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Conservation Laws for Classes of Nonlinear Evolution Equations Solvable by the Spectral Transform

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Abstract. The existence of conservation laws for novel classes of nonlinear evolution equations (with linearly x-dependent coefficients) solvable by the spectral transform is investigated. A remarkably explicit representation is moreover obtained for the conserved quantities of the "old" classes of nonlinear evolution equations (with x-independent coefficients; including the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the nonlinear Schrödinger equation, etc.).

1. Introduction

A characteristic feature of the class of nonlinear evolution equations solvable via the spectral transform (see, for instance, [1–20]) is the existence of an infinite number of conserved quantities (see, for instance, [1–28]). Indeed the discovery of an infinite number of local conservation laws for the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations has preceeded, and paved the way, to the introduction of the inverse scattering transform (or, as we prefer to call it, the *spectral transform*); it has played a crucial rôle in certain fundamental developments of the theory of these equations, such as the hamiltonian formulation (see for instance [4–5]); and it has been the subject of many papers, including several recent ones (see, for instance, [17–19], [22–28], and the paper by Wadati in [20]).

Recently we have introduced an extension of the approach based on the spectral transform that enlarges the classes of nonlinear evolution equations solvable by this technique [29–32]. A characteristic feature of these extended classes is that they are no more associated to isospectral flows. One finds accordingly that the existence of an infinite number of local conservation laws holds no more. It is however still possible to exhibit conserved quantities, although their practical usefulness is doubtful, since each of them is a linear combination of an infinite number of the "old" conserved quantities. It is moreover possible to

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investigate the time-dependence of the "old" conserved quantities; the practicality and interest of the corresponding results is highlighted by the discovery of instances in which they are periodic in time (with a period determined by the structure of the evolution equation, independently of the initial conditions).

The main purpose of this paper is to describe the results outlined above. But in the process of deriving them we have also obtained a remarkably explicit, and simple, representation of the old conserved quantities. Thus the results of this paper are relevant not only to the novel classes of nonlinear evolution equations solvable via the extension of the spectral transform method recently introduced [29–32], but also to the equations belonging to the standard classes [1–28], including for instance the KdV, mKdV, nonlinear Schrödinger (NLS) and Sine-Gordon (SG) equations.

In the following Section we report tersely the main results concerning the novel classes of nonlinear evolution equations, so as to make this paper, at least notationally, self-consistent. The results are then derived in Section 3 for the class of nonlinear evolution equations solvable by the spectral transform associated to the Schrödinger spectral problem, and in Section 4 for the class of nonlinear evolution equations solvable by the spectral transform associated to the (generalized) Zakharov-Shabat spectral problem. Section 5 contains the explicit analysis of three examples, namely generalized versions (with linearly x-dependent coefficients) of the KdV, mKdV and NLS equations. A concluding Section 6 contains some final remarks.

2. Preliminaries and Notation

The (novel) class of nonlinear evolution equations solvable via the spectral transform associated to the Schrödinger linear problem can be written as follows [29]:

$$u_t(x,t)=\alpha(L,t)\,u_x(x,t)+\beta(L,t)\,[xu_x(x,t)+2u(x,t)]\,. \tag{2.1}$$

Here $u(x,t)$ is the field to be determined, characterized by the initial condition

$$u(x,t_0)=u_0(x)\,. \tag{2.2}$$

In this paper we always restrict attention to fields vanishing (sufficiently fast!) as $|x|\to\infty$; it is of course sufficient that this condition be imposed at the initial time t_0. In (2.1) $\alpha(z,t)$ and $\beta(z,t)$ are essentially arbitrary functions, although the most interesting cases correspond to a low-order polynomial (or rational) dependence on z; $\beta\equiv 0$ yields back the "old" class of nonlinear evolution equations (including the KdV equation, see below; note that the condition $\beta=0$ is also necessary and sufficient to guarantee the invariance of (2.1) under space translations). The integro-differential operator L is defined by

$$Lf(x)=f_{xx}(x)-4u(x,t)f(x)+2u_x(x,t)\int_x^{+\infty}dx'f(x')\,, \tag{2.3}$$

f being here an arbitrary function (vanishing as $x\to+\infty$). Note that L depends on u itself, and this causes (2.1) to be a nonlinear evolution equation. Clearly in these formulae, as in the rest of the paper, subscripted variables indicate (partial) differentiation.

