap I_k = \emptyset$ for $j \neq k\}$, and $T(\emptyset) = 1$.]

T6) Hermiticity. On D_0, $T(X)^* = \bar{T}(X)$.

T7) Time Translation Covariance. For every $f \in \mathscr{S}(\mathbb{R}^{\nu n})$, every $a^0 \in \mathbb{R}$, every $\psi \in D_0$, $U(a^0, \mathbf{0}) \bar{T}(f) U(a^0, \mathbf{0})^{-1} = \bar{T}(f_a)\psi$ where $f_a(x_1, \ldots, x_n) = f(x_1 - a, \ldots, x_n - a)$, $a = (a^0, \mathbf{0})$.

Remark. Most of the ensueing construction is independent of the underlying Hilbert space structure and could be done for any system of distributions having only the linear properties of $(\Omega, \bar{T}(X_1) \ldots \bar{T}(X_n)\Omega)$.

In relativistically invariant theories additional conditions are imposed:

T8) U extends to a continuous unitary representation of the Poincaré group, leaving the vacuum invariant, mapping D_0 into itself and *T7)* is extended in the usual way.

I.2. Axioms for Schwinger Functions

A system of Schwinger functions is defined to be a sequence $\{S_n\}_{n \in \mathbb{N}}$ of tempered distributions such that:

S1) $S_0 \in \mathbb{C}$. For $n \geqq 1$, $S_n \in \mathscr{S}'(\mathbb{R}^{\nu n})$. Here $\mathbb{R}^{\nu n}$ is regarded as $(\mathscr{E}^\nu)^n$, $\mathscr{E}^\nu$ being the ν-dimensional Euclidean space in which a special orthogonal basis $(e_0, e_1, \ldots, e_{\nu-1})$ has been chosen. A point y in $\mathscr{E}^\nu$ will be specified by its coordinates in this basis $(y^0, y^1, \ldots, y^{\nu-1})$, usually denoted $(y^0, \mathbf{y})$. The scalar product is given by $\sum_{j=0}^{\nu-1} y^j y'^j = y^0 y'^0 + \mathbf{y}\mathbf{y}'$.

S2) S_n is symmetric, i.e. for every permutation π of $\{1, \ldots, n\}$ and every $f \in \mathscr{S}(\mathbb{R}^{\nu n})$,

$$S_n(f) = S_n(f_\pi),$$

where $f_\pi(y_1, \ldots, y_n) = f(y_{\pi 1}, \ldots, y_{\pi n})$.

S3) S_n is invariant under Euclidean time-translations, i.e. for all $a = (a^0, \mathbf{0})$, and for all $f \in \mathscr{S}(\mathbb{R}^{\nu n})$, one has

$$S_n(f) = S_n(f_a).$$

With the functional notation this reads

$$S_n(y_1, \ldots, y_n) = S_n(a + y_1, \ldots, a + y_n), \qquad (a = (a^0, \mathbf{0})).$$

S4) Extended Osterwalder-Schrader Positivity. Let $\{f_n\}_{n=1,2,\ldots,N}$ be any finite sequence of functions such that

a) $f_0 \in \mathbb{C}$; for $n \geqq 1$, $f_n \in \mathscr{S}(\mathbb{R}^{\nu n})$.

b) For each $n \geqq 1$, the support of f_n is contained in

$$\{(y_1, \ldots, y_n) | y_j^0 \geqq 0, \forall j = 1, \ldots, n\}.$$

Then

$$\sum_{0 \leqq m, n \leqq N} \int \overline{f_m((-y_m'^0, \mathbf{y}_m'), \ldots, (-y_1'^0, \mathbf{y}_1))} f_n((y_1^0, \mathbf{y}_1), \ldots, (y_n^0, \mathbf{y}_n))$$
$$d^\nu y_1' \ldots d^\nu y_m' d^\nu y_1 \ldots d^\nu y_n S_{m+n}(y_1', \ldots, y_m', y_1, \ldots, y_n) \geqq 0 .$$

In other words if for any $g \in \mathscr{S}(\mathbb{R}^{\nu n})$ we denote

$$(\Theta g)(y_1, \ldots, y_n) = \overline{g((-y_n^0, \mathbf{y}_n), \ldots, (-y_1^0, \mathbf{y}_1))}, \tag{4}$$

the above condition reads

$$\sum_{n,m} S_{m+n}((\Theta f_m) \otimes f_n) \geqq 0 . \tag{5}$$

Remark. The only difference between these hypotheses and the O.S. axioms is that O.S. positivity is replaced by extended O.S. positivity. As a consequence many steps in the construction of Section III.1 are repetitions of those appearing in [OS 1, OS 2, G 1]. They are included for the sake of logical continuity. For related, partly overlapping assumptions and developments see also [H, F, DF, GJ 2, Y]. In particular Condition *S*4) appears in [H, DF, Y].

*S*5) *Full Euclidean Invariance.*
The following property will play a crucial role in the reconstruction of time ordered products.

*S*6) *Growth Condition.* There are constants *K*, *L*, *s* such that for all *n* and all $f_j \in \mathscr{S}(\mathbb{R}^\nu)$, $j = 1, \ldots, n$,

$$|S_n(f_1 \otimes \ldots \otimes f_n)| \leqq K^n n!^L \prod_{j=1}^n |f_j|_s .$$

[$|\cdot|_s$ denotes the Schwartz norm

$$|f|_s = \sup_{x, 0 \leqq |\alpha|, |\beta| \leqq s} |x^\beta D^\alpha f(x)| .]$$

I.3. Results

Two systems respectively satisfying the "*T*" and "*S*" axioms are naturally related if they satisfy the following condition:

R) If $f \in \mathscr{S}(\mathbb{R}^{\nu n})$, $f_\lambda(x_1, \ldots, x_n) = f(\lambda x_1^0, \mathbf{x}_1, \ldots, \lambda x_n^0, \mathbf{x}_n)$ for real $\lambda > 0$, and if the map $\lambda \to f_\lambda$ can be extended to a continuous map of the angle $\{\lambda \in \mathbb{C}, |\lambda| > 0, \operatorname{Re}\lambda \geqq 0, \operatorname{Im}\lambda \geqq 0\}$ into $\mathscr{S}(\mathbb{R}^{\nu n})$, analytic in the interior of this angle, then

$$\int i^n f_i(x_1, \ldots, x_n) S_n(x_1, \ldots, x_n) d^\nu x_1 \ldots d^\nu x_n$$
$$= \int f(x_1, \ldots, x_n)(\Omega, \bar{T}(x_1, \ldots, x_n)\Omega)\, d^\nu x_1 \ldots d^\nu x_n . \tag{6}$$

Remark. The conventions of this paper differ from those of [EEF] in that

$$S(x_1, \ldots, x_n) = S_{EEF}(-x_1^0, \mathbf{x}_1, \ldots, -x_n^0, \mathbf{x}_n) .$$

We are now in a position to state

Theorem 1. *Given a system of (anti-)time ordered products satisfying T1)–T7) one can construct a unique system of Schwinger functions satisfying S1)–S4) such that R) holds.*

Conversely, one has

Theorem 2. *Given a system of Schwinger functions satisfying S1)–S4) and the growth condition S6) one can construct a unique system of (anti-)time ordered products satisfying T1)–T7) such that R) holds.*

Corollary 3. *The inclusion of T8) in the assumptions of Theorem 1 implies that S5) holds in its conclusion, and conversely for Theorem 2.*

Comment. Our assumptions are formulated for the case of one neutral scalar field $A(x)=\bar{T}(x)=T(x)$, but it is straightforward to extend all our considerations to the case of any number of fields with arbitrary charge, spin, and statistics. [In fact, in the application to φ_3^4 we shall need the fields φ, φ^2, φ^3, φ^4 as basic objects.] The symmetry conditions $T4$) and $S2$) can be recovered by considering all fields to be the components of a single object.

In Section IV we show that the Schwinger functions of the φ_3^4 theory satisfy the conditions $S1$)–$S6$).

II. Proof of Theorem 1

II.1. Euclidean Time-Ordered Operators

We assume that a system satisfying $T1$)–$T7$) is given. It will be useful to construct, as intermediate objects, operator valued distributions $\mathcal{O}(x_1, \ldots, x_n)$ which can formally be thought of as $\bar{T}(ix_1^0, \mathbf{x}_1, \ldots, ix_n^0, \mathbf{x}_n)$. Let D_1 be the intersection of the domains of the closures of all finite products $\bar{T}(f_1)\ldots\bar{T}(f_N)$, (initially defined on D_0), where $f_j \in \mathcal{S}(\mathbb{R}^{\nu p_j})$. The operators $\mathcal{O}$ shall satisfy:

$\mathcal{O}1$) If $\psi \in D_1$, and if $f \in \mathcal{S}(\mathbb{R}^{\nu n+1})$ has its support in

$$\{(x_1, \ldots, x_n, v) \in \mathbb{R}^{\nu n+1} \mid 0 \leqq x_j^0 < v, j=1, \ldots, n\} \tag{7}$$

then

$$\int \mathcal{O}(x_1, \ldots, x_n)\, e^{-vH} \psi f(x_1, \ldots, x_n, v)\, dx_1 \ldots dx_n dv$$

is well defined, belongs to D_1 and depends continuously on f.

$\mathcal{O}2$) For any $f \in \mathcal{S}(\mathbb{R}^{\nu n})$ and $w = u + iv \in \mathbb{C}$ with $v > 0$, such that

$$\operatorname{supp.} f \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^{\nu n} \mid \forall j, 0 \leqq x_j^0 < v\},$$

and for every $\psi \in D_1$, $\mathcal{O}(f) e^{iwH} \psi$ is defined, belongs to D_1, depends continuously on f and holomorphically on w.

$\mathcal{O}3$) *Symmetry.* $\mathcal{O}(x_1, \ldots, x_n) = \mathcal{O}(x_{\pi 1}, \ldots, x_{\pi n})$ for every permutation π.

[We again introduce the notation $\mathcal{O}(X)$ as for the $\bar{T}$, and we define $\mathcal{O}(\emptyset) = 1$].

$\mathcal{O}4)$ *Time Translation.* For $0 \leqq x_j^0 \leqq v$, $(1 \leqq j \leqq n)$ and $t \geqq 0$, and all $\psi \in D_1$,

$$e^{-tH}\mathcal{O}(x_1, \ldots, x_n)e^{-vH}\psi = \mathcal{O}(x'_1, \ldots, x'_n)e^{-(v+t)H}\psi$$

with $x_j'^0 = x_j^0 + t$, $\mathbf{x}'_j = \mathbf{x}_j$, $1 \leqq j \leqq n$, with obvious notations.

$\mathcal{O}5)$ *Factorization.* In the open set $\{(y_1, \ldots, y_n) \in \mathbb{R}^{vn} | y_j^0 < y_k^0$ for every $j = 1, \ldots, p$ and every $k = p+1, \ldots, n\}$,

$$\mathcal{O}(y_1, \ldots, y_n) = \mathcal{O}(y_1, \ldots, y_p)\mathcal{O}(y_{p+1}, \ldots, y_n)$$

as distributions, on D_1.

$\mathcal{O}6)$ Let $X_1, \ldots, X_r$ be any partition of $\{1, \ldots, N\}$ and let $Z_1, \ldots, Z_l$ be any partition of $\{N+1, \ldots, N+p\}$. Then, for every $f \in \mathcal{S}(\mathbb{R}^{v(N+p)})$ having support in

$$\{(x_1, \ldots, x_N, z_{N+1}, \ldots, z_{N+p}) | j \in X_t, k \in X_s, t < s \Rightarrow 0 \leqq x_j^0 \leqq x_k^0 \leqq b\},$$

the vector

$$\Phi = \int e^{iw_0 H}\mathcal{O}(X_1) \ldots e^{iw_{r-1}H}\mathcal{O}(X_r)e^{(iw_r - b)H}\bar{T}(Z_1) \ldots \bar{T}(Z_l)\Omega$$
$$f(x_1, \ldots, x_N, z_{N+1}, \ldots, z_{N+p})d^v x_1 \ldots d^v z_{N+p}$$

is well-defined, is in D_1, depends holomorphically on $w_0, \ldots, w_r$ for $\operatorname{Im} w_j > 0$, and for every β there is a constant C_β such that (for $\operatorname{Im} w_j \geqq 0$),

$$\|D_w^\beta \Phi\| \leqq C_\beta(1+|w|)^L |f|_{L+|\beta|}$$

where L depends only on $N+p$, and where $|w| = \sum |w_j|$.

The operators $\mathcal{O}$ are naturally related to the operators $\bar{T}$ by the following condition:

R') Let $g \in \mathcal{S}(\mathbb{R}^{vn+1})$ have support in (7) and denote, for $\lambda > 0$, $g_\lambda(x_1, \ldots, x_n, v) = g(\lambda x_1^0, \mathbf{x}_1, \ldots, \lambda x_n^0, \mathbf{x}_n, \lambda v)$. Assume that the map $\lambda \to g_\lambda$ extends to a continuous map of

$$\{\lambda \in \mathbb{C} \,|\, |\lambda| > 0, \operatorname{Re}\lambda \geqq 0, \operatorname{Im}\lambda \geqq 0\} \tag{8}$$

into $\mathcal{S}(\mathbb{R}^{vn+1})$, holomorphic in the interior of (8). Then for all $\psi \in D_1$,

$$\int \mathcal{O}(x_1, \ldots, x_n)e^{-vH}\psi g_i(x_1, \ldots, x_n, v) i^{n+1} d^v x_1 \ldots d^v x_n dv$$
$$= \int \bar{T}(x_1, \ldots, x_n)e^{ivH}\psi g(x_1, \ldots, x_n, v) d^v x_1 \ldots d^v x_n dv\,. \tag{9}$$

Remark. The r.h.s. of (9) makes sense.

Proof. With our choice of D_1, it is clear that $e^{iwH}\psi$ exists for every $\psi \in D_1$ and every $w = u + iv \in \mathbb{C}$ with $v \geqq 0$ and is in D_1. It is $\mathscr{C}^\infty$ in w and holomorphic when $v > 0$. Moreover there exists, for each N an L and a C such that for any partition $X_1, \ldots, X_r$ of $\{1, \ldots, N\}$, and any $f \in \mathcal{S}(\mathbb{R}^{vN})$,

$$\left\|\int f(x_1, \ldots, x_N)e^{iw_0 H}\bar{T}(X_1)e^{iw_1 H} \ldots e^{iw_{r-1}H}\bar{T}(X_r)\Omega\right.$$
$$\left. dx_1 \ldots dx_N\right\| \leqq C(1+|w|)^L |f|_L$$

with $w_j = u_j + iv_j$, $v_j \geqq 0$, $(0 \leqq j < r)$. This clearly implies

$$\|D_w^\beta \int f(x_1, \ldots, x_N) e^{iw_0 H} \bar{T}(X_1) \ldots e^{iw_{r-1} H} \bar{T}(X_r) \Omega dx_1 \ldots dx_N\| \leqq C_{|\beta|}(1+|w|)^L |f|_{L+|\beta|}. \quad \text{q.e.d.} \tag{10}$$

The construction of $\mathcal{O}(X)$ is obtained by an induction on $|X|$. Note that $\mathcal{O}(X)$ has been defined for $X = \emptyset$ as $\mathcal{O}(\emptyset) = 1$. We assume that $\mathcal{O}(X)$ has been defined for all X with $|X| \leqq n-1$, with all the properties $\mathcal{O}1)$–$\mathcal{O}6)$, R') and we construct $\mathcal{O}(X)$ for $X = \{1, \ldots, n\}$. More precisely, let $Z = \{n+1, \ldots, n+p\}$ and $z = (z_{n+1}, \ldots, z_{n+p})$, $Z = Z_1 \cup \ldots \cup Z_r$, (with $Z_j \cap Z_k \neq \emptyset$ for $j \neq k$). We shall define, for all $f \in \mathcal{S}(\mathbb{R}^{\nu n + 1 + \nu p})$, with support

$$\operatorname{supp} f \subset \{(x_1, \ldots, x_n, v, z), 0 \leqq x_j^0 \leqq v, j \in X\} \tag{11}$$

the vector

$$\int \mathcal{O}(X) e^{-vH} \bar{T}(Z_1) \ldots \bar{T}(Z_r) \Omega f(x_1, \ldots, x_n, v, z) \, d^\nu x_1 \ldots d^\nu x_n dv d^\nu z_{n+1} \ldots d^\nu z_{n+p}. \tag{12}$$

(We shall concentrate on the case $n > 1$, the case $n = 1$ being a simpler version, left to the reader. We also assume $Z \neq \emptyset$; the case $Z = \emptyset$ is a trivial variant.)

There is in fact no freedom left in defining $\mathcal{O}$: for a radially analytic test-function g, $\mathcal{O}(g)$ is determined by R'); for a test function vanishing at coinciding points, it is determined by the factorization property $\mathcal{O}5$) and by already constructed operators. We shall see that any test function can be written as a sum of functions of the two preceding types.

We first indicate how to decompose an arbitrary function. For this we use two auxiliary functions which we now define. Denote $y = n^{-1}(x_1^0 + \ldots + x_n^0)$, $\xi_j = x_j^0 - y$, $1 \leqq j \leqq n$, (so that $\sum_{j \in X} \xi_j = 0$) and let $\xi = (\xi_1, \ldots, \xi_{n-1})$.

The function α_L. L is a positive integer and, for all $t \in \mathbb{R}$,

$$\alpha_L(t) = \theta(t) t^L e^{(i-1)t} = \frac{L!}{2\pi} \int e^{-ipt} (1 - i - ip)^{-L-1} dp. \tag{13}$$

More generally, for every complex λ with $|\lambda| > 0$ and $0 \leqq \arg \lambda \leqq \frac{\pi}{2}$,

$$\alpha_{\lambda, L}(t) = \theta(t) \lambda^L t^L e^{(i-1)\lambda t} = \frac{L!}{2\pi} \int e^{-ipt} (1 - i - i\lambda^{-1} p)^{-L-1} \lambda^{-1} dp, \tag{14}$$

and, in particular

$$\alpha_{i, L}(t) = \theta(t) i^L t^L e^{-(i+1)t} = \frac{L!}{2\pi} \int e^{-ipt} (1 - i - p)^{-L-1} (-i dp). \tag{15}$$

Note that

$$\lambda \left(1 - i + \lambda^{-1} \frac{\partial}{\partial t}\right)^{L+1} \alpha_{\lambda, L}(t) = L! \delta(t). \tag{16}$$

The Function $w: w(\xi, y)$ is a real function over $\mathbb{R}^n$ such that

1) w is homogeneous and of degree 0, i.e. for all $\varrho > 0$, $w(\varrho\xi, \varrho y) = w(\xi, y)$.
2) w is $\mathscr{C}^\infty$ in $\mathbb{R}^n \setminus \{0\}$.
3) The support of w is contained in the cone

$$\{(\xi, y) | 0 \leqq y; 0 \leqq y + n\xi_j \quad \text{for all} \quad j = 1, \ldots, n\},$$

(we recall that $\xi_n = -\xi_1 - \ldots - \xi_{n-1}$). Expressed in the variables $x_j^0 = y + \xi_j$, this reads

$$\left\{(x_1^0, \ldots, x_n^0) | \forall j, x_j^0 - \frac{n-1}{n^2} \sum_{k=1}^{n} x_0^k \geqq 0\right\}.$$

Note that this implies $x_j^0 \geqq \frac{n-1}{n} y$ hence $x_j^0 \leqq \frac{n^2 - (n-1)^2}{n} y \leqq 2y$.

4) $0 \leqq w \leqq 1$.
5) $D_\xi^\beta w(0, y) = 0$ for all $|\beta| > 1$ and all $y \neq 0$: $w(0, y) = 1$ for all $y > 0$.

It is easy to construct such functions: take $w(\xi, 1)$ to be any $\mathscr{C}^\infty$ function of $\xi = (\xi_1, \ldots, \xi_{n-1})$ such that $0 \leqq w(\xi, 1) \leqq 1$, $w(0, 1) = 1$, $D_\xi^\beta w(0, 1) = 0$ for all $|\beta| \geqq 1$, and

$$\operatorname{supp} w(\xi, 1) \subset \{\xi : 0 \leqq n\xi_j + 1 \quad \text{for all} \quad j = 1, \ldots, n-1, 0 \leqq 1 - n\xi_1 - \ldots - n\xi_{n-1}\}.$$

Note that 0 is an interior point of this set. Then define

$$w(\xi, y) = w(\xi/y, 1) \quad \text{for} \quad y > 0, \quad \text{and} \quad w(\xi, y) = 0 \quad \text{for} \quad y \leqq 0.$$

The Decomposition of f. Fix L (to be determined later). Consider any $f \in \mathscr{S}(\mathbb{R}^{\nu n + 1 + \nu p})$ with support in $\{(x_1, \ldots, x_n, v, z) | 0 \leqq x_j^0 \leqq v, \forall j \in X\}$.

Define φ by

$$\varphi(\xi, \mathbf{x}, y, v, z) = -(L!)^{-2} \left(1 - i - i\frac{\partial}{\partial y} - 2i\frac{\partial}{\partial v}\right)^{L+1} \left(1 - i - i\frac{\partial}{\partial v}\right)^{L+1}$$
$$f((\xi_1 + y, \mathbf{x}_1), \ldots, (\xi_n + y, \mathbf{x}_n), v, z), \tag{17}$$

where ξ, y have been defined previously.

By Equation (16) it follows that f can be written as

$$f(x_1, \ldots, x_n, v, z) = \int_{\mathbb{R}^2} da\,db\, \varphi(\xi, \mathbf{x}, a, b, z)$$
$$\alpha_{i,L}(y-a)\alpha_{i,L}(v - 2y - b + 2a). \tag{18}$$

φ is in $\mathscr{S}(\mathbb{R}^{\nu(n+p)+1})$ as a function of all its arguments, and has support in

$$\{(\xi, \mathbf{x}, a, b, z) | \xi_j + a \geqq 0, (1 \leqq j \leqq n), 0 \leqq a \leqq b\}.$$

Furthermore,

$$|\varphi|_R \leqq \text{Const}\, |f|_{R+2L+2},$$

(the constant depending only on R and L).

We decompose f into two parts, $f = f_0 + f_1$ by defining

$$f_1(x, v, z) = \int da\,db\,\alpha_{i,L}(y-a)\alpha_{i,L}(v - 2y - b + 2a)$$
$$\left[\sum_{|\beta|=0}^{M+1} \frac{\xi^\beta}{\beta!} D_\xi^\beta \varphi(0, \mathbf{x}, a, b, z)\right] w(\xi, y-a). \tag{19}$$

$M \geqq 0$ will be chosen later.

Definition of $\mathcal{O}(f_0)$. Denote again $y=n^{-1}(x_1^0+\ldots+x_n^0)$, $\xi_j=x_j^0-y$, $(1\leqq j\leqq n)$, $\xi=(\xi_1,\ldots,\xi_{n-1})$. Let $\mathscr{E}$ be the subspace of $\mathscr{S}(\mathbb{R}^{\nu n+1+\nu p})$ consisting of functions having their support in the set (11), and let $\mathscr{E}_0\subset\mathscr{E}$ be the set of functions in $\mathscr{E}$ which vanish in a neighborhood of

$$\{(x_1,\ldots,x_n,v,z)|\xi=0\}.$$

Lemma 4. *A function* $g\in\mathscr{E}_0$ *can be written as* $g=\sum_{A,B}g_{A,B}$, $A\cup B=X$, $A\cap B=\emptyset$, $A,B\neq\emptyset$, *with* $\operatorname{supp}g_{A,B}\subset\{(x_1,\ldots,x_n,v,z)|x_j^0<x_k^0,\forall j\in A,k\in B\}$.

The proof is deferred to the end of the subsection.

We now *define* for $g\in\mathscr{E}_0$

$$\begin{aligned}&\int\mathcal{O}(X)e^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega g(x,v,z)dx\,dv\,dz\\&\quad=\sum_{A,B}\int\mathcal{O}(A)\mathcal{O}(B)e^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega g_{A,B}(x,v,z)dxdvdz\,.\end{aligned}\tag{20}$$

This definition is forced by the requirement $\mathcal{O}5$) (Factorization). The r.h.s. of this equation is well defined according to the induction hypothesis and is bounded in norm by Const $|g_{A,B}|_R$ (the constant and R depend only on $n+p$). It is easy to see by standard methods, (see e.g. [EG 1,2]), that there is a $K\geqq0$ such that (for all $g\in\mathscr{E}_0$) the expression (20) is bounded in norm by Const. $|g|_K$.

The function f_1 defined by (19) is not $\mathscr{C}^\infty$ but it belongs to the completion of $\mathscr{S}(\mathbb{R}^{\nu n+1+\nu p})$ in the norm $|\,|_M$, and

$$|f_1|_M\leqq\text{Const.}\,|\varphi|_{2M+1}\leqq\text{Const.}\,|f|_{2M+2L+3}\,,$$

provided we have chosen $L\geqq M+1$. Its support is contained in

$$\{(x_1,\ldots,x_n,v,z_{n+1},\ldots,z_{n+p})|\forall j\in X,0\leqq x_j^0\leqq v\}.$$

Hence f_0 has the same properties and, furthermore, vanishes together with its derivatives of order $\leqq M+1$ when $\xi=0$. As a consequence if we replace f by f_0 in (12), the corresponding vector is well defined by our previous discussion, (formula (20) with $g=f_0$) and bounded in norm by Const. $|f_0|_M\leqq$ Const. $|f|_{2M+2L+3}$, provided we choose $M\geqq K$.

Definition of $\mathcal{O}(f_1)$. The function f_1 can be re-written as

$$f_1(x,v,z)=\int dadb\varphi_0(\xi,y-a,v-b,\mathbf{x},a,b,z)\,,$$

where $\varphi_0=\varphi_{0,1}$,

$$\begin{aligned}\varphi_{0,\lambda}(\xi,y,v,\mathbf{x},a,b,z)=&\left[\sum_{|\beta|=0}^{M+1}\frac{\lambda^{|\beta|}\xi^\beta}{\beta!}D_\xi^\beta\varphi(0,\mathbf{x},a,b,z)\right]w(\xi,y)\alpha_{i,L}(\lambda y)\\&\alpha_{i,L}(\lambda v-2\lambda y)\,.\end{aligned}$$

This function is radially analytic (with respect to the variables ξ,y,v) and has support in (11). Hence the requirements $\mathcal{O}4$) and R') force the definition:

$$\begin{aligned}&\int\mathcal{O}(X)e^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega\varphi_0(\xi,y-a,v-b,\mathbf{x},a,b,z)dxdvdz\\&\quad=\int e^{-aH}\bar{T}(X)e^{(iv-b+a)H}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega\\&\quad n\varphi_{0,-i}(\xi,y,v,\mathbf{x},a,b,z)i^{-n-1}d\xi dydvd\mathbf{x}dz\,.\end{aligned}\tag{21}$$

The r.h.s. of (21) is a well-defined vector: considered as a test function in the variables $\xi, y, v, \mathbf{x}, z$, depending on the parameters a and b, $\varphi_{0,-i}$ belongs to the completion of $\mathscr{S}$ in the norm $|\,|_M$, (with the condition $L \geqq M+1$), and is an admissible test-function if M is sufficiently large. Moreover,

$$\sup_{a,b}(1+a+b)^R|\varphi_0(\ldots,a,b)|_M \leqq \text{Const.}\,|\varphi|_{2M+1+R}$$
$$\leqq \text{Const.}\,|f|_{2M+R+2L+3}\,. \tag{22}$$

Thus we can integrate (21) over a and b and *define*

$$\begin{aligned}&\int \mathcal{O}(X)e^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega f_1(x,v,z)dx\,dv\,dz\\ &\quad=\int e^{-aH}\bar{T}(X)e^{(iv-b+a)H}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega\\ &\quad n\varphi_{0,-i}(\xi,y,v,\mathbf{x},a,b,z)i^{-n-1}d\xi\,dy\,dv\,d\mathbf{x}\,dz\,da\,db\,.\end{aligned} \tag{23}$$

This completes the definition of the vector (12), which is bounded in norm by Const. $|f|_{4M+8}$.

Verification of the Properties $\mathcal{O}1)$–$\mathcal{O}6)$. The vector constructed above is independent of $L \geqq M+1$ because it is easy to check (by using the induction hypothesis) that if f happens to be radially analytic in the variables ξ, y and v, then so are f_0 and f_1 and the above definition yields exactly

$$\begin{aligned}&\int \bar{T}(X)e^{ivH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega f((-ix_1^0,\mathbf{x}_1),\ldots,(-ix_n^0,\mathbf{x}_n),-iv,z)\\ &\quad(-i)^{n+1}d^\nu x_1\ldots d^\nu x_n\,dv\,dz_{n+1}\ldots dz_{n+p}\,.\end{aligned}$$

(We omit the details of this verification. An analogous verification is sketched in Section III.) Thus $\mathcal{O}1)$ and $R')$ are satisfied.

Having defined vectors of the type (12), we now wish to define

$$\int \mathcal{O}(X)e^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega g(x_1,\ldots,x_n,z_{n+1},\ldots,z_{n+p})d^\nu x_1\ldots d^\nu z_{n+p}\,,$$

where $v>0$ and $g\in\mathscr{S}(\mathbb{R}^{\nu(n+p)})$ has its support in

$$\{(x_1,\ldots,x_n,z):\forall j\in X, 0\leqq x_j^0\leqq v\}\,.$$

This can be rewritten, formally, as

$$\begin{aligned}&\int \mathcal{O}(X)e^{(it-v)H}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega g(x_1,\ldots,x_n,z_{n+1},\ldots,z_{n+p})\\ &\quad(Q!)^{-1}\left(1-i+\frac{\partial}{\partial t}\right)^{Q+1}\alpha_Q(t)dt\,dx\,dz\\ &=\int \mathcal{O}(X)e^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega g(x_1,\ldots,x_n,z'_{n+1},\ldots,z'_{n+p})\\ &\quad(Q!)^{-1}\left(1-i+\frac{\partial}{\partial t}\right)^{Q+1}\alpha_Q(t)dt\,dx\,dz\end{aligned}$$

where $z'_k=(z_k^0-t,\mathbf{z}_k)$, $(n+1\leqq k\leqq n+p)$. By integrating by parts over t this becomes

$$\begin{aligned}&\int \mathcal{O}(X)e^{(-v+it)H}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega\\ &\quad K_Q g(x_1,\ldots,x_n,z_{n+1},\ldots,z_{n+p})\alpha_Q(t)dt\,dx\,dz\,,\end{aligned}$$

where

$$K_Q g(x_1, \ldots, x_n, z_{n+1}, \ldots, z_{n+p})$$
$$=(Q!)^{-1}\left(1-i-\frac{\partial}{\partial t}\right)^{Q+1} g(x_1, \ldots, x_n, z'_{n+1}, \ldots, z'_{n+p})|_{t=0}.$$

By analytic continuation in t, this becomes

$$\int \mathcal{O}(X)e^{-(v+t)H}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega K_Q g(x,z)\alpha_{i,Q}(t)idt\,dx\,dz,$$

and, for Q sufficiently large, this has been defined previously.

The same formulae evidently lead to a consistent definition of

$$\int \bar{T}(Y_1)\ldots\bar{T}(Y_s)e^{i\sigma H}\mathcal{O}(X)^{-vH}\bar{T}(Z_1)\ldots\bar{T}(Z_r)\Omega$$
$$h(\{y\}_{Y_1\cup\ldots\cup Y_s}, x, v, z)dx\,dx\,dv\,dz.$$

This shows that all such vectors do belong to D_1.

This proves $\mathcal{O}2$). The properties $\mathcal{O}3$), $\mathcal{O}4$), $\mathcal{O}5$), $\mathcal{O}6$) follow from the construction, from the corresponding properties for the $\bar{T}$, and by analytic continuation. We omit the details of the verification, which are entirely straightforward.