The spectral transform S associated to u consists of the following data

$$S:\{R(k),\ -\infty<k<+\infty\,;\, p_n, c_n^2, n=1,2,\ldots,N\}, \tag{2.4}$$

which are defined by the spectral problem characterized by the Schrödinger equation

$$\psi_{xx}(k,x)=[u(x)-k^2]\,\psi(k,x) \tag{2.5}$$

and by the boundary conditions

$$\begin{cases} \psi(k,x)\xrightarrow[x\to+\infty]{} \exp(-ikx)+R(k)\exp(ikx) \\ \psi(k,x)\xrightarrow[x\to-\infty]{} T(k)\exp(-ikx) \end{cases} \tag{2.6}$$

$$\int_{-\infty}^{+\infty} dx\,[\psi(ip_n,x)]^2=1\,, \tag{2.7}$$

$$\psi(ip_n,x)\xrightarrow[x\to+\infty]{} c_n\exp(-p_nx),\ p_n>0\,. \tag{2.8}$$

There is then a one-to-one correspondence between a function $u(x)$ (vanishing as $|x|\to\infty$) and its spectral transform S: the spectral transform is associated to u through the direct spectral problem outlined above, Equations (2.5–2.8); while u is uniquely determined by S through the inverse spectral problem, characterized by the equations

$$M(x)=\sum_{n=1}^{N} c_n^2\exp(-p_nx)+(2\pi)^{-1}\int_{-\infty}^{+\infty} dk\,R(k)\exp(ikx)\,, \tag{2.9}$$

$$K(x,x')+M(x+x')+\int_{x}^{+\infty} dx''\,K(x,x'')\,M(x'+x'')=0,\ x'\geqq x\,, \tag{2.10}$$

$$u(x)=-2dK(x,x)/dx\,. \tag{2.11}$$

In these last equations we have omitted to indicate any time-dependence. But of course if u is time-dependent, also time-dependent are all the quantities defined above; indeed the solvability of the class of nonlinear evolution equations (2.1) obtains because the corresponding time-evolution of the spectral transform is simple, being characterized by the equations

$$\begin{aligned} &R_t(k,t)+k\beta(-4k^2,t)\,R_k(k,t)\\ &\quad=2ik\alpha(-4k^2,t)\,R(k,t)\,, \end{aligned} \tag{2.12}$$

$$p_t(t)=p(t)\,\beta[4p^2(t),t]\,. \tag{2.13}$$

We have omitted to indicate explicitly, in the last equation, the label n characterizing the different discrete eigenvalues; nor have we written the equation for the time evolution of $c_n(t)$, since it is complicated and it is not needed in the following. We write instead the evolution equation satisfied by the transmission coefficient $T(k,t)$, that will play a crucial rôle in the following:

$$T_t(k,t)+k\beta(-4k^2,t)\,T_k(k,t)$$
$$=-2ikT(k,t)\int_{-\infty}^{+\infty} dx\tilde{\beta}(-4k^2,L,t)\,[xu_x(x,t)+2u(x,t)]\,. \tag{2.14}$$

In this formula, and always in the following, for any given function $f(z,t)$ we define $\tilde{f}(z_1,z_2,t)$ by the formula

$$\tilde{f}(z_1,z_2,t)\equiv[f(z_1,t)-f(z_2,t)]/(z_1-z_2)\,. \tag{2.15}$$

The formula (2.14), that is reported here for the first time, can be obtained in close analogy to the derivation of (2.12) described in the first paper of [29], using the results of the second paper of [13] (a misprint in this paper should be corrected; $-\lambda$ should appear in place of $+\lambda$ in front of the third term in the r.h.s. of Equation (3.2.19)).