Proof of Lemma 4. We consider, in the space of the variables ξ, the "sphere" Ξ consisting of all ξ verifying

$$\sum_{1\leqq j<k\leqq n}(\xi_j-\xi_k)^2=1$$

$\left(\text{Note that } \sum_{1\leqq j<k\leqq n}(\xi_j-\xi_k)^2=n\sum_{1\leqq j\leqq n}\xi_j^2\right)$.

For any $\xi\in\Xi$, there is at least one pair of distinct j and k, such that $\xi_j-\xi_k \geqq(\frac{1}{2}n(n-1))^{-1/2}>2^{1/2}n^{-1}$. Thus if $\alpha=\min\{\xi_j:1\leqq j\leqq n\}$ and $\beta=\max\{\xi_j:1\leqq j\leqq n\}$, we have

$$2^{1/2}n^{-1}<\beta<a\leqq 1.$$

The real numbers ξ_j, $1\leqq j\leqq n$, subdivide the interval $[\alpha,\beta]$ into at most $n-1$ open intervals. At least one of these subintervals say (γ,δ) has a length

$$\delta-\gamma\geqq(n-1)^{-1}2^{1/2}n^{-1}>2^{1/2}n^{-2}.$$

Let

$$A=\{j:1\leqq j\leqq n, \xi_j\leqq\gamma\},\qquad B=\{k:1\leqq k\leqq n, \xi_k\geqq\delta\}.$$

Then ξ belongs to the open set of the sphere defined by

$$\Omega_{A,B}=\{\xi:\forall j\in A, \forall k\in B, \xi_k-\xi_j>2^{1/2}n^{-2}\}$$

itself contained in

$$\Omega'_{A,B}=\{\xi:\forall j\in A, \forall k\in B, \xi_k-\xi_j>n^{-2}\}.$$

Let $\{\chi_{A,B}\}_{\substack{A\cup B=X\\ A\cap B=\emptyset\\ A,B\neq\emptyset}}$ be a family of $\mathscr{C}^\infty$ functions on Ξ with

$$\text{supp.}\,\chi_{A,B}\subset\Omega'_{A,B}\quad\text{and}\quad\sum_{A,B}\chi_{A,B}=1.$$

[Example: denote $F_{A,B}(\xi)=\prod_{\substack{j\in A\\k\in B}}\theta(\xi_k-\xi_j-n^{-2})\exp-(\xi_k-\xi_j-n^{-2})^{-1}$ and $\chi_{A,B}(\xi)=F_{A,B}(\xi)\left[\sum_{C,D}F_{C,D}(\xi)\right]^{-1}$. This is well defined since, on $\Omega_{C,D}$, $F_{C,D}(\xi)>\exp(-n^4(2^{1/2}-1)^{-1})$.]

Then, for general $\xi\neq 0$, denote $\psi_{A,B}(\xi)=\chi_{A,B}\left(\xi\left(n\sum_{j=1}^{n}\xi_j^2\right)^{-1/2}\right)$. If $g\in\mathscr{E}_0$, $g=\sum_{A,B}g_{A,B}$ with $g_{A,B}=\psi_{A,B}(\xi)g\in\mathscr{E}_0$. This completes the proof.

II.2. The Construction of Schwinger Functions

The previous construction has led, in particular, to a definition of $(\Omega,\mathcal{O}(X)\Omega)$ as a continuous linear functional over the subspace of $\mathscr{S}(\mathbb{R}^{\nu n})$ consisting of functions with support in $\{(x_1,\dots,x_n)|x_j^0\geqq 0, j=1,\dots,n\}$. However this functional is invariant under the translations of the form $(x_1,\dots,x_n)\to(x_1+a,\dots,x_n+a)$ where $a^0>0$, $\mathbf{a}=0$, and can be uniquely extended to a continuous linear functional over the whole of $\mathscr{S}(\mathbb{R}^{\nu n})$, by using the translation invariance and a partition of the unit. The tempered distribution S_n so defined (also denoted S if n is unambiguous) is symmetric, invariant under time translations, and coincides with $(\Omega,\mathcal{O}(X),\Omega)$ when integrated with test-functions with support in $\{(x_1,\dots,x_n)\in\mathbb{R}^{\nu n}|\forall j, x_j^0\geqq 0\}$. Thus it satisfies the properties $S1$), $S2$), $S3$) and R).

In order to study further properties of S, we use some well-known regularization procedures.

The preceding construction of the distributions $(\Omega,\mathcal{O}(X_1)\dots\mathcal{O}(X_r)\Omega)$ does not really depend on the Hilbert space structure but only on the linear properties of the distributions $(\Omega,\bar{T}(X_1)\dots\bar{T}(X_1)\Omega)$. Thus it can be straight-forwardly adapted to the case of a "linear system of n-point functions", i.e. a set of tempered distributions $\langle\!\langle\bar{T}(X_1)\dots\bar{T}(X_s)\rangle\!\rangle$ over $\mathbb{R}^{\nu n}$ (here $X_1\cup\dots\cup X_s=X=\{1,\dots,n\}$, $X_j\cap X_k=\emptyset$ if $j\neq k$, $1\leqq s\leqq n$), having all the linear properties of the $(\Omega,\bar{T}(X_1)\dots\bar{T}(X_s)\Omega)$. (These "linear systems" are discussed in [EGS]). Among the "axioms" which are imposed on such a linear system is the spectrum condition: let

$$\langle\!\langle\bar{T}(X_1)\dots\bar{T}(X_r)\rangle\!\rangle^{\sim}(p_1,\dots,p_n)\delta\Big(\sum_{1\leqq j\leqq n}p_j\Big)$$
$$=(2\pi)^{-n\nu}\int\Big(\exp i\sum_{1\leqq j\leqq n}p_j x_j\Big)\langle\!\langle\bar{T}(X_1)\dots\bar{T}(X_r)\rangle\!\rangle(x_1,\dots,x_n)d^\nu x_1\dots d^\nu x_n. \quad (24)$$

[Here $p_j x_j=p_j^0x_j^0-\mathbf{p}_j\mathbf{x}_j$.]

Denote, for every proper subset Y of X, $p_Y=\sum_{k\in Y}p_k$. Let $I_1=X_1,\dots,I_s=\bigcup_{1\leqq j\leqq s}X_j$. Then the support of the distribution (24) is contained in

$$\left\{(p_1,\dots,p_n)\middle|\sum_{j=1}^{n}p_j=0,\quad\text{and, for every}\quad s=1,\dots,r-1, p_{I_s}\in\bar{V}^+(M_{I_s})\right\}.$$

Here $\bar{V}^+(M)$ denotes $\{p\in\mathbb{R}^\nu | p^0 \geqq 0, p\cdot p \geqq M^2\}$. (See Remark 1 at the end of this section.) For every proper subset J of X, M_J is a fixed real number $\geqq 0$, called the threshold in the channel J; $M_J = M_{X\setminus J}$. If $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle = (\Omega, \bar{T}(X_1)\ldots\bar{T}(X_r)\Omega)$, the thresholds are, in general, 0 because of the vacuum contribution. If $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle$ is taken to be $(\Omega, \bar{T}(X_1)\ldots\bar{T}(X_r)\Omega)^T$, and if the theory has a unique vacuum and a mass gap, then for all J, $M_J \geqq \mu$, where μ is a fixed minimum mass >0.

Given a linear system of n-point functions, it is possible to define, by suitable linear combinations, the corresponding "generalized retarded functions" $\langle\!\langle R_{\mathscr{S}}\rangle\!\rangle$. Their Fourier transforms $\langle\!\langle R_{\mathscr{S}}\rangle\!\rangle^\sim$, defined by

$$\langle\!\langle R_{\mathscr{S}}\rangle\!\rangle^\sim(p_1,\ldots,p_n)\delta\Big(\sum_j p_j\Big) = (2\pi)^{-\nu n}\int e^{ipx}\langle\!\langle R_{\mathscr{S}}\rangle\!\rangle(x_1,\ldots,x_n)dx^{\nu n}$$

are the boundary values from certain tubes of a single function H holomorphic in a certain domain in the "complex momentum space", $\Big\{(k_1,\ldots,k_n)\in\mathbb{C}^{\nu n}\Big|\sum_j k_j = 0\Big\}$. The domain of analyticity of H contains the following set:

$$\Big\{(k_1,\ldots,k_n)\in\mathbb{C}^{\nu n}|\sum_j k_j = 0; \quad \text{for all } j, \quad \mathrm{Im}\,\mathbf{k}_j = 0; \quad \text{and},$$
$$\forall J\subset X, (k_J^0)^2 \notin M_J^2 + \mathbb{R}^+\Big\}.$$

Thus, if the thresholds are all strictly positive, the domain of holomorphy of H contains the set $\tilde{E}_n$ of all "Euclidean momenta",

$$\tilde{E}_n = \Big\{(k_1,\ldots,k_n)\in\mathbb{C}^{\nu n}|\sum_j k_j = 0 \quad \text{and for all } j, \quad \mathrm{Im}\,\mathbf{k}_j = 0, \mathrm{Re}\,k_j^0 = 0\Big\}.$$

In this case, (all $M_J \geqq \mu > 0$), the distributions $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle$ can be unambiguously recovered from the function H by a well-defined linear procedure. This allows the following regularization method.

Let

$$H^{\mathrm{reg}}(k) = \left[\prod_{j=1}^n(-k_j^2 + L^2)^{-R}\right]H(k)$$

where $R \geqq 0$ is an integer and $L > \max_J M_J$. This function has all the linear properties of an n-point momentum-space analytic function and the above mentioned linear procedure, applied to H^{reg}, yields a new linear system of n-point functions, denoted $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle^{\mathrm{reg}}$. They verify

$$\left[\prod_{j=1}^n(\Box_{x_j} + L^2)^R\right]\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle^{\mathrm{reg}} = \langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle \tag{25}$$

For sufficiently large R, all $\langle\!\langle \bar{T}\ldots\rangle\!\rangle^{\mathrm{reg}}$ become continuous and even finitely differentiable, polynomially bounded functions. Furthermore if $w(z_1,\ldots,z_n)$ denotes the value at $(z_1,\ldots,z_n)\in\mathbb{C}^{n\nu}$ of the analytic Wightman function which has as its boundary values the various

$$w_\pi = \langle\!\langle \bar{T}(\{\pi_1\})\ldots\bar{T}(\{\pi_n\})\rangle\!\rangle,$$

and if w^{reg}, w_π^{reg} denotes the corresponding regularized objects, it is easy to see that w^{reg} is continuous and has finitely many continuous derivatives at the boundaries of the permuted forward tubes. In particular w^{reg} defines a piecewise continuous function on the Euclidean world. Applying the characteristic property (R), it is then clear that

$$S^{\text{reg}}(y_1, \ldots, y_n) = w^{\text{reg}}((iy_1^0, \mathbf{y}_1), \ldots, (iy_n^0, \mathbf{y}_n)), \tag{26}$$

and that

$$S(y_1, \ldots, y_n) = \prod_{j=1}^{n} (-\Delta_{y_j} + L^2)^R S^{\text{reg}}(y_1, \ldots, y_n). \tag{27}$$

The last equation holds in the sense of tempered distributions, and, of course S and S^{reg} are given by $\langle\!\langle \mathcal{O}(X) \rangle\!\rangle$ and $\langle\!\langle \mathcal{O}(X) \rangle\!\rangle^{\text{reg}}$, respectively. It is well-known [Sy], (and re-proved in [EGS] and [EEF]), that

$$\begin{aligned} S^{\text{reg}}(y_1, \ldots, y_n) = \int \exp\left(-i \sum_j q_j y_j\right) H^{\text{reg}}((-iq_1^0, \mathbf{q}_1), \ldots, (-iq_n^0, \mathbf{q}_n)) \\ i^{n-1} \delta\left(\sum_{j=1}^{n} q_j\right) dq_1 \ldots dq_n \end{aligned} \tag{28}$$

so that, applying (27), we obtain

$$\begin{aligned} S(y_1, \ldots, y_n) = \int \exp\left(-i \sum_j q_j y_j\right) H((-iq_1^0, \mathbf{q}_1), \ldots, (-iq_n^0, \mathbf{q}_n)) \\ i^{n-1} \delta\left(\sum_{j=1}^{n} q_j\right) dq_1 \ldots dq_n, \end{aligned} \tag{29}$$

(only valid for strictly positive thresholds). Here $q_j y_j = q_j^0 y_j^0 - \mathbf{q}_j \mathbf{y}_j$.

Now assume that $0 < x_{\pi 1}^0 < \ldots < x_{\pi r}^0$, and $0 < x_{\sigma 1}'^0 < \ldots < x_{\sigma s}'^0$, where $r + s = n$ and π and σ are respectively permutations of $(1, \ldots, r)$ and $(1, \ldots, s)$. Then

$$w_1^{\text{reg}}((\mu x_{\sigma s}'^0, \mathbf{x}_{\sigma s}') \ldots, (\mu x_{\sigma 1}'^0, \mathbf{x}_{\sigma 1}'^0), (\lambda x_{\pi 1}^0, \mathbf{x}_{\pi 1}), \ldots, (\lambda x_{\pi r}^0, \mathbf{x}_{\pi r})),$$

initially defined for $\lambda > 0$, $\mu > 0$, can be continued as an analytic function of λ and μ in $\{\lambda \mid |\lambda| \neq 0, 0 < \arg \lambda < \frac{\pi}{2}\} \times \{\mu \mid |\mu| \neq 0, 0 < -\arg \mu < \frac{\pi}{2}\}$, continuous on the boundary of this domain, (with values in the piecewise continuous functions over $\mathbb{R}^{\nu n}$). For $\lambda = i$, $\mu = -i$ this continuation yields

$$S^{\text{reg}}((-x_{\sigma s}'^0, \mathbf{x}_{\sigma s}'), \ldots, (-x_{\sigma 1}'^0, \mathbf{x}_{\sigma 1}'), (x_{\pi 1}^0, \mathbf{x}_{\pi 1}), \ldots, (x_{\pi r}^0, \mathbf{x}_{\pi r})).$$

Let $f \in \mathscr{S}(\mathbb{R}^{\nu r})$ and $g \in \mathscr{S}(\mathbb{R}^{\nu s})$ have their supports in $\{(x_1, \ldots, x_r) \mid \forall j,\ x_j^0 \geqq 0\}$ and $\{(x_1, \ldots, x_s) \mid \forall j,\ x_j^0 \geqq 0\}$, respectively. Defining, for $\lambda > 0$, $f_\lambda(x_1, \ldots, x_r) = f((\lambda x_1^0, \mathbf{x}_1), \ldots, (\lambda x_r^0, \mathbf{x}_r))$, *and similarly* g_λ, we suppose that $\lambda \to f_\lambda$, $\lambda \to g_\lambda$ can be extended to holomorphic maps of $\{\lambda \in \mathbb{C} \setminus \{0\} \mid 0 < \arg \lambda < \frac{\pi}{2}\}$, continuous on $\{\lambda \in \mathbb{C} \setminus \{0\} \mid 0 \leqq \arg \lambda \leqq \frac{\pi}{2}\}$, into $\mathscr{S}(\mathbb{R}^{\nu r})$ and $\mathscr{S}(\mathbb{R}^{\nu s})$, respectively. Then

$$\begin{aligned} \int w^{\text{reg}}((\mu x_{\sigma s}'^0, \mathbf{x}_{\sigma s}'), \ldots, (\mu x_{\sigma 1}'^0, \mathbf{x}_{\sigma 1}'), (\lambda x_{\pi 1}^0, \mathbf{x}_{\pi 1}), \ldots, (\lambda x_{\pi r}^0, \mathbf{x}_{\pi r})) \\ \theta(x_{\sigma s}'^0 - x_{\sigma(s-1)}'^0) \ldots \theta(x_{\sigma 2}'^0 - x_{\sigma 1}'^0)\, \theta(x_{\pi 2}^0 - x_{\pi 1}^0) \ldots \theta(x_{\pi r}^0 - x_{\pi(r-1)}^0) \\ \bar{g}_{\bar{\mu}}(x_1', \ldots, x_s')\, f_\lambda(x_1, \ldots, x_r) \lambda^r \mu^s dx_1' \ldots dx_s'\, dx_1 \ldots dx_r \end{aligned} \tag{30}$$

defines a holomorphic function over $\{\lambda\in\mathbb{C}\setminus\{0\}\,|\,0<\arg\lambda<\frac{\pi}{2}\}\times\{\mu\in\mathbb{C}\setminus\{0\}\,|\,0<-\arg\mu<\frac{\pi}{2}\}$, (with continuity at the boundaries except $\lambda=0$ or $\mu=0$). But, for $\lambda>0$ and $\mu>0$, this integral is independent of λ and μ. Hence it remains constant for complex λ and μ and, by continuity, we get, for $\lambda=\bar{\mu}=i$,

$$\begin{aligned}&\int S^{\mathrm{reg}}((-x'^0_{\sigma s},\mathbf{x}'_{\sigma s}),\ldots,(-x'^0_{\sigma 1},\mathbf{x}'_{\sigma 1}),(x^0_{\pi 1},\mathbf{x}_{\pi 1}),\ldots,(x^0_{\pi r},\mathbf{x}_{\pi r}))\\&\qquad\theta(x'^0_{\sigma s}-x'^0_{\sigma(s-1)})\ldots\theta(x'^0_{\sigma 2}-x'^0_{\sigma 1})\,\theta(x^0_{\pi 2}-x^0_{\pi 1})\ldots\theta(x^0_{\pi r}-x^0_{\pi(r-1)})\\&\qquad f_i(x_1,\ldots,x_r)\overline{g_i(x'_1,\ldots,x'_s)}\,i^{r-s}d^\nu x'_1\ldots d^\nu x'_s d^\nu x_1\ldots d^\nu x_r\\&=\int w^{\mathrm{reg}}_1(x'_{\sigma s},\ldots,x'_{\sigma 1},x_{\pi 1},\ldots,x_{\pi r})\,\theta(x'^0_{\sigma s}-x'^0_{\sigma(s-1)})\ldots\theta(x'^0_{\sigma 1})\\&\qquad\theta(x^0_{\pi 2}-x^0_{\pi 1})\ldots\theta(x^0_{\pi r}-x^0_{\pi(r-1)})\,f(x_1,\ldots,x_r)\bar{g}(x'_1,\ldots,x'_s)\,dx_1\ldots dx'_s\,.\end{aligned}$$

Using the symmetry of S^{reg}, and summing on both sides over σ and π, we obtain

$$\begin{aligned}&\int S^{\mathrm{reg}}((-x'^0_s,\mathbf{x}'_s),\ldots,(-x'^0_1,\mathbf{x}'_1),x_1,\ldots,x_r)\overline{g_i(x'_1,\ldots,x'_s)}\\&\qquad f_i(x_1,\ldots,x_r)dx_1\ldots dx'_s\\&=\int\langle\!\langle T(x'_1,\ldots,x'_s)\,\bar{T}(x_1,\ldots,x_r)\rangle\!\rangle^{\mathrm{reg}}\bar{g}(x'_1,\ldots,x'_s)f(x_1,\ldots,x_r)\\&\qquad dx_1\ldots dx'_s\,.\end{aligned}\tag{31}$$

By using (27) and (25) we obtain the

Lemma 5.

$$\begin{aligned}&\int S((-x'^0_s,\mathbf{x}'_s),\ldots,(-x'^0_1,\mathbf{x}'_1),x_1,\ldots,x_r)\bar{g}_i(x'_1,\ldots,x'_s)f_i(x_1,\ldots,x_r)dx_1\ldots dx'_s\\&=\int\langle\!\langle T(x'_1,\ldots,x'_s)\,\bar{T}(x_1,\ldots,x_r)\rangle\!\rangle\,\bar{g}(x'_1,\ldots,x'_s)f(x_1,\ldots,x_r)dx_1\ldots dx'_s\,.\end{aligned}\tag{32}$$

Proof. We have already shown that this formula holds for all linear systems with strictly positive thresholds. We extend it to all linear systems of n-point functions. Note that both sides of Equation (32) have a well defined meaning even in the case of zero thresholds, since the construction of S as a tempered distribution remains valid. Starting from a linear system of n-point functions with possibly zero thresholds, we approximate it by a new system, with strictly positive thresholds, by the following method. For every $z\in\mathbb{C}^\nu$ with $(z\cdot z)\notin A^2+\mathbb{R}^+$, we define

$$F(z;A)=\exp(iA^{-1}[(z\cdot z)-A^2]^{1/2}+1)\,.\tag{33}$$

Here A is strictly positive; the function $\zeta\to(\zeta-A^2)^{1/2}$ is defined in $\mathbb{C}\setminus(A^2+\mathbb{R}^+)$ by the condition $\mathrm{Im}(\zeta-A^2)^{1/2}>0$. We also denote

$$F^\pm(x;A)=\lim_{\substack{y\in V^+\\y\to 0}}F(x\mp iy;A)\,,$$

$$\begin{aligned}&F^c(x;A)=\theta(x^0)F^+(x;A)+\theta(-x^0)F^-(x;A)\,,\\&F^{ac}(x;A)=\theta(-x^0)F^+(x;A)+\theta(x^0)F^-(x;A)\,.\end{aligned}\tag{34}$$

Note that $|F(z;A)|\leqq e$ for all z such that $(z\cdot z)\notin A^2+\mathbb{R}^+$ and $F(z;A)$ has continuous boundary values at the boundaries of this domain. For any pair (j,k) with $j<k,j$

and $k \in \{1, \ldots, n\}$, we can define a linear system of n-point functions, as explained in [EGS, § 6.3], by

$$\begin{aligned}\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle' &= F^+(x_j - x_k)\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle \\ &\quad \text{if } j\in X_a,\ k\in X_b,\ a<b, \\ &= F^-(x_j - x_k)\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle \\ &\quad \text{if } j\in X_a, k\in X_b, b<a, \\ &= F^{ac}(x_j - x_k)\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle \\ &\quad \text{if } j \text{ and } k \text{ belong to the same } X_a.\end{aligned}$$

The reasons for which this is possible and leads to a linear system are given in [EGS, § 6]. By repeating this procedure, we obtain a new linear system of n-point functions given by

$$\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle_A^{\text{new}} = \Big(\prod_{j<k} F^{(+,-,ac)}(x_j - x_k)\Big)\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle. \tag{35}$$

In particular

$$w_A^{\text{new}}(z_1, \ldots, z_n) = \Big(\sum_{j<k} \exp(iA^{-1}[(z_j - z_k)^2 - A^2]^{1/2} + 1)\Big) w(z_1, \ldots, z_n). \tag{36}$$

This decreases exponentially at infinity in any direction strictly contained in a permuted tube. In fact the Fourier transform

$$\tilde{F}^{\pm}(p; A) = \int e^{ipx} F^{\pm}(x; A) d^\nu x$$

has its support in $\bar{V}^{\pm}(A^{-1})$, and hence the thresholds of the new system are all above A^{-1}.

When A tends to ∞, $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle_A^{\text{new}}$ tends, in the sense of tempered distributions, to $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle$ and, (since the construction of the $\langle\!\langle \mathcal{O}(X_1)\ldots\rangle\!\rangle$ depends continuously on the $\langle\!\langle \bar{T}(X_1)\ldots\bar{T}(X_r)\rangle\!\rangle$), $S_A^{\text{new}} \to S$ in the sense of tempered distributions. This limiting process yields (32) in the general case. Moreover since (29) holds for S_A^{new} and H_A^{new}, and since $S_A^{\text{new}} \to S$ in $\mathcal{S}'$ as $A \to \infty$, the Fourier transform of S_A^{new} also tends (in the sense of $\mathcal{S}'$) to that of S. On the other hand $H_A^{\text{new}}(k)$ tends to $H(k)$ uniformly in every compact of the tubes associated with the $\langle\!\langle R_{\mathcal{S}}\rangle\!\rangle$; the union of these tubes always contains

$$\Big\{k = (k_1, \ldots, k_n)\,\Big|\, \sum_{j=1}^{n} k_j = 0;\ \forall j, k_j^0 = iq_j^0, \mathbf{k}_j = \mathbf{q}_j,$$

q_j^0 and $\mathbf{q}_j$ real, and for every proper subset J of $\{1, \ldots, n\}$,

$$q_J^0\Big(\equiv \sum_{j\in J} q_j^0\Big) \neq 0\Big\}.$$

At all such q, the Fourier transform of S thus coincides with $H(-iq^0, \mathbf{q})\delta(\sum q_j)$.

Remarks. 1. In discussing "linear systems of n-point functions" we have used notations adapted to the relativistic case. However the regularization procedures described above also hold in the non relativistic case and, in particular, Equation (32) remains valid.

2. As noted before, in a field theory with unique vacuum and a mass gap, the time ordered functions have zero thresholds but the truncated time ordered functions have strictly positive thresholds. In this case, Equation (29) is satisfied not by S but its truncated version, S^T.

We return to the study of a field theory equipped with (anti-) time ordered products in a Hilbert space $\mathscr{H}$, i.e. we assume all the hypotheses T1)–T7). Then the distributions S verify the positivity condition S4).

Proof. In the l.h.s. of Equation (5), we first replace each f_n by a corresponding $g_{n,i}$, given by

$$g_{n,\lambda}(y_1,\dots,y_n)=\int_0^\infty \varrho(a;-i\lambda\mu^{-1})\frac{d\mu}{\mu}f_n((\mu y_1^0,\mathbf{y}_1),\dots,(\mu y_n^0,\mathbf{y}_n)).$$

$\varrho(a;\zeta)$ is defined in the Appendix. Here $|\lambda|>0, 0\leqq\arg\lambda\leqq\frac{\pi}{2}$. According to (32), the left-hand side of (5) is then equal to

$$\left\|\sum_n \int g_{n,1}(x_1,\dots,x_n)\bar{T}(x_1,\dots,x_n)\Omega\, dx_1\dots dx_n\right\|^2$$

and is positive. If we let a tend to $+\infty$, $g_{n,i}$ tends to f_n in $\mathscr{S}(\mathbb{R}^{\nu n})$ and the inequality (5) is obtained in the limit $a\to\infty$ (see Appendix).

III. Proof of Theorem 2

In this section, we start from a set of "Schwinger functions" satisfying $S1),\dots,S4)$ and construct first the operators $\mathcal{O}(y_1,\dots,y_n)$. Then the growth condition S6) is used to define the distribution $(\Omega,\mathcal{O}(X_1)e^{iw_1H}\dots e^{iw_{r-1}H}\mathcal{O}(X_r)\Omega)$ from which the anti-time-ordered distributions can be obtained by a purely linear operation (although the vector formalism will be used for notational simplicity).

III. 1. Construction of the Operators $\mathcal{O}(Y)$

Let $\{S_n\}$ be a sequence of tempered distributions satisfying $S1)$ to $S4)$. Let $\mathscr{S}$ denote the vector space of finite sequences $\{f_n\}$ (with arbitrary length) with $f_n\in\mathscr{S}(\mathbb{R}^{\nu n})$, equipped with the natural (direct topological sum) topology. $\mathscr{S}_+$ will denote the subspace of $\mathscr{S}$ consisting of the finite sequences $\{f_n\}$ such that, for each $n\geqq 1$

$$\operatorname{supp.} f_n\subset\{(y_1,\dots,y_n): y_1^0\geqq 0,\dots,y_n^0\geqq 0\}.$$

For each real t, we denote L_t the operator defined on $\mathscr{S}$ by

$$(L_t f)_n(y_1,\dots,y_n)=f_n((y_1^0-t,\mathbf{y}_1),\dots,(y_n^0-t,\mathbf{y}_n)).$$

When t is $\geqq 0$, L_t maps $\mathscr{S}_+$ into itself, and $t\to L_t$, $(t\geqq 0)$ is a continuous semi-group of continuous operators on $\mathscr{S}_+$.

If $f=\{f_n\}$ is an element of $\mathscr{S}$ we denote $S(f)=\sum_{n=0}^{\infty} S_n(f_n)$. If g is also an element of $\mathscr{S}$, we denote

$$\Theta g=\{\Theta g_n\}, (\Theta g_n)(y_1,\dots,y_n)=\overline{g_n((-y_n^0,\mathbf{y}_n),\dots,(-y_1^0,\mathbf{y}_1))}.$$

We also write

$$(g\otimes f)_n(y_1,\ldots,y_n)=\sum_{p+q=n} g_p(y_1,\ldots,y_p)f_q(y_{p+1},\ldots,y_{p+q}),$$

and note the following algebraic rules:

$$L_s(g\otimes f)=L_sg\otimes L_sf \text{ for all } s\in\mathbb{R},\ f \text{ and } g \text{ in } \mathscr{S},$$

$$L_s\Theta f=\Theta L_{-s}f \text{ for all } s\in\mathbb{R},\ f \text{ in } \mathscr{S}.$$

The map from $\mathscr{S}_+\times\mathscr{S}_+$ into $\mathbb{C}$ defined by

$$(g,f)\to S(\Theta g\otimes f)$$

is continuous, sesquilinear, and by $S4$), positive in the sense that

$$S(\Theta f\otimes f)\geqq 0 \quad \text{for all} \quad f\in\mathscr{S}_+.$$

Let $\mathscr{N}$ be the subspace of all $f\in\mathscr{S}_+$ such that for all $g\in\mathscr{S}_+$,

$$S(\Theta f\otimes g)=0.$$

Because of the Schwarz inequality

$$|S(\Theta f\otimes g)|^2\leqq S(\Theta f\otimes f)S(\Theta g\otimes g),$$

so that

$$\mathscr{N}=\{f\in\mathscr{S}_+:S(\Theta f\otimes f)=0\}.$$

For all $s\in\mathbb{R}$, f and g in $\mathscr{S}$ we have

$$S(\Theta g\otimes L_sf)=S(L_{-s}(\Theta g\otimes L_sf))=S(L_{-s}\Theta g\otimes f)=S(\Theta L_sg\otimes f). \tag{37}$$

Hence for $s\geqq 0$, $L_s\mathscr{N}\subset\mathscr{N}$.

The space $\mathscr{S}_+/\mathscr{N}$ is a separated pre-Hilbert space and can be completed into a Hilbert space $\mathscr{H}$. We denote Ψ the canonical map of $\mathscr{S}_+$ into $\mathscr{H}$, and, in particular, $\Psi(\{f_0=1,\ldots,f_n=0,\ldots\})=\Omega$. By definition $\Psi(\mathscr{S}_+)$ is dense in $\mathscr{H}$ and for f,g in $\mathscr{S}_+$:

$$(\Psi(g),\Psi(f))=S(\Theta g\otimes f).$$

The map Ψ is continuous from $\mathscr{S}^+$ to $\mathscr{H}$. Since $L_t\mathscr{N}\subset\mathscr{N}$ for any $t\geqq 0$, we can define, for each $t\geqq 0$, an operator P_t on $\Psi(\mathscr{S}_+)$ with the properties

$$P_t\Psi(g)=\Psi(L_tg) \quad \text{for all} \quad g\in\mathscr{S}_+,$$

$$P_{t+s}=P_tP_s \quad \text{for all} \quad s\geqq 0, t\geqq 0,$$

$$(\Psi(g),P_t\Psi(f))=(P_t\Psi(g),\Psi(f)) \quad (\text{all } t\geqq 0, f \text{ and } g \text{ in } \mathscr{S}_+),$$

$$\|P_t\Psi(f)\|\leqq(1+t)^{B(f)}C(f) \quad (\text{all } t\geqq 0, f\in\mathscr{S}_+).$$

Moreover $t\to P_t\Psi(f)$ is a continuous map of $[0,\infty)$ into $\mathscr{H}$ for every fixed $f\in\mathscr{S}_+$.