If $\beta=0$, both p and T are time-independent; the first of these well-known properties displays the isospectral character of the flow, and the second property provides a convenient starting point for the derivation of conserved quantities, as we shall discuss in the following Section in the more general context of the flow (2.1) with $\beta\neq 0$.

We end this terse survey noting that (2.12) is explicitly solved by the formulae

$$R(k,t)=\exp\left[2i\int_{t_0}^{t} dt'\,\chi\alpha(-4\chi^2,t')\right]R_0[k_0(t,k)]\,, \tag{2.16}$$

$$\chi\equiv\chi[t',k_0(t,k)]\,, \tag{2.17}$$

the function $\chi(t,k_0)$ being defined by the (ordinary) differential equation

$$\chi_t(t,k_0)=\chi(t,k_0)\,\beta[-4\chi^2(t,k_0),t] \tag{2.18}$$

and by the initial condition

$$\chi(t_0,k_0)=k_0\,, \tag{2.19}$$

while the function $k_0(t,k)$ in (2.16) and (2.17) is defined from χ through the (implicit) formula

$$\chi(t,k_0)=k\,. \tag{2.20}$$

Of course R_0 is the initial value of R:

$$R_0(k)=R(k,t_0)\,. \tag{2.21}$$

Proceeding in close parallelism to the treatment given above, we report now the corresponding formulae for the (novel) class of nonlinear evolution equations solvable via the spectral transform associated to the (generalized) Zakharov-

Shabat linear problem [29]. Now the class of nonlinear evolution equations reads

$$\sigma_3 v_t(x,t)+\gamma(L,t)\,v(x,t)+\mu(L,t)\,xv(x,t)=0\,. \tag{2.22}$$

Here and below the matrices σ_3, σ_+ and σ_- are defined by

$$\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\sigma_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\sigma_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \tag{2.23}$$

and the field to be determined,

$$v(x,t)\equiv r(x,t)\,\chi_+ + q(x,t)\,\chi_-\,, \tag{2.24}$$

is a two-component vector (or spinor), represented by this formula in terms of the two eigenvectors $\chi_\pm$ of σ_3:

$$\sigma_3\chi_\pm=\pm\chi_\pm,\sigma_\pm\chi_\pm=0,\sigma_\pm\chi_\mp=\chi_\pm\,. \tag{2.25}$$

In addition to the nonlinear evolution Equation (2.22) there is of course an initial condition,

$$v(x,t_0)=v_0(x)\,. \tag{2.26}$$

The integro-differential operator L is now defined by the formula

$$Lu(x)=(2i)^{-1}\left\{\sigma_3 u_x(x)+2v(x,t)\int_x^{+\infty}dx'\,[q(x',t)\,u_+(x') - r(x',t)\,u_-(x')]\right\}, \tag{2.27}$$

$u(x)$ being here a generic vector,

$$u(x)=u_+(x)\,\chi_+ + u_-(x)\,\chi_-\,. \tag{2.28}$$

The nonlinear evolution equation (2.22) involves the vector field $v(x,t)$, or equivalently the two scalar fields $q(x,t)$ and $r(x,t)$; cases in which it can be reduced to an equation for a single scalar field are indicated below. The functions $\gamma(z,t)$ and $\mu(z,t)$ are essentially arbitrary; the most interesting cases correspond to low-order polynomial (or rational) dependences on z, and moreover to some restrictions on their reality and/or parity, to allow the reduction mentioned above. The "old" class of nonlinear evolution equations (including the mKdV, NLS and Sine-Gordon equations; see below) corresponds to $\mu\equiv 0$, this condition being also necessary and sufficient to imply the invariance of (2.22) under space translations.

The spectral transform S associated to v consists now of the following data:

$$S:\{\alpha^{(\pm)}(k),\ -\infty<k<+\infty\,;k_n^{(\pm)},\varrho_n^{(\pm)}\}\,, \tag{2.29}$$

which are defined by the (generalized) Zakharov-Shabat spectral problem characterized by the equation

$$\psi_x(x,k)=[q(x)\,\sigma_+ + r(x)\,\sigma_- - ik\sigma_3]\,\psi(x,k) \tag{2.30}$$

and the asymptotic conditions

$$\begin{cases} \lim_{x\to+\infty} [\exp(ikx\sigma_3)\Psi(x,k)] = 1 + \sigma_-\alpha^{(+)}(k) + \sigma_+\alpha^{(-)}(k) \\ \lim_{x\to-\infty} [\exp(ikx\sigma_3)\Psi(x,k)] = \frac{1}{2}(1+\sigma_3)\beta^{(+)}(k) + \frac{1}{2}(1-\sigma_3)\beta^{(-)}(k), \end{cases} \tag{2.31}$$

$$\int_{-\infty}^{+\infty} dx(\psi(x,k_n^{(\pm)}), (\sigma_+ + \sigma_-)\psi(x,k_n^{(\pm)})) = 1, \tag{2.32}$$

$$\begin{cases} \lim_{x\to+\infty} [\exp(\mp ik_n^{(\pm)}x)\psi(x,k_n^{(\pm)})] = \gamma_n^{(\pm)}\chi_\mp, \\ \lim_{x\to-\infty} [\exp(\pm ik_n^{(\pm)}x)\psi(x,k_n^{(\pm)})] = \delta_n^{(\pm)}\chi_\pm, \end{cases} \tag{2.33}$$

$$\varrho_n^{(\pm)} = i\gamma_n^{(\pm)}\delta_n^{(\pm)}. \tag{2.34}$$

In (2.31), Ψ is a matrix of rank 2 whose columns are solutions of (2.30), satisfying appropriate boundary conditions (set out by (2.31) itself); in (2.29) and (2.32–2.33), the sign label on the discrete eigenvalues $k_n^{(\pm)}$ is defined by the rule

$$\pm \mathrm{Im}(k_n^{(\pm)}) > 0. \tag{2.35}$$

There is a one-to-one correspondence between a vector $v(x)$ and its spectral transform S: the spectral transform is associated to v by the direct spectral problem outlined above, Equations (2.30–2.35); while v is uniquely determined by S through the inverse spectral problem, characterized by the equations

$$M(x) = \frac{1}{2}(1+\sigma_3)m^{(-)}(x) + \frac{1}{2}(1-\sigma_3)m^{(+)}(x), \tag{2.36}$$

$$m^{(\pm)}(x) = \mp i\sum_n \varrho_n^{(\pm)} \exp(\pm ik_n^{(\pm)}x) + (2\pi)^{-1} \int_{-\infty}^{+\infty} dk\alpha^{(\pm)}(k)\exp(\pm ikx), \tag{2.37}$$

$$K(x,x') + M(x+x') + \int_x^{+\infty} dx'' K(x,x'')(\sigma_+ + \sigma_-)M(x''+x') = 0, \quad x' \geqq x, \tag{2.38}$$

$$q(x) = -2K_{11}(x,x), \qquad r(x) = -2K_{22}(x,x). \tag{2.39}$$

In these last equations we have omitted to indicate any time-dependence. But of course if v is time-dependent, also time-dependent are all the quantities defined above; indeed the solvability of the class of nonlinear evolution equations (2.22) obtains because the corresponding time-evolution of the spectral transform is simple, being characterized by the equations

$$\alpha_t^{(\pm)}(k,t) + \frac{1}{2}i\mu(k,t)\alpha_k^{(\pm)}(k,t) \pm \gamma(k,t)\alpha^{(\pm)}(k,t) = 0, \tag{2.40}$$

$$k_t^{(\pm)}(t) = \frac{1}{2}i\mu[k^{(\pm)}(t), t]. \tag{2.41}$$

We have omitted to indicate explicitly, in the last equation, the label n characterizing the different discrete eigenvalues; nor have we written the equation for the time evolution of $\varrho_n^{(\pm)}(t)$, since it is complicated and it is not needed in the following. We report instead the evolution equation satisfied by the quantities