Following Osterwalder and Schrader we conclude that, for all $t\geqq 0$ and all $f\in\mathscr{S}_+$,

$$(\Psi(f),P_t\Psi(f))\leqq\|\Psi(f)\|(\Psi(f),P_{2t}\Psi(f))^{\frac{1}{2}}$$

$$\leqq\ldots\leqq\|\Psi(f)\|^{1+\frac{1}{2}+\ldots+2^{-r}}(\Psi(f),P_{2^{r+1}t}\Psi(f))^{2^{-r-1}},$$

and, passing to the limit:

$$(\Psi(f), P_t\Psi(f)) \leqq \|\Psi(f)\|^2.$$

It follows that P_t can be extended by continuity to all of $\mathscr{H}$ and defines a continuous contraction semi-group. In particular

$$P_t = e^{-tH}$$

where H is a positive self adjoint operator whose domain contains $\Psi(\mathscr{S}_+)$. (In fact $\Psi(\mathscr{S}_+)$ contains all vectors of the form

$$e^{-tH}\Psi(g) = \Psi(L_t g), \qquad t>0.$$

These are a dense set of analytic vectors for H, and thus a core for H.) In particular

$$P_t\Omega = \Omega, \quad H\Omega = 0.$$

If Φ_1 and Φ_2 are vectors in $\mathscr{H}$, we can define

$$(\Phi_1, e^{i(u+iv)H}\Phi_2)$$

provided $v \geqq 0$; this is a holomorphic function of $u+iv$ for $v>0$, continuous for $v=0$; if Φ_2 (or Φ_1) is in $\Psi(\mathscr{S}_+)$ this function is even C^∞ in the closed upper half plane. We also note that for $t \geqq 0$, e^{-tH} is invertible, its inverse e^{tH} having as its domain precisely $e^{-tH}\mathscr{H}$.

Let $g \in \mathscr{S}$. The projected support of g, denoted Proj. supp. g is the closed subset of $\mathbb{R}$ defined as the closure of

$$\{t \in \mathbb{R} : \exists m>0, \exists (y_1, \ldots, y_m) \in \operatorname{supp}. g_m, \exists 1 \leqq j \leqq m \text{ with } y_j^0 = t\}$$

For any $t \in \mathbb{R}$, the condition $g \in L_t\mathscr{S}_+$ is equivalent to

$$t \leqq \inf \text{Proj. supp.}\, g$$

and the condition $\Theta g \in L_{-t}\mathscr{S}_+$ is equivalent to

$$\sup \text{Proj. supp.}\, g \leqq t.$$

Let f, g, and h belong to $\mathscr{S}_+$, with

$$0 < \text{supp. Proj. supp.}\, g = T < \infty.$$

For any $s \geqq T$,

$$\begin{aligned} S(\Theta h \otimes g \otimes L_s f) &= S(L_{-s}(\Theta h \otimes g \otimes L_s f)) \\ &= S(L_{-s}\Theta h \otimes L_{-s} g \otimes f) = S(\Theta L_s h \otimes L_{-s} g \otimes f) \\ &= S(\Theta(\Theta L_{-s} g \otimes L_s h) \otimes f) = S(\Theta(L_s \Theta g \otimes L_s h) \otimes f). \end{aligned}$$

This shows that, for fixed g and s, the map from $\mathscr{S}_+$ to $\mathscr{S}_+$

$$f \rightarrow g \otimes L_s f$$

maps $\mathscr{N}$ into itself. We therefore *define* an operator $\Psi(\mathscr{S}_+) \rightarrow \Psi(\mathscr{S}_+)$ by:

$$\mathcal{O}_s(g)\Psi(f) = \Psi(g \otimes L_s f). \tag{38}$$

Note that the right hand side is a continuous function of (s, g, f); it is even $\mathscr{C}^\infty$ in s for $s \geqq T$. Furthermore the adjoint of this operator also maps $\Psi(\mathscr{S}_+)$ into itself. It is given, on $\Psi(\mathscr{S}_+)$, by

$$\mathcal{O}_s(g)^* \Psi(h) = \Psi(L_s \Theta g \otimes L_s h) = \mathcal{O}_s(L_s \Theta g)\Psi(h).$$

Clearly, for $t \geqq 0$,

$$\mathcal{O}_s(g) e^{-tH} = \mathcal{O}_{s+t}(g) \quad \text{on} \quad \Psi(\mathscr{S}_+).$$

It is possible to write $\mathcal{O}_s(g) = \mathcal{O}(g) e^{-sH}$, where $\mathcal{O}(g)$ is defined on $e^{-sH}\Psi(\mathscr{S}_+)$. Indeed $e^{-sH}\Psi(\mathscr{S}_+) = \Psi(L_s \mathscr{S}_+)$.

Any $u \in L_s \mathscr{S}_+$ can be written uniquely as $L_s v$, $v \in \mathscr{S}_+$ and $u \in \mathscr{N} \Leftrightarrow \Psi(u) = 0 \Leftrightarrow \forall g$, $(\Psi(g), e^{-sH}\Psi(v)) = 0 \Rightarrow \Psi(v) = 0$. Thus $u \in \mathscr{N} \Leftrightarrow v \in \mathscr{N}$ and we can define

$$\mathcal{O}(g)\Psi(u) = \mathcal{O}_s(g)\Psi(v) = \Psi(g \otimes u). \tag{39}$$

We shall verify that these operators satisfy the properties $\mathcal{O}1$)–$\mathcal{O}6$), with D_1 replaced by a new domain D_2 to be defined later.

It will be convenient to use the notation

$$\int \mathcal{O}(y_1, \ldots, y_n) f(y_1, \ldots, y_n) dy_1 \ldots dy_n$$

for $\mathcal{O}(g)$ when $g = \{0, \ldots, f, 0, \ldots\}$, and similarly

$$\mathcal{O}_s(y_1, \ldots, y_n) = \mathcal{O}(y_1, \ldots, y_n) e^{-sH}$$

The construction of these operators does not depend on the symmetry property $S2$). If $S2$) holds, for every permutation π of $(1, \ldots, n)$,

$$\mathcal{O}(y_1, \ldots, y_n) = \mathcal{O}(y_{\pi 1}, \ldots, y_{\pi n}), \text{ (this is } \mathcal{O}3)).$$

This will allow us to use the abbreviated notation $\mathcal{O}(Y)$ as before.

It is also clearly possible to define,

$$G_f(iv_0, \ldots, iv_{r-1}) = \int e^{-v_0 H} \mathcal{O}(Y_1) e^{-v_1 H} \mathcal{O}(Y_2) \ldots e^{-v_{r-1} H} \mathcal{O}(Y_r) \Omega \\ f(y_1, \ldots, y_N) dy_1 \ldots dy_N. \tag{40}$$

Here $v_0, \ldots, v_{r-1}$, are all $\geqq 0$; $Y_1 = \{1, \ldots, n_1\}$, $Y_2 = \{n_1 + 1, \ldots, n_2\}, \ldots, Y_r = \{n_{r-1} + 1, \ldots, n_r = N\}$, are disjoint non empty subsets of $\{1, \ldots, N\}$ with union $\{1, \ldots, N\}$. $f \in \mathscr{S}(\mathbb{R}^{\nu n})$ has its support in

$$\{(y_1, \ldots, y_N) \in \mathbb{R}^{\nu N} : y_j^0 \leqq y_k^0 \quad \text{if} \quad j \in Y_t \quad \text{and} \quad k \in Y_u \quad \text{with} \quad t < u\}. \tag{41}$$

The precise definition of (40) is $\Psi(g)$ where $g_p = \delta_{pN} g_N$ and

$$g_N(y_1, \ldots, y_N) = f((y_1^0 - v_0, \mathbf{y}_1), \ldots, (y_{n_1}^0 - v_0, \mathbf{y}_{n_1}), \\ (y_{n_1+1}^0 - v_0 - v_1, \mathbf{y}_{n_1+1}), \ldots, (y_N^0 - v_0 - \ldots - v_{r-1}, \mathbf{y}_N)) \tag{42}$$

It will prove convenient to use an adaptation of the method of [OS2] for regularizing the behavior of $S_n(x_1, \ldots, x_n)$ at large distances, based on the following remark.

If $\{S_n'\}$ and $\{S_n''\}$ are two systems of distributions satisfying $S1$), $S3$), $S4$), and if it is possible to define the pointwise product $S_n'''(x_1, \ldots, x_n) = S_n'(x_1, \ldots, x_n) S_n''(x_1, \ldots, x_n)$, (e.g. as a limit of $S'_n \cdot (S''_n * \varrho)$ as $\varrho \to \delta$), then $\{S'''_n\}$ also satisfies $S1$), $S3$), $S4$). Indeed, for every finite sequence $\{f_n\}$ such that $f_n \in \mathcal{S}(\mathbb{R}^{vn})$ and $\operatorname{supp} f_n \subset \{(x_1, \ldots, x_n) | 0 \leqq x_j^0, 1 \leqq j \leqq n\}$,

$$\sum_{n,m} S'''_{n+m}(\Theta f_m \otimes f_n) = \sum_{m,n} \int ([\mathcal{O}'(x_1, \ldots, x_m) \otimes \mathcal{O}''(x_1, \ldots, x_m)] \Omega' \otimes \Omega'', \\ [\mathcal{O}'(y_1, \ldots, y_n) \otimes \mathcal{O}''(y_1, \ldots, y_n)] \Omega' \otimes \Omega'') \overline{f_m(x_1, \ldots, x_m)} f_n(y_1, \ldots, y_n) \\ dx_1 \ldots dx_m dy_1 \ldots dy_n .$$

An example $\{\Xi_n\}$ of a sequence of distributions satisfying $S1$), $S3$), and almost $S4$) is given by

$$\Xi_N(x_1, \ldots, x_N) = F_N(x_1^0 - x_N^0), \quad N \geqq 2,$$

where, for every real t,

$$F_N(t) = \int_0^\infty e^{-p|t|} h(p)^N \sigma(p) dp,$$

and where h and σ are real functions, $\sigma \geqq 0$, so chosen that the above integral is absolutely convergent and continuous in t on $[0, \infty)$.

Indeed one finds, for all finite sequences $\{f_n\}$ as above, with $f_0 = 0$,

$$\sum_{m,n} \Xi_{m+n}(\Theta f_m \otimes f_n) = \sum_{m,n} \int_0^\infty dp \sigma(p) \overline{c_m(p)} c_n(p) = \int_0^\infty dp \sigma(p) \left| \sum_n c_n(p) \right|^2,$$

where

$$c_n(p) = \int e^{-px_n^0} h(p)^n f_n(x_1, \ldots, x_n) \, dx_1 \ldots dx_n .$$

More specifically, we choose $\sigma(p) = e^{-p}$, $h(p) = p^R$, $R \geqq 0$, so that

$$F_N(t) = (RN)!(|t| + 1)^{-RN-1} .$$

With this choice, we denote $\mathcal{H}_0$, $\Psi'_0(f)$, $\mathcal{O}_0(x_1, \ldots, x_n) \equiv \mathcal{O}_0(x_1^0, \ldots, x_n^0)$, H_0 the objects obtained from $\{\Xi_n\}$ by the preceding construction.

We return to the original sequence $\{S_n\}$ satisfying $S1$), ..., $S4$) and the growth condition $S6$). For every $N > 1$ and $f \in \mathcal{S}(\mathbb{R}^{vn_j})$, $n_j \geqq 1$, $\operatorname{Proj.supp.} f_j \leqq \operatorname{Proj.supp.} f_{j+1}$, $(j = 1, \ldots, r-1)$, and $(v_1, \ldots, v_{r-1}) \in [0, \infty)^{r-1}$, we denote:

$$S_N(f_1, \ldots, f_r; iv_1, \ldots, iv_{r-1}) = S_N(f_1 \otimes L_{v_1} f_2 \otimes \ldots \otimes L_{v_1 + \ldots + v_{r-1}} f_r)$$

and

$$\hat{S}_N(f_1, \ldots, f_r; iv_1, \ldots, iv_{r-1}) = \int S_N(x_1, \ldots, x_N) F_N(y_1 - y_r) \\ (f_1 \otimes L_{v_1} f_2 \otimes \ldots \otimes L_{v_1 + \ldots + v_{r-1}} f_r)(x_1, \ldots, x_N) \, dx_1 \ldots dx_N , \tag{43}$$

where $y_j = n_j^{-1} (x^0_{n_1 + \ldots + n_{j-1} + 1} + \ldots + x^0_{n_1 + \ldots + n_j})$, $\quad 1 \leqq j \leqq r$.

If, in particular $0 \leqq \operatorname{Proj.supp.} f_1 \leqq T_1$, the expression (43) is equal to

$$(\Psi'(L_{T_1} \Theta f_1), e^{-v_1 H'} \mathcal{O}'(L_{-T_1} f_2) \ldots e^{-v_{r-1} H'} \Psi'(L_{-T_1} f_r)),$$

with $H' = H \otimes 1 + 1 \otimes H_0$, and

$$\mathcal{O}'(f) = \int \mathcal{O}(x_1, \ldots, x_n) \otimes \mathcal{O}_0 \left(\frac{1}{n} \sum_{j=1}^n x_j^0 \right) f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n ,$$

and similarly, for $\Psi'(f)$.

We shall restrict our attention to the case when all n_j are bounded by a fixed integer Q. In that case it follows from the growth condition S6) and from a repeated use of the theorem in [OS2, appendix], that there exist constants C, K, S, independent of N, such that, for all $N>1$, $r>1$, $(N \leqq Qr)$, $f_1, \ldots, f_r$, (with supports as above),

$$|\hat{S}_N(f_1, \ldots, f_r; 0, \ldots, 0)| \leqq C^N (N!)^K \Big(\prod_{1<j<r} |f_j|_S \Big) \left| \frac{f_1 \otimes f_r}{(1+|y_1-y_r|)^{NR+1}} \right|_S$$

Since the l.h.s. is invariant under simultaneous translations of all arguments, we can evaluate it by first performing a time translation such that Proj. supp. $f_1 \leqq 0 \leqq$ Proj. supp. f_r. In that situation, the r.h.s. can be rewritten as

$$C^N (N!)^K \sup_{\substack{x \\ |\alpha_j| \leqq S \\ |\beta_j| \leqq S \\ (\alpha_j \text{ restricted})}} \left[\prod_{1<j<r} |x^{\alpha_j} D^{\beta_j} f_j(x)| \right] \left| x^{\alpha_1} x^{\alpha_r} D^{\beta_1} D^{\beta_r} \frac{(f_1 \otimes f_r)(x)}{(1+y_r-y_1)^{NR+1}} \right| .$$

Since $|D^{\beta_1} D^{\beta_r} (1+y_r-y_1)^{-NR-1}| \leqq (NR+2S)^{2S} (1+y_r-y_1)^{-NR-1}$ and since

$$Qy_1 \leqq x_j^0 \leqq Qy_r, \quad \text{i.e.} \quad |x_j^0| \leqq Q(1+y_r-y_1),$$

if we choose $R=S$, there are new constants independent of N, such that

$$|\hat{S}_N(f_1, \ldots, f_r; 0, \ldots, 0)| \leqq C'^N (N!)^{K'} \prod_{j=1}^{r} \sup_{\substack{x \\ |\alpha| \leqq S \\ |\beta| \leqq S}} |\mathbf{x}^\alpha D^\beta f_j(x)| .$$

Since these new norms are invariant under time translations, this inequality remains valid when the f_j are (time-) translated back to their original position, and, for all $v_j \geqq 0$, $(1 \leqq j \leqq r-1)$,

$$|\hat{S}_N(f_1, \ldots, f_r; v_1, \ldots, v_{r-1})| \leqq C'_N (N!)^{K'} \prod_{j=1}^{r} \wr f_j \wr_S ,$$

$$\wr f \wr_S = \sup_{\substack{x \\ |\alpha| \leqq S \\ |\beta| \leqq S}} |\mathbf{x}^\alpha D^\beta f_j(x)| .$$

This bound and the positivity which $\{\hat{S}_N\}$ inherits from $\{S_N\}$ allow us to follow the method of [G1, OS2] and to obtain an analytic function

$$(w_1, \ldots, w_{r-1}) \rightarrow \hat{S}_N(f_1, \ldots, f_r; w_1, \ldots, w_{r-1})$$

continuing $\hat{S}_N(f_1, \ldots, f_r; iv_1, \ldots, iv_{r-1})$ in the topological product of $r-1$ upper half planes.