$\beta^{(\pm)}(k,t)$, that will play a crucial rôle in the following:

$$\beta_t^{(\pm)}(k,t)+\tfrac{1}{2}i\mu(k,t)\beta_k^{(\pm)}(k,t)\pm\theta(k,t)\beta^{(\pm)}(k,t)=0\,, \tag{2.42}$$

$$\theta(k,t)=-\tfrac{1}{2}i\int_{-\infty}^{+\infty}dx(v(x,t),(\sigma_+-\sigma_-)\tilde{\mu}(k,L,t)xv(x,t))\,. \tag{2.43}$$

In writing the last formula we have used the notation (2.15).

If $\mu\equiv 0$, both $k_n^{(\pm)}$ and $\beta^{(\pm)}$ are time-independent; the first of these well-known properties displays the isospectral character of the flow, and the second property provides a convenient starting point for the derivation of conserved quantities, as we shall discuss in Section 4 in the more general context of the flow (2.22) with $\mu\neq 0$.

The linear partial differential Equation (2.40) is explicitly solved by the formula

$$\alpha^{(\pm)}(k,t)=\exp\left\{\mp\int_{t_0}^{t}dt'\gamma[\chi(t',k_0(k,t)),t']\right\}\alpha_0^{(\pm)}[k_0(k,t)]\,, \tag{2.44}$$

the function $\chi(t,k_0)$ being defined by the (ordinary) differential equation

$$\chi_t(t,k_0)=\tfrac{1}{2}i\mu[\chi(t,k_0),t] \tag{2.45}$$

and by the initial condition

$$\chi(t_0,k_0)=k_0\,, \tag{2.46}$$

while the function $k_0(k,t)$ is defined from χ through the (implicit) formula

$$\chi(t,k_0)=k\,. \tag{2.47}$$

Of course $\alpha_0^{(\pm)}$ is the initial value of $\alpha^{(\pm)}$

$$\alpha_0^{(\pm)}(k)=\alpha^{(\pm)}(k,t_0)\,. \tag{2.48}$$

Finally we report the conditions relevant to the reduction of (2.22) to a nonlinear evolution equation for a single scalar field:

case i): $r(x,t)=\varepsilon q(x,t)$; $\alpha^{(+)}(k,t)=\varepsilon\alpha^{(-)}(-k,t)$,
$\beta^{(+)}(k,t)=\beta^{(-)}(-k,t)$; $k_n^{(+)}(t)=-k_n^{(-)}(t)$, $\varrho_n^{(+)}(t)=-\varepsilon\varrho_n^{(-)}(t)$;
$\gamma(z,t)=-\gamma(-z,t)$, $\mu(z,t)=-\mu(-z,t)$;

case ii): $r(x,t)=\varepsilon q^*(x,t)$; $\alpha^{(+)}(k,t)=\varepsilon\alpha^{(-)*}(k^*,t)$,
$\beta^{(+)}(k,t)=\beta^{(-)*}(k^*,t)$; $k_n^{(+)}(t)=k_n^{(-)*}(t)$, $\varrho_n^{(+)}(t)=\varepsilon\varrho_n^{(-)*}(t)$;
$\gamma(z,t)=-\gamma^*(z^*,t)$, $\mu(z,t)=-\mu^*(z^*,t)$.

The last conditions reported in each case, specifying the limitations on the functions $\gamma(z,t)$ and $\mu(z,t)$ that characterize the structure of each particular equation of the class (2.22), are necessary and sufficient to guarentee the compatibility of the other equations with the time evolution, namely if they are given initially, they are always maintained.

In case i), the class of nonlinear evolution equations (2.22) can be written directly for the single field q, and it reads

$$q_t(x,t)=\alpha(M,t)q_x(x,t)+\beta(M,t)[xq_x(x,t)+q(x,t)]\,, \tag{2.49}$$

where α and β are related to the functions γ and μ through

$$\gamma(z,t) = -2iz\alpha[(2iz)^2, t], \quad \mu(z,t) = -2iz\beta[(2iz)^2, t], \tag{2.50}$$

and the (integro-differential scalar) operator M is defined by the formula

$$Mf(x) = f_{xx}(x) + 4\varepsilon q^2(x,t)f(x) + 4\varepsilon q_x(x,t) \int_x^{+\infty} dx' q(x',t) f(x'). \tag{2.51}$$

For instance, the mKdV corresponds to $\alpha(z,t) = z$ and $\beta(z,t) = 0$.

In case ii), one can similarly write, in place of the vector Equation (2.22), the scalar (complex) equation

$$iq_t(x,t) = \eta(S,t)q(x,t) + v(S,t)xq(x,t), \tag{2.52}$$

where now the *real* functions η and v are related to γ and μ by the formula

$$\gamma(z,t) = -i\eta(z,t), \qquad \mu(z,t) = -iv(z,t), \tag{2.53}$$

while the integro-differential scalar operator S is defined as follows:

$$Sf(x) = (2i)^{-1}\left\{-f_x(x) + 2\varepsilon q(x,t) \int_x^{+\infty} dx'[q(x',t)f^*(x') - q^*(x',t)f(x')]\right\}. \tag{2.54}$$

(Warning: this operator does not commute with complex constants, namely $Scf(x) \neq cSf(x)$ if c is not real). For instance, the NLS equation corresponds to $\eta(z,t) = -(2iz)^2$ and $v(z,t) = 0$ $(\varepsilon = -)$.

Note that the definitions (2.50) and (2.53) imply automatically that γ and μ satisfy the conditions displayed above respectively for case i) and ii).

We end this Section mentioning that a more general class of nonlinear evolution equations may be solved via the spectral transform associated to the matrix Schrödinger spectral problem (see the second paper of [13]); the extension of such a class to include non-translation-invariant contributions such as those considered above has however not yet been reported, and therefore we postpone to a separate paper also the discussion of the conservation laws in such more general context.

3. Conserved Quantities for the Class of Nonlinear Evolution Equations Solvable by the Spectral Transform Associated to the Schrödinger Linear Problem

It is convenient to associate a function $\varphi(k,t)$ to $T(k,t)$ through the definition

$$T(k,t) = T(-k,t)\exp[4i\varphi(k,t)]. \tag{3.1}$$

It is then immediately seen, inserting (3.1) in (2.14), that $\varphi(k,t)$ satisfies the equation

$$\varphi_t(k,t) + k\beta(-4k^2,t)\varphi_k(k,t) = -k \int_{-\infty}^{+\infty} dx\tilde{\beta}(-4k^2, L, t)[xu_x(x,t) + 2u(x,t)]. \tag{3.2}$$

It is moreover easy to see that $\varphi(k,t)$ is odd in k, and it has the asymptotic expansion

$$\varphi(k,t) = \sum_{n=0}^{N} \varphi^{(n)}(t)(2k)^{-2n-1} + 0(k^{-2N-3}). \tag{3.3}$$

In fact the sum in the r.h.s. of (3.3) may well converge in the limit $N \to \infty$ (see below for an explicit example); but the weaker statement that (3.3) provides an asymptotic expansion is sufficient for the following developments.

Let us now insert (3.3) in (3.2), employing moreover the power expansion of $\beta(z,t)$

$$\beta(z,t)=\sum_{m}\beta_m(t)z^m\,. \tag{3.4}$$

Such a representation of $\beta(z,t)$ must of course always be possible, in order that the nonlinear evolution Equation (2.1) make sense; actually the simpler, and more interesting, instances are the cases in which the sum in the r.h.s. of (3.4) contains only a finite number of terms. Note however that at this stage we need not limit our consideration to such cases; nor are we assuming that the sum in the r.h.s. of (3.4) extends only over positive values of m.