To do this in a systematic way we consider a sequence $f_1, \ldots, f_r$ such that $0 \leqq T_{j-1} \leqq$ Proj. supp. $f_j \leqq T_j$. Then

$$\begin{aligned} &(\Psi'(g), e^{-v_0 H'} \mathcal{O}'(f_1) e^{-v_1 H'} \mathcal{O}'(f_2) \ldots e^{-v_{r-1} H'} \Psi'(f_{r-1})) \\ &\quad = (\mathcal{O}'(L_{T_j} \Theta f_j) e^{-v_{j-1} H'} \ldots e^{-v_1 H'} \mathcal{O}'(L_{T_j} \Theta f_1) e^{-(v_0+T_j) H'} \Psi'(g), \qquad (44) \\ &\quad e^{-v_j H'} \mathcal{O}'(L_{-T_j} f_{j+1}) \ldots e^{-v_{r-1} H'} \Psi'(L_{-T_j} f_r)) . \end{aligned}$$

The r.h.s. continues to make sense if $e^{-v_jH'}$ is replaced by $e^{iw_jH'}$ with $\operatorname{Im} w_j > 0$. Thus the l.h.s. is analytically continuable in each v_j while the others are kept fixed so that as a function of $iv_0, \ldots, iv_{r-1}$, it has an analytic continuation in

$$C_r^{(1)} = \{(w_0, \ldots, w_{r-1}) \,|\, |\mathrm{Arg} - iw_j| < \Theta_j, \textstyle\sum \Theta_j \leqq \frac{\pi}{2}\}.$$

Moreover the methods of [G1, OS2] show that in a smaller domain $D_r^{(1)}$ it is bounded by Const. $\|\Psi'(g)\|$ and thus it defines a vector valued holomorphic function on $D_r^{(1)}$ denoted

$$G'_{f_1 \otimes \ldots \otimes f_r}(w_0, \ldots, w_{r-1}) = e^{iw_0H'}\mathcal{O}'(f_1)\ldots e^{iw_{r-1}H'}\Psi'(f_{r-1})$$

The formula (44) can be analytically continued to:

$$\begin{aligned}&(\Psi'(g), e^{iw_0H'}\mathcal{O}'(f_1)\ldots e^{iw_{r-1}H'}\Psi'(f_r))\\ &\quad = (\mathcal{O}'(L_{T_j}\Theta f_j)e^{-i\bar{w}_{j-1}H'}\mathcal{O}'(L_{T_j}\Theta f_{j-1})\ldots e^{-i\bar{w}_1H'}\mathcal{O}'(L_{T_j}\Theta f_1)e^{-(i\bar{w}_0+T_j)H'}\Psi'(g),\\ &\quad\quad e^{iw_jH'}\mathcal{O}'(L_{-T_j}f_{j+1})\ldots e^{iw_{r-1}H'}\Psi'(L_{-T_j}f_r))\end{aligned}$$

(Here $\Psi'(g)$ is also supposed to be of the form $\mathcal{O}'(g_0)e^{-s_1H'}\ldots e^{-s_pH'}\Psi'(g_p)$). Hence one can iterate the procedure and eventually obtain the analyticity of the vector G' in the topological product of r upper half planes. To obtain bounds on its norm in this domain, it is necessary to apply the Schwarz inequality at each iteration. This involves, as in [OS2] a doubling of the number of variables and also a doubling of the number of f_j's which also have to be time-translated. However, since the norms $\wr f_j \wr_S$ occurring in the initial bounds are invariant under these time-translations, the result is the same as in [OS2]:

The vector $G'_{f_1 \otimes \ldots \otimes f_n}(w_0, \ldots, w_{r-1})$ is analytic in the product of upper half-planes and is bounded there by

$$\|G'_{f_1 \otimes \ldots \otimes f_r}(w_0, \ldots, w_{r-1})\| < B'_N \prod_{j=1}^{r} \wr f_j \wr_S (1+|w|)^{K'_N} \left(\prod_{j=1}^{r-1} v_j^{-K'_N}\right),$$

where $|w| = \sum_{j=0}^{r-1} |w_j|$ and $v_j = \operatorname{Im} w_j$. Using again [OS2, Appendix] this implies for every f with support in (41)

$$\|G'_f(w_0, \ldots, w_{r-1})\| < B''_N \wr f \wr_{S'(N)} (1+|w|)^{K'_N} \left(\prod_{j=1}^{r-1} v_j^{-K'_N}\right).$$

Going back to the original functions (cf. (40)), we get:

There are constants K_N and B_N such that, for all $(w_0, \ldots, w_{r-1})$ with $w_j = u_j + iv_j$, $v_j > 0$

$$\|G_f(w_0, \ldots, w_{r-1})\| < B_N |f|_{K_N} \left(\sup_{a \leqq j \leqq r-1} v_j^{-K_N}\right)(1+|w|)^{K_N},$$

However if f is sufficiently regular, differentiation in $w_0, \ldots, w_r$ can be transferred to f so that

$$\|D_w^\alpha G_f(w_0, \ldots, w_{r-1})\| < A'_N |f|_{K_N+|\alpha|} \left(\sup_j v_j^{-K_N}\right)(1+|w|)^{K_N},$$

and, reintegrating K_N+1 times we find new constants such that

$$\|D_w^\alpha G_f(w_0, \ldots, w_{r-1})\| < A_N |f|_{2K_N+|\alpha|+1}(1+|w|)^{K_N}.$$

(In particular if $f \in \mathscr{S}$, G is $\mathscr{C}^\infty$ in $w_0, \ldots, w_{r-1}$ in the topological product of r closed upper half planes). The vector $G_f(w_0, \ldots, w_{r-1})$ will also be denoted

$$\int e^{iw_0 H}\mathcal{O}(Y_1)e^{iw_1 H}\ldots e^{iw_{r-1}H}\mathcal{O}(Y_r)\Omega \quad f(y_1, \ldots, y_N)dy_1\ldots dy_N.$$

This notation is justified since it is easy to show that this vector is in the domain of the closure of $\mathcal{O}(g)$ (provided $g \otimes f$ is sufficiently smooth and has the correct support), and that we have ($\mathcal{O}(g)$ being identified with its closure, as we shall always do in the sequel),

$$\begin{aligned} &e^{iwH}\mathcal{O}(g)\{\int e^{iw_0 H}\mathcal{O}(Y_1)\ldots e^{iw_{r-1}H}\mathcal{O}(Y_r)\Omega\, f(y_1, \ldots, y_N)dy_1\ldots dy_N\} \\ &\quad = \int e^{iwH}\mathcal{O}(\xi_1, \ldots, \xi_R)e^{iw_0 H}\mathcal{O}(Y_1)\ldots e^{iw_{r-1}}\mathcal{O}(Y_r)\Omega \\ &\qquad g_R(\xi_1, \ldots, \xi_R)\, f(y_1, \ldots, y_N)d\xi_1\ldots d\xi_R\, dy_1\ldots dy_N. \end{aligned}$$

Also in the domain of the closure of $\mathcal{O}(g)$ are the vectors obtained from the preceding by integrating over the w_j along paths with suitable test-functions etc. The intersection of the domains of the closures of all finite products $\mathcal{O}(f_1)e^{iw_1 H}\ldots e^{iw_{r-1}H}\mathcal{O}(f_r)$ is denoted D_2. It is straightforward to verify that the system of operators $\mathcal{O}$ thus constructed satisfies the requirements $\mathcal{O}1$)–$\mathcal{O}6$) with the domain D_2.

III. 2. Inductive Construction of $\bar{T}(X)$

Starting from the operators $\mathcal{O}(X)$ obtained in III.1, the construction of the $\bar{T}(X)$ will be carried out by induction on $|X|$. In fact, the requirements of causal factorization $T5$), translational invariance $T7$) and the relation R') will leave no freedom in this construction. The procedure very closely parallels that of II.1.

The induction hypothesis postulates that the operators $\bar{T}(X)$, for $|X| \leqq n-1$, have already been constructed so as to satisfy the above requirements, and that any finite product of such operators can be applied to vectors of the domain D_2, which it maps into itself. (The domain D_2 has been defined at the end of III.1).

Let $X = \{1, \ldots, n\}$ and $Z = Z_1 \cup \ldots \cup Z_r = \{n+1, \ldots, n+p\}$. Denote $x = (x_1, \ldots, x_n)$, $z = (z_{n+1}, \ldots, z_{n+p})$. For any $f \in \mathscr{S}(\mathbb{R}^{\nu(n+p)+1})$ with support in

$$\{(x, v, z) | \text{for every } j \in Z_a, k \in Z_b, \text{with } a < b, z_j^0 \leqq z_k^0\}, \tag{45}$$

for every $(w_0, \ldots, w_{r-1}) \in \mathbb{C}^r$ with $\operatorname{Im} w_j \geqq 0$ for all j, we propose to define the vector:

$$\begin{aligned} &\int f(x_1, \ldots, x_n, v, z_{n+1}, \ldots, z_{n+p})\,\bar{T}(X)e^{i(w_0+v)H}\mathcal{O}(Z_1) \\ &\quad e^{iw_1 H}\ldots e^{iw_{r-1}H}\mathcal{O}(Z_r)\Omega\, dx_1\ldots dx_n\, dv\, dz_{n+1}\ldots dz_{n+p}. \end{aligned} \tag{46}$$

We concentrate on the case $n > 1$, $Z \neq \emptyset$. The other cases are straightforward simplifications. Let $\mathscr{F}$ denote the subspace of $\mathscr{S}(\mathbb{R}^{\nu(n+p)+1})$ consisting of functions having their support in (45). Denote, as in II.1,

$$y = n^{-1}(x_1^0 + \ldots + x_n^0), \quad \xi_j = x_j^0 - y, \quad (1 \leqq j \leqq n), \quad \xi = (\xi_1, \ldots, \xi_{n-1})$$

and define $\mathscr{F}_0$ as the space of functions $f \in \mathscr{F}$ vanishing in a neighborhood of

$$\{(x_1, \ldots, x_n, v, z) : \xi = 0\}.$$

Any function $f \in \mathscr{F}$ can be written in the form:

$$f(x_1, \ldots, x_n, v, z) = \int_{\mathbb{R}^2} da\,db\,\varphi(\xi, \mathbf{x}, a, b, z) \quad \alpha_L(y-a)\alpha_L(v-2y-b+2a). \tag{47}$$

Here, α_L is the function defined by (13) and, as previously, we find φ is given by

$$\varphi(\xi, \mathbf{x}, y, v, z) = (L!)^{-2}\left(1 - i + \frac{\partial}{\partial y} + 2\frac{\partial}{\partial v}\right)^{L+1}\left(1 - i + \frac{\partial}{\partial v}\right)^{L+1}$$
$$f((\xi_1 + y, \mathbf{x}_1), \ldots, (\xi_n + y, \mathbf{x}_n), v, z). \tag{48}$$

The constant L will be chosen later.

Just as in II.1, we decompose f into $f = f_0 + f_1$, with

$$f_1(x, v, z) = \int da\,db\,\alpha_L(y-a)\alpha_L(v-2y-b+2a)$$
$$\cdot\left[\sum_{|\beta|=0}^{M+1} \frac{\xi^\beta}{\beta!} D_\xi^\beta \varphi(0, \mathbf{x}, a, b, z)\right] w(\xi, y-a), \tag{49}$$

where w is the auxiliary function used in II.1. and $M \geqq 0$ will be chosen later. Again,

$$|f_1|_M \leqq \text{Const.}\,|f|_{2M+2L+3}, \qquad (L \geqq M+1).$$

f_0 has the same property and vanishes together with its derivatives of order $\leqq M+1$ when $\xi = 0$. Hence it is in the closure of $\mathscr{F}_0$ in the norm $|\,|_M$.

For any g in $\mathscr{F}_0$ the vector (46) with f replaced by g is well defined by virtue of the induction hypothesis and the requirement of causal factorization. The proof is identical to the corresponding one in II.1. The resulting vector is bounded in norm by:

$$\text{Const.}\,|g|_K(1 + |w|)^K, \qquad |w| = \sum_j |w_j|.$$

As a consequence, if we choose $M \geqq K$, the vector obtained by replacing f by f_0 in (46) is well-defined and bounded in norm by Const. $|f|_{2M+2L+3}(1+|w|)^M$. On the other hand, if we denote

$$\varphi_{0,\lambda}(\xi, y, v, \mathbf{x}, a, b, z) = \left[\sum_{|\beta|=0}^{M+1} \frac{(\lambda\xi)^\beta}{\beta!} D_\xi^\beta \varphi(0, \mathbf{x}, a, b, z)\right]$$
$$\cdot w(\xi, y)\alpha_{\lambda, L}(y)\alpha_{\lambda, L}(v-2y),$$

translational invariance requires, for sufficiently large M and L

$$\int \bar{T}(X) e^{i(w_0+v)H} \mathcal{O}(Z_1) e^{iw_1 H} \ldots e^{iw_{r-1}H} \mathcal{O}(Z_r)\Omega$$
$$\varphi_{0,1}(\xi, y-a, v-b, \mathbf{x}, a, b, z)\, dx\, dv\, dz$$
$$= \int e^{iaH} \bar{T}(X) e^{i(w_0+v+b-a)H} \mathcal{O}(Z_1) e^{iw_1 H} \ldots e^{iw_{r-1}H} \mathcal{O}(Z_r)\Omega$$
$$n\varphi_{0,1}(\xi, y, v, \mathbf{x}, a, b, z)\, dv\, d\xi\, dy\, d\mathbf{x}\, dz. \tag{50}$$

The radial analyticity of $\varphi_{0,1}$ and the condition R') require that (50) should be equal to:

$$\int e^{iaH}\mathcal{O}(X)e^{-vH}e^{i(w_0+b-a)H}\mathcal{O}(Z_1)\ldots e^{iw_{r-1}H}\mathcal{O}(Z_r)\Omega \\ ni^{n+1}\varphi_{0,i}(\xi,y,v,\mathbf{x},a,b,z)\,dv\,d\xi\,dy\,d\mathbf{x}\,dz\,. \tag{51}$$

This is a well defined vector, depending continuously on a and b, and bounded in norm by

$$\text{Const.}\,(1+|a|+|b|)^{-3}\,|f|_{3M+2L+6}(1+|w|)^M$$

We can integrate over a and b and define:

$$\begin{aligned}&\int \bar{T}(X)e^{i(w_0+v)H}\mathcal{O}(Z_1)e^{iw_1H}\ldots\mathcal{O}(Z_r)\Omega\, f_1(x,v,z)\,dx\,dv\,dz\\ &=\int da\,db\,e^{iaH}\mathcal{O}(X)e^{(-v+iw_0+ib-ia)H}\mathcal{O}(Z_1)e^{iw_1H}\ldots\mathcal{O}(Z_r)\Omega\\ &\quad n\varphi_{0,i}(\xi,y,v,\mathbf{x},a,b,z)i^{n+1}\,d\xi\,dy\,dv\,d\mathbf{x}\,dz\end{aligned} \tag{52}$$

The definition of (46) is now the sum of (52) and of the known result for f_0; L must be $\geqq M+1$, and the resulting vector is bounded in norm by Const. $|f|_{3M+2L+6}(1+|w|)^M$.

It is now easy to check that, if f happens to be radially analytic in the variables $x_1^0,\ldots,x_n^0,v$ and has support in $\{x,v,z:\forall j\in X, 0\leqq x_j^0\leqq v\}$, this definition is such that R') is satisfied. We only give a brief sketch of this verification.

Suppose that $f\in\mathcal{F}$ is of the form (47) and

$$f_\lambda(x,v,z)=f((\lambda x_1^0,\mathbf{x}_1),\ldots,(\lambda x_n^0,\mathbf{x}_n),\lambda v,z),\qquad(\lambda>0)$$

can be analytically continued in λ in the angle $0<\arg\lambda<\frac{\pi}{2}$, with continuity at the boundary. Then the same is true (by virtue of (48)) of

$$\varphi_\lambda(\xi,\mathbf{x},y,v,z)=\varphi(\lambda\xi,\mathbf{x},\lambda y,\lambda v,z),$$

(which also has support in $\{(x_1,\ldots,x_n,v,z)|\forall j\in X, 0\leqq x_j^0\leqq v\}$), and of

$$\begin{aligned}f_{1,\lambda}(x,v,z)&=\lambda^2\int da\,db\,\alpha_{\lambda,L}(y-a)\alpha_{\lambda,L}(v-2y-b+2a)\\ &\quad\cdot\left[\sum_{|\beta|=0}^{M+1}\frac{(\lambda\xi)^\beta}{\beta!}D_\xi^\beta\varphi(0,\mathbf{x},\lambda a,\lambda b,z)\right]w(\xi,y-a)\,.\end{aligned}$$

As a consequence the l.h.s. of (52) should be equal to

$$\begin{aligned}&\int\mathcal{O}(X)e^{(iw_0-v)H}\mathcal{O}(Z_1)\ldots\mathcal{O}(Z_r)\Omega f_{1,i}(x,v,z)i^{n+1}\,dx\,dv\,dz\\ &=-\int da\,db\,e^{-aH}\mathcal{O}(X)e^{(iw_0-v-b+a)H}\mathcal{O}(Z_1)\ldots\mathcal{O}(Z_r)\Omega\\ &i^{n+1}n\alpha_{i,L}(y)\alpha_{i,L}(v-2y)\left[\sum_{|\beta|=0}^{M+1}\frac{(i\xi)^\beta}{\beta!}D_\xi^\beta\varphi(0,\mathbf{x},ia,ib,z)\right]w(\xi,y)\,d\xi\,d\mathbf{x}\,dy\,dz\,dv\,.\end{aligned}$$

Noting that $D_\xi^\beta\varphi(0,\mathbf{x},\lambda a,\lambda b,z)$ can be analytically continued in λ (and has support in $\{b-a\geqq 0\}$), we can rotate the contours in a and b and obtain exactly (52). As to the part of (45) corresponding to f_0, it satisfies the contour-rotation condition R') by virtue of the induction hypothesis.

Having defined expressions of the form (45), we can now suppress the integration over v by the same method as in II.1 i.e. by essentially using

$$\psi = \int \alpha_Q(v) e^{ivH} (Q!)^{-1} (1 - i - iH)^{Q+1} \psi \, dv$$

and

$$H\psi = \left(-\frac{d}{dt}\right) e^{-tH} \psi|_{t=0}.$$

This allows us to use the definition

$$\begin{aligned} &\int \bar{T}(X) e^{iw_0 H} \mathcal{O}(Z_1) e^{iw_1 H} \ldots e^{iw_{r-1} H} \mathcal{O}(Z_r) \Omega \\ &\quad g(x_1, \ldots, x_n, z_{n+1}, \ldots, z_{n+p}) dx_1 \ldots dx_n \, dz_{n+1} \ldots dz_{n+p} \\ &= \int \bar{T}(X) e^{i(w_0 + v)H} \mathcal{O}(Z_1) e^{iw_1 H} \ldots e^{iw_{r-1} H} \mathcal{O}(Z_r) \Omega \\ &\quad K_Q g(x, z) \alpha_Q(v) dv \, dx_1 \ldots dx_n dz_{n+1} \ldots dz_{n+p}. \end{aligned} \tag{53}$$

$$K_Q g(x, z) = (Q!)^{-1} \left(1 - i + i\frac{\partial}{\partial t}\right)^{Q+1} g(x, z_t)|_{t=0},$$

where z_t is given by: $\mathbf{z}_{t,j} = \mathbf{z}_j$, $z_{t,j}^0 = z_j^0 - t$, $(j = n+1, \ldots, n+p)$. We again omit straightforward verifications.

III.3. Poincaré Covariance

Up to this point, only the invariance under time-translations of the S_n has been used. If the S_n are invariant under space and time translations, it is clear that the generalized (anti-)time ordered functions have the same property.

Let $\hat{m}^{0\mu}$, $\mathbf{m}^{0\mu}$, p_μ, $(\mu = 1, \ldots, \nu - 1)$, be the differential operators defined on $\mathcal{S}$ by

$$(\hat{m}^{0\mu} f)_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} \left(x_j^\mu \frac{\partial}{\partial x_j^0} - x_j^0 \frac{\partial}{\partial x_j^\mu}\right) f_n(x_1, \ldots, x_n),$$

$$(\mathbf{m}^{0\mu} f)_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} \left(x_j^\mu \frac{\partial}{\partial x_j^0} - x_j^0 \frac{\partial}{\partial x_{j\mu}}\right) f_n(x_1, \ldots, x_n),$$

$$(p_\mu f)_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j^\mu}(x_1, \ldots, x_n).$$

$\left(\text{Note that } \frac{\partial}{\partial x_{j\mu}} = -\frac{\partial}{\partial x_j^\mu}.\right)$ These operators satisfy

$$\mathbf{m}^{0\mu} L_t f = L_t \mathbf{m}^{0\mu} f + t p_\mu L_t f.$$

Assuming the $\{S_n\}$ to be invariant under the full Euclidean group implies

$$S(\hat{m}^{0\mu} f) = 0 \quad \text{for all} \quad \mu = 1, \ldots, \nu - 1 \quad \text{and all} \quad f.$$

Let $X = \{1, \ldots, n\}$ and $f \in \mathcal{S}(\mathbb{R}^{\nu n})$ with support in $\{(x_1, \ldots, x_n) | x_j^0 \geqq 0 \text{ for all } j\}$. Suppose that f is radially analytic in the angle $\{\lambda | 0 < \arg\lambda < \frac{\pi}{2}\}$. Then

$$(\Omega, \bar{T}(\mathbf{m}^{0\mu} f) \Omega) = i^n S_n((\mathbf{m}^{0\mu} f)_i)$$

and, since $(\mathbf{m}^{0\mu}f)_i = -i\hat{m}^{0\mu}f_i$, this vanishes. By the density of radially analytic functions and translational invariance of $(\Omega, \bar{T}(X)\Omega)$ it follows that, for all $f \in \mathscr{S}(\mathbb{R}^{\nu n})$,

$$(\Omega, \bar{T}(\mathbf{m}^{0\mu}f)\Omega) = 0\,.$$

Suppose now that $f = f_1 \otimes L_{v_1} f_2 \otimes \ldots \otimes L_{v_1 + \ldots + v_{r-1}} f_r$ and that all f_j have compact support. Then, for sufficiently large positive $v_1, \ldots, v_{r-1}$,

$$\begin{aligned} 0 = (\Omega, \bar{T}(\mathbf{m}^{0\mu}f)\Omega) &= \sum_{j=1}^{r} (\Omega, \bar{T}(f_1)\bar{T}(L_{v_1}f_2)\ldots\bar{T}(\mathbf{m}^{0\mu}L_{v_1+\ldots+v_{j-1}}f_j)\ldots \\ &\quad \ldots\bar{T}(L_{v_1+\ldots+v_{r-1}}f_r)\Omega) \\ &= \sum_{j=1}^{r} (\Omega, \bar{T}(f_1)e^{iv_1H}\ldots\bar{T}(f_{j-1})e^{iv_{j-1}H}[\bar{T}(\mathbf{m}^{0\mu}f_j) \\ &\quad + \left(\sum_{k=1}^{j-1} v_k\right)\bar{T}(p_\mu f_j)]e^{iv_jH}\ldots\bar{T}(f_r)\Omega)\,. \end{aligned}$$

By analytic continuation this remains true for all real $v_1, \ldots, v_{r-1}$ so that

$$\sum_{j=1}^{r-1} (\Omega, \bar{T}(f_1)\ldots\bar{T}(\mathbf{m}^{0\mu}f_j)\ldots\bar{T}(f_r)\Omega) = 0$$

for all $f_1, \ldots, f_r$ with compact support. This proves the Lorentz invariance of all $(\Omega, \bar{T}(X_1)\ldots\bar{T}(X_r)\Omega)$. This in turn implies, as it is well known, the relativistic form of the causal factorization property.

IV. Application to the φ_3^4 Quantum Field Theory

The purpose of this section is to show that perturbation theory holds for the weakly coupled φ^4 quantum field theory in 3 space-time dimensions. This statement will be substantiated below, but we first want to give a short account of existing work. The existence of the φ_3^4 interaction as a quantum field theory satisfying all the Wightman axioms was proved in [FO, MS1, MS2], based on the earlier work in [G2, GJ, Fe]. We shall assume that the reader is more or less acquainted with the definitions and results of [FO] on which our analysis below will be based. The papers [FO, MS1] contain proofs of the differentiability of the Schwinger functions in the bare parameters, i.e. in the coupling constant and the bare mass. In [MS2] it was even shown that the Schwinger functions are Borel summable in the coupling constant so that the theory is uniquely determined by perturbation theory.

Two interesting further developments in a direction similar to our aims are described in [FR] and [C]. In [FR], the question of field equations is addressed and it is shown that generalized Schwinger functions for the φ and φ^3 fields can be defined in *non-coinciding* points as tempered distributions. Similarly, in order to prove the non-triviality of the S-matrix for the φ_3^4 theory at small coupling, it was shown in [C] that *truncated* generalized Schwinger functions for the φ, φ^2 and φ^3 fields exist as tempered distributions, coinciding points included.

Since some confusion might arise about the definition of generalized Schwinger functions, we repeat here the canonical definition known from perturbation theory for the case of the φ^4 theory.

We consider the cutoff Euclidean interaction

$$\sum_{j=1}^{4} \int d^3x :\varphi_\kappa^j:(x)\lambda_j g_j(x), \tag{54}$$

where φ_κ is some free Euclidean field with bare mass m_0 and cutoff κ. The renormalization procedure tells us how to associate to the "Euclidean Lagrangian" (54) a renormalized one, called $\mathscr{L}_\kappa$. It is well-known that in the case of the superrenormalizable theories $\mathscr{L}_\kappa$ can be found in the form of a polynomial in the λ_i with coefficients which tend to infinity as the cutoff κ is removed. The natural definition of the truncated generalized Schwinger functions of the fields $\varphi_I^{\nu_k}(x)$ $(k=1,\dots,n, \nu_k=1,2,3,4)$ is *not* the truncation of

$$\lim_{g_j(x)\to 1}\lim_{\kappa\to\infty} \left\langle \prod_{k=1}^{n}\left\{\frac{\delta}{\lambda_{\nu_k}\delta g_{\nu_k}(x_k)}\mathscr{L}_\kappa\right\} e^{-\mathscr{L}_\kappa}\right\rangle_0 \langle e^{-\mathscr{L}_\kappa}\rangle_0^{-1}, \tag{55}$$

where $\langle\,\rangle_0$ is the free Euclidean expectation with bare mass m_0. We shall rather adopt the following definition.

Let $f_{j,k}\in\mathscr{S}(\mathbb{R}^3)$, $j=1,\dots,4$, $k=1,\dots,n$, let $g_j\in\mathscr{S}(\mathbb{R}^3)$, $j=1,\dots,4$. Then the generalized truncated Schwinger functions $S^T_{\nu_1,\dots,\nu_n}(x_1,\dots,x_n)$ are defined as distributions by the formula

$$\begin{aligned}&\sum_{\substack{\nu_k=1,\dots,4\\ k=1,\dots,n}} \int \prod_{k=1}^{n} d^3x_k f_{\nu_k,k}(x_k) S^T_{\nu_1,\dots,\nu_n}(x_1,\dots,x_n)\\ &= \lim_{\substack{g_j(x)\to 1\\ j=1,\dots,4}}\lim_{\kappa\to\infty}\sum_{\substack{\nu_k=1,\dots,4\\ k=1,\dots,n}}\prod_{k=1}^{n}\frac{\partial}{\partial\mu_{\nu_k,k}}\\ &\quad \log\langle e^{-\mathscr{L}_\kappa(g,f,\lambda,\mu)}\rangle_0\Big|_{\substack{\mu_{\nu_k,k}=0\\ k=1,\dots,n}},\end{aligned} \tag{56}$$

where $\mathscr{L}_\kappa(g,f,\lambda,\mu)$ is the renormalized Euclidean Lagrangian associated to

$$\sum_{j=1}^{4}\int d^3x :\varphi_\kappa^j:(x)\,[\lambda_j g_j(x)+\sum_{k=1}^{n}\mu_{j,k}f_{j,k}(x)]\,. \tag{57}$$

We shall describe below why the *existence* of the objects described in (56) is an easy variant of results contained in the earlier work on φ_3^4.

The Equations (55) and (56) define distributions which are equal on non-coinciding points, but only (56) defines a natural extension to coinciding arguments in the sense of local field theory. When (56) happens to define a locally integrable function (as is the case for the φ_I and φ_I^2 field in the φ_3^4 theory, or for all Wick powers in the $P(\varphi)_2$ theories) then the natural extension to coinciding arguments of (55) (as an integrable function) agrees with (56) everywhere. This observation (which follows from the very construction of the models) was basic to the papers [EEF, OSe, O].

The main point of the preceding discussion is to emphasize that the definition (56) is suitable from an axiomatic point of view. In particular, it is for the untruncated *generalized Schwinger functions* $S_{\nu_1\ldots\nu_n}$, obtained from the $S^T_{\nu_1\ldots\nu_n}$ in the standard way, that we shall prove the *extended O.S. positivity*, and the distributional bounds.

Theorem 6. *The generalized Schwinger functions are, as distributions, infinitely differentiable with respect to* $\lambda_1, \ldots, \lambda_4$ *and* $m_0 > 0$ *in a region of the form* $0 \leqq \lambda_4 m_0^{-1} < \varepsilon$, $|\lambda_j| m_0^{j/2-3} < \varepsilon \lambda_4 m_0^{-1}$, $j = 1, 2, 3$, *for some* $\varepsilon > 0$. *The derivatives are (finite) sums of truncated generalized Schwinger functions, integrated over some of their arguments.*

For an illustration of this last statement, see [EEF, Lemma 6a, 6b].

Theorem 7. *The family of generalized Schwinger functions satisfies the conditions S1)–S6).*

The proofs of these two theorems will be sketched below. Given the very detailed accounts of [FO, MS1, 2] we refrain from repeating the whole construction of the φ_3^4 theory for just proving one additional estimate, which seems to us implicitly contained in the aforementioned papers.

The conclusions of Theorems 6 and 7, combined with the results of the preceding sections imply the existence of time-ordered products for the fields φ^ν; $\nu = 1, \ldots, 4$.

Theorem 8. *The generalized analytic momentum space functions* $H_{\nu_1\ldots\nu_n}(k; m_0, \lambda_1, \ldots, \lambda_4)$ *of the* φ_3^4 *theory are* $\mathscr{C}^\infty$ *functions in* $m_0, \lambda_1, \ldots, \lambda_4$ *in the region described in Theorem 6 and analytic in k in the n point axiomatic domain with single particle poles at* $k_j^2 = m^2(m_0, \lambda_1, \ldots, \lambda_4)$ *and thresholds above* $2m_0^2 - 0(\varepsilon)$.

Proof. By the analysis of the preceding sections and by Theorem 2, Equation (29) $H_{\nu_1\ldots\nu_n}$ is for imaginary time components of k the Fourier transform of $S^T_{\nu_1\ldots\nu_n}$. The proof of differentiability follows then in the same way as for Theorem 7 of [EEF]. The existence of an isolated one particle singularity has been shown for the φ_3^4 theory by Burnap [B].

(Note: in perturbation theory, the renormalization is often performed so that the physical mass coincides with the parameter m, given in advance, which occurs in the free propagators. In constructive theory the bare mass m_0 is given; renormalization is performed by introducing only those counterterms necessary to compensate the divergences of the primitively divergent graphs: since these are in finite number for superrenormalizable theories $\mathscr{L}_\kappa$ is then a polynomial in the λ_j. This determines the physical mass m, as a function of m_0 and the λ_j). We can now repeat almost verbatim the analysis done in [EEF], and we restate three relevant results.

Theorem 9. (=Theorem 8 in [EEF]). *There is an* $\varepsilon_1, 0 < \varepsilon_1 < \varepsilon$ *and for each* $a > 0$ *a* $\mathscr{C}^\infty$ *function* $\lambda = (\lambda_1, \ldots, \lambda_4) \to m_0(a, \lambda)$ *on* $0 \leqq \lambda_4 a^{-1} < \varepsilon_1$, $|\lambda_j| a^{j/2-3} < \varepsilon_1 \lambda_4 a^{-1}$, $j = 1, 2, 3$ *such that one has in the above region*

$$m(m_0(a, \lambda), \lambda) = a.$$

(Note that in 3 dimensions, the mass satisfies due to scaling $m(m_0, \lambda_1, \ldots, \lambda_4) = m_0 u(\lambda_1 m_0^{1/2-3}, \ldots, \lambda_4 m_0^{4/2-3})$.)