With this insertion two formulas are obtained, resulting from the identification of the coefficients of the positive respectively negative (odd) powers of k. The first formula reads

$$\varphi^{(n)}(t)=(-)^{n+1}[2(2n+1)]^{-1}C^{(n)}(t)\,, \tag{3.5}$$

having defined

$$C^{(n)}(t)=\int_{-\infty}^{+\infty}dxL^n[xu_x(x,t)+2u(x,t)]\,,\qquad n=0,1,2,\ldots\,. \tag{3.6}$$

Note that this equation has nothing to do with the time-evolution problem; it expresses merely a property of the Schrödinger spectral problem (2.5–2.6), namely that the "phase" $\varphi(k)$, related to the transmission coefficient $T(k)$ by (3.1), has an asymptotic expansion,

$$\varphi(k)=\sum_{n=0}(2k)^{-2n-1}(-)^{n+1}[2(2n+1)]^{-1}C^{(n)}\,, \tag{3.7}$$

whose coefficients are given, in terms of the "potential" u, by the explicit formula (3.6) (with the operator L defined by (2.3)). Incidentally, an explicit check of this formula in the special case of the reflectionless potential

$$u(x)=-2p^2/\cosh^2[p(x-\xi)] \tag{3.8}$$

(corresponding to one soliton) is provided by the expression of $T(k)$ appropriate to this case,

$$T(k)=(k+ip)/(k-ip)\,, \tag{3.9}$$

yielding

$$\varphi(k)=\operatorname{arctg}(p/k)=\sum_{n=0}^{\infty}(-)^n(2n+1)^{-1}(p/k)^{2n+1}\,, \tag{3.10}$$

namely, through (3.7),

$$C^{(n)}=-2(2p)^{2n+1}\,; \tag{3.11}$$

while this same result also obtains from (3.6), since a simple computation, with $u(x)$ given by (3.8), yields

$$C^{(0)} = \int_{-\infty}^{+\infty} dx[xu_x(x) + 2u(x)] = -4p \tag{3.12}$$

and

$$C^{(n)} = 4p^2 C^{(n-1)}, \tag{3.13}$$

the last formula being implied through (3.6) by the relation

$$L[xu_x(x) + 2u(x)] = 4p^2[xu_x(x) + 2u(x)] - 4pu_x(x) \tag{3.14}$$

following from (3.8), and by the well-known general formula

$$\int_{-\infty}^{+\infty} dxL^n u_x(x) = 0, \qquad n = 0, 1, 2, \ldots. \tag{3.15}$$

The second formula mentioned above obtains from the identification of the coefficients of the negative powers of k when (3.3) is inserted in (3.2); it can now be written, using (3.5), in terms of the quantities $C^{(n)}(t)$ defined by (3.6). It reads:

$$C_t^{(n)}(t) = (2n+1) \sum_m \beta_m(t) C^{(n+m)}(t), \qquad n = 0, 1, 2, \ldots, \tag{3.16}$$

implying of course that all the quantities $C^{(n)}$ are time-independent if $\beta = 0$, i.e. for the "old" class of nonlinear evolution equations. We have therefore found a compact expression, Equation (3.6), for the constants of motion of the class of equations (2.1) with $\beta = 0$ (this class includes, as already mentioned, the KdV equation, corresponding to $\alpha(z, t) = z$, and all the so-called higher KdV equations). These constants coincide of course with those already known, for instance

$$C^{(0)} = \int_{-\infty}^{+\infty} dxu(x, t), \tag{3.17a}$$

$$C^{(1)} = -3 \int_{-\infty}^{+\infty} dx[u(x, t)]^2, \tag{3.17b}$$

$$C^{(2)} = 5 \int_{-\infty}^{+\infty} dx\{[u_x(x, t)]^2 + 2[u(x, t)]^3\}. \tag{3.17c}$$

If instead $\beta \neq 0$, namely in the case of the novel class of evolution Equations (2.1), the quantities $C^{(n)}$ are no more constant, their time-evolution being determined by (3.16). This is an infinite system of linear first-order ordinary differential equations; but as we presently show, it is actually possible to solve it, at least formally. Before doing this, let us however pause to consider the simplest case, when the function $\beta(z, t)$ is a (t-independent) polynomial (of degree, say, M), namely (see (3.4)) $\beta_m(t) = \beta_m$ for $0 \leqq m \leqq M$, $\beta_m(t) = 0$ for $m < 0$ and for $m > M$. Then the system (3.16) becomes triangular, implying immediately that its solutions have, provided $\beta_0 \neq 0$, the structure

$$C^{(n)}(t) = \sum_{l=n}^{\infty} c_{nl} \exp[(2l+1)\beta_0 t], \qquad n = 0, 1, 2, \ldots, \tag{3.18}$$

the (time-independent!) quantities c_{nl} being determined by the β_m's and by the initial values $C^{(n)}(t_0)$ of the $C^{(n)}$'s. This formula implies that in this case (which also yields the simpler nonlinear evolution equations contained in the class (2.1)) all the quantities $C^{(n)}(t)$ are periodic in t with the same period T, $C^{(n)}(t+T)=C^{(n)}(t)$, iff β_0 is pure imaginary, $\beta_0=2\pi i/T$. Thus for this simple class of nonlinear evolution equation the phenomenon of periodicity of the quantities $C^{(n)}$ never occurs if the consideration is restricted to real equations (as is generally the case in applications). The results given below (or, for that matter, the structure of the system (3.16)) imply that the phenomenon of periodicity of all quantities $C^{(n)}(t)$ can occur, for real and autonomous equations (i.e., for real time independent β in (2.1)) only if in the r.h.s. of (3.4) there are nonvanishing coefficients β_m both for positive and negative values of m.

Another case worth mentioning is when the function β is a polynomial (with time-independent coefficients) in z^{-1} (say, in (3.4), $\beta_m(t)=\beta_m$ for $-M\leqq m\leqq 0$, $\beta_m(t)=0$ for $m>0$ and for $m<-M$). Then the system (3.16) is again triangular, and its solutions have, provided $\beta_0\neq 0$, the simple structure

$$C^{(n)}(t)=\sum_{l=0}^{n} c_{nl}\exp[(2l+1)\beta_0 t]\,,\qquad n=0,1,2,\ldots, \tag{3.19}$$

again with the constants c_{nl} determined by the coefficients β_m and by the initial values $C^{(n)}(t_0)$ (for instance in this case $c_{00}=C^{(0)}(t_0)\exp(-\beta_0 t_0)$). Thus in this case the time evolution of the quantities $C^{(n)}(t)$ is very simple indeed (and again of course it is periodic iff β_0 is pure imaginary). On the other hand the nonlinear evolution equations have a more complicated structure than those yielded by a β that is polynomial in z.

Let us now proceed to discuss the formal solution of the system (3.16) in the general case. It is provided by the formula

$$C^{(n)}(t)=\int_{-\infty}^{+\infty} dx[\theta(t,t_0;L_0)]^{2n+1}L_0^n[xu_x(x,t_0)+2u(x,t_0)]\,,\quad n=0,1,2,\ldots, \tag{3.20}$$

where L_0 is the operator L of Equation (2.3), but with $u(x,t)$ replaced by $u(x,t_0)$, and the function $\theta(t,t_0;\lambda_0)$ is defined by the first-order nonlinear ordinary differential equation

$$\theta_t(t,t_0;\lambda_0)=\theta(t,t_0;\lambda_0)\beta[\lambda_0\theta^2(t,t_0;\lambda_0),t] \tag{3.21}$$

and by the boundary condition

$$\theta(t_0,t_0;\lambda_0)=1\,, \tag{3.22}$$

implying (see (2.18–2.19))

$$\theta(t,t_0;-4k_0^2)=\chi(t,k_0)/k_0\,. \tag{3.23}$$

Indeed (3.23) insures that the quantities $C^{(n)}(t)$ of Equation (3.20) satisfy the initial condition (i.e. Eq. (3.6) for $t=t_0$), while the fact that (3.20) satisfies (3.16) can be formally verified by straightforward substitution, using (3.21) and (3.4) (assuming of course that the priority of the operations of t-differentiation and x-integration can be exchanged, and treating moreover the operator L_0 as if it were an ordinary variable; this can be formally