Theorem 10. (=Theorem 10 in [EEF]).

(i) *For φ_3^4 theories with physical mass $m>0$ and bare coupling constants λ_j satisfying $|\lambda_j m^{j/2-3}| \leqq \varepsilon_1 \lambda_4 m^{-1}$, $j=1,2,3$, $0 \leqq \lambda_4 m^{-1} < \varepsilon_1$ the function*

$$G(k_1, \ldots, k_n; m, \lambda) = \left\{ \prod_{j=1}^{n} (k_j^2 - m^2) \right\} H_{1\ldots1}(k_1, \ldots, k_n; m_0(m, \lambda), \lambda_1, \ldots, \lambda_4)$$

is $\mathscr{C}^\infty$ in the λ_j and holomorphic in $k_1, \ldots, k_n$ (with $\sum k_j = 0$) in the axiomatic domain, with thresholds above $2m - 0(\varepsilon_1)$ and no single particle poles (at $k_j^2 = m^2$).

(ii) *The Taylor expansion of $G(k; m, \lambda)$ in λ_j at $\lambda_j = 0, j=1, \ldots, 4$, for k taken in the axiomatic domain (as described above), is given by standard renormalized perturbation theory.*

Theorem 11. (=Theorem 12 in [EEF]). *At non-overlapping points of the real mass-shell the S-matrix elements of a φ_3^4 theory with fixed physical mass $m>0$ and coupling constants as in Theorem 10 are $\mathscr{C}^\infty$ in λ_j, $j=1, \ldots, 4$ in that region as tempered distributions in the momenta. Their Taylor expansion at $\lambda_j = 0, j=1, \ldots, 4$ is given by standard renormalized perturbation theory.*

In particular, we recover the non-triviality of the S-matrix for $\lambda_4 > 0$ sufficiently small, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, [C].

Proof of Theorem 6. This theorem is in a sense already contained in the proofs of the existence and differentiability in λ of the ordinary Schwinger functions. We therefore only show how these proofs have to be read in order to arrive at the statement of Theorem 6. We shall adopt the terminology and notations of [FO], but [MS1, MS2] could serve just as well, and we assume the reader is familiar with [FO].

To explain our point, we first refer to § 2.1 of [FO], (graph norms). A graph is a Wick monomial G of degree $E_i(G)$ with a kernel $K(G)(z_1, \ldots, z_{Ei(G)})$, where we consider all "legs" as "initial legs". For all $\delta > 2\alpha > 0$, a norm $\|G\|_{\delta,\alpha}$ is defined ([FO, Eq. (2.6)]). The importance of these norms stems from the fact that all bounds and convergence statements are made with respect to them for suitable δ and α. As an example, in Theorem 4.6 of [FO] we have the bound

$$|Z(\Lambda)^{-1} \langle G \rangle_{1, t_m, \chi_\Lambda g}| \leqq \prod_{\Delta} (n(\Delta)!) \|G\|_{\delta,\alpha}, \tag{58}$$

where $n(\Delta)$ is the number of arguments of $K(G)$ localized in the unit cube Δ (the support of $K(G)$ is assumed here to be a product of such unit cubes), and $Z(\Lambda)$ is the "partition function". In the case at hand, the space cutoff is not the function g, but the family of functions, cf. Equation (57),

$$\left\{ \lambda_j g_j + \sum_{k=1}^{n} \mu_{j,k} f_{j,k}, j=1, \ldots, 4 \right\},$$

and it is clear from the definition of the norms $\| \ \|_{\delta,\alpha}$ that in our case we will have a bound on the l.h.s. of (58) of the form

$$\prod_{\Delta}(n(\Delta)!)\|G\|^0_{\delta,\alpha}\sum_j\left\{|\lambda_j||g_j|_s+\sum_{k=1}^{n}|\mu_{jk}||f_{jk}|_s\right\}^{O(\Sigma n(\Delta))}O(1),$$

where $\|G\|^0_{\delta,\alpha}$ is bounded by the norm $\|G\|_{\delta,\alpha}$ for a g whose Fourier transform is bounded by $|k|^{-1}$ at infinity (cf. [Fe, page 97]), and $| |_s$ is a fixed Schwartz norm.

The differentiability follows now as in Corollary 4.6b of [FO]. A more detailed outline of the proof is given in [MS2, page 270], where it is sketched how the derivatives combine to sums of graphs whose $\|G\|^0_{\delta,\alpha}$ norms are finite. For the case of the φ^3 derivatives this is done in great detail in [FR, pages 215–217], where it is shown how the crucial perturbation formula [FO, Eqs. (2.37)–(2.39), (3.2), (3.3)] is used in this procedure. The next important point of the proof is to realize that the $\| \ \|^0_{\delta,\alpha}$ norms of the graphs produced through this procedure are bounded, for $S^T_{\nu_1\ldots\nu_n}$ by $\prod_{j=1}^{n}|f_{\nu_j,j}|_s\, n!^{K'}$ for some universal constant K' and Schwartz norm $| |_s$. This is due to the superrenormalizability of the theory, and proved through the mechanism of "estimating big graphs as products of small graphs" which was discovered by Glimm [G2].

We therefore get a bound

$$\begin{aligned}&\left|\int S^T_{\nu_1\ldots\nu_n}(x_1,\ldots,x_n)f_1(x_1)\ldots f_n(x_n)d^3x_1\ldots d^3x_n\right|\\&\leqq Jn!^K\prod_{j=1}^{n}\left\{(L|f_j|_s)\left(L+\sum_{i=1}^{n}|f_i|_s\right)^M\right\}\end{aligned}\tag{59}$$

for a universal Schwartz norm $| |_s$ and universal constants J, K, L, M, provided the coupling constants are in a region of the form described in the statement of Theorem 6 (this is used to bound the exponential of $-\mathscr{L}_\kappa$).

Similarly, we get for the derivatives the equality

$$\begin{aligned}&\prod_{j=1}^{4}\left(\frac{\partial}{\partial\lambda_j}\right)^{n_j}\int S^T_{\nu_1\ldots\nu_n}(x_1,\ldots,x_n)f_1(x_1)\ldots f_n(x_n)d^3x_1\ldots d^3x_n\\&=\int S^T_{\nu_1\ldots\nu_n,\underbrace{1\ldots1}_{n_1},\ldots\underbrace{4,\ldots4}_{n_4}}(x_1,\ldots,x_n,y_1,\ldots,y_{\Sigma n_j})f_1(x_1)\ldots f_n(x_n)d^3x_1\ldots d^3y_{\Sigma n_j}\end{aligned}\tag{60}$$

and the bound

$$J\left(n+\sum_{j=1}^{4}n_j\right)!^K\left(\prod_{j=1}^{n}(L|f_j|_s)\right)\left(L+\sum_{i=1}^{n}|f_i|_s\right)^{M\left(n+\sum_{j=1}^{4}n_j\right)}.$$

Since a distribution is a linear functional, we find, writing $f_j=(f_j/(|f_j|_s))|f_j|_s$,

$$\begin{aligned}&\left|\prod_{j=1}^{4}\left(\frac{\partial}{\partial\lambda_j}\right)^{n_j}\int S^T_{\nu_1\ldots\nu_n}(x_1,\ldots,x_n)f_1(x_1)\ldots f_n(x_n)d^3x_1\ldots d^3x_n\right|\\&\leqq J\left(n+\sum_{j=1}^{4}n_j\right)!^K L^n(L+n)^{M\left(n+\sum_{j=1}^{4}n_j\right)}\prod_{j=1}^{n}|f_j|_s.\end{aligned}\tag{61}$$

The Equations (60), (61) are more than enough to prove Theorem 6, in fact, we have already proven the growth condition of Theorem 7.

Proof of Theorem 7. All axioms are obviously satisfied for the generalized (extended) Schwinger functions defined by (56), as in [FO, MS1] except for the growth condition which we just proved and the extended O.S. positivity which we prove now.

Time-ordered products can be defined for a cutoff theory, since for $\kappa<\infty$ (as in [Fe]), and a cutoff of the form $g_i(x^0,\mathbf{x})=\chi_{[-T,T]}(x^0)\cdot h_i(\mathbf{x})$, we have a Hamiltonian theory in which sharp time fields are defined. Hence we have extended O.S. positivity in this case by Theorem 1. This carries over to the limits, of which we have already shown existence. The proof of Theorem 7 is complete.

Appendix

Radially Analytic Functions

Let f belong to $\mathscr{S}(\mathbb{R}^N)$. For all real $\lambda>0$, $f_\lambda(x)=f(x)$ defines another element of $\mathscr{S}(\mathbb{R}^N)$ and the map

$$(f,\lambda)\to f_\lambda$$

from $\mathscr{S}(\mathbb{R}^N)\times(0,\infty)$ to $\mathscr{S}(\mathbb{R}^N)$ is continuous and $\mathscr{C}^\infty$ in λ. We shall say that f is radially analytic and continuous in the angle $\{\lambda\in\mathbb{C}:|\lambda|>0,\,\theta_1\leqq\arg\lambda\leqq\theta_2\}$ (when $\theta_1\leqq 0\leqq\theta_2$) if there exists a continuous map $(f,\lambda)\to f_\lambda$ from $\mathscr{S}(\mathbb{R}^N)\times$(this angle) into $\mathscr{S}(\mathbb{R}^N)$, holomorphic in λ in the interior of the angle, such that, for λ real >0, $f_\lambda(x)=f(\lambda x)$.

It is easy to construct examples of radially analytic functions. Denote, for instance, for any real $a>0$,

$$\varrho(a,\lambda)=c(a)\exp(-a(\lambda^{-1/2}+\lambda^{1/2})),\qquad c(a)^{-1}=\int_0^\infty\frac{d\lambda}{\lambda}\exp(-a(\lambda^{-1/2}+\lambda^{1/2}))$$

where $\lambda\to\lambda^{1/2}$ is the holomorphic function over $\mathbb{C}\backslash\mathbb{R}_-$ equal to $|\lambda|^{1/2}$ for $\lambda>0$. Note that the restriction of this function to any closed half plane of the form $\{\lambda:\mathrm{Re}\,e^{-i\theta}\lambda\geqq 0\}$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ is $\mathscr{C}^\infty$ and vanishes at 0 with all its derivatives.

Let F be a continuous function on $(0,\infty)$ with

$$|F(\lambda)|<A(\lambda^{1/2}+\lambda^{-1/2})^L,\qquad L\geqq 0.$$

Then (with new integration variables $\lambda=e^{2\varphi}$, $y=ch\varphi$)

$$\int_0^\infty\frac{d\lambda}{\lambda}F(\lambda)\exp-a(\lambda^{1/2}+\lambda^{-1/2})=2\int_{-\infty}^{\infty}e^{-2ach\varphi}F(e^{2\varphi})d\varphi\,,$$

$$2\int_0^\infty e^{-2ach\varphi}F(e^{2\varphi})d\varphi=2\int_1^\infty e^{-2ay}F((y+(y^2-1)^{1/2})^2)dy(y^2-1)^{-1/2}$$

and the last expression shows that, since $|F(\lambda)| < A(2y)^L$, the integrals converge absolutely. In particular

$$\begin{aligned} c(a)^{-1} &= 4 \int_1^\infty e^{-2ay}(y^2-1)^{-1/2} dy \\ &> 4 \int_{2a}^{4a} e^{-y} \frac{dy}{y} \\ &> a^{-1} e^{-2a}(1-e^{-2a}). \end{aligned}$$

Thus, if $a \geqq 1$, $c(a)^{-1} \geqq (2a)^{-1} e^{-2a}$, i.e. $c(a) \leqq 2ae^{2a}$.

On the other hand, if $\varphi_0 > 0$,

$$\begin{aligned} &\left| c(a) \int_{\varphi_0}^\infty F(e^{2\varphi}) e^{-2ach\varphi} d\varphi \right| \\ &\leqq 2ae^{2a} A \int_{\varphi_0}^\infty (2ch\varphi)^L e^{-2ach\varphi} d\varphi \\ &\leqq 2^{L+1} aA \int_{\varphi_0}^\infty d\varphi \exp\left(-a\varphi^2 - a\frac{\varphi^4}{12} + L\varphi \right) \\ &\leqq 2^{L+1} aA e^{-a\varphi_0^2} \int_{\varphi_0}^\infty d\varphi \exp\left(-a\frac{\varphi^4}{12} + L\varphi \right). \end{aligned}$$

If $a \geqq 1$ this is less than

$$\begin{aligned} &2^{L+1} Aae^{-a\varphi_0^2} \textstyle\int_0^\infty d\varphi \exp\left(-a\frac{\varphi^4}{12} + La^{1/4}\varphi \right) \\ &= 2^{L+1} Aa^{3/4} e^{-a\varphi_0^2} (12)^{1/4} \textstyle\int_0^\infty \exp -(\theta^4 - (12)^{1/4} L\theta)\, d\theta. \end{aligned}$$

From this it follows that

$$\int_{\substack{|\lambda-1|>\varepsilon \\ \lambda>0}} \varrho(a,\lambda) F(\lambda) \frac{d\lambda}{\lambda}$$

tends to zero as $a \to \infty$. On the other hand $\int_0^\infty \varrho(a,\lambda) \frac{d\lambda}{\lambda} = 1$. Since F is continuous, for every $\eta > 0$ we can find $a > 0$ so large that

$$\left| \int_0^\infty (F(\lambda) - F(1)) \varrho(a,\lambda) \frac{d\lambda}{\lambda} \right| < \eta.$$

Hence $\frac{1}{\lambda} \varrho(a,\lambda)$ is, when $a \to \infty$, an approximation of $\delta(\lambda - 1)$.

Suppose that $\varphi \in \mathscr{S}(\mathbb{R}^N)$. Then

$$f(x) = \int_0^\infty \varrho(a,\lambda) \frac{d\lambda}{\lambda} \varphi(\lambda^{-1} x)$$

defines a new function also belonging to $\mathscr{S}(\mathbb{R}^N)$. In fact

$$x^\beta D^\alpha f(x) = \int_0^\infty \varrho(a,\lambda)\frac{d\lambda}{\lambda}\lambda^{|\beta|-|\alpha|}(\lambda^{-1}x)^\beta D^\alpha \varphi(\lambda^{-1}x),$$

so that we can apply the preceding extimate with $L=|\beta|+|\alpha|$ and obtain an inequality of the type

$$\sup_x |x^\beta D^\alpha f(x)| \leqq C(\alpha,\beta,a)\sup_x |x^\beta D^\alpha \varphi(x)|.$$

Furthermore, as $a\to\infty$, f tends to φ in the strong topology of $\mathscr{S}$, and uniformly if φ varies in a bounded set of $\mathscr{S}(\mathbb{R}^N)$. For $\lambda>0$, denoting $f_\lambda(x)=f(\lambda x)$, we have

$$f_\lambda(x) = \int_0^\infty \varrho(a,\lambda\mu)\frac{d\mu}{\mu}\varphi(\mu^{-1}x). \tag{A 1}$$

The right hand side of this equation has an analytic continuation in λ in the open set $\mathbb{C}\backslash\mathbb{R}^-$, and the map $(\lambda,\varphi)\to f_\lambda$ is a continuous map of $(\mathbb{C}\backslash\mathbb{R}^-)\times\mathscr{S}(\mathbb{R}^N)$ into $\mathscr{S}(\mathbb{R}^N)$, holomorphic in λ. Thus f is radially continuous and analytic in the angle $\{\lambda:|\lambda|>0, \theta_1\leqq\arg\lambda\leqq\theta_2\}$ whenever $-\pi<\theta_1<\theta_2<\pi$.

This shows that radially analytic functions are dense in $\mathscr{S}(\mathbb{R}^N)$. Note that if φ has its support in a closed set $F\subset\mathbb{R}^N$ then for all $\lambda\in\mathbb{C}\backslash\mathbb{R}^-$, f_λ as defined by (A 1) has its support in the closed cone generated by F.

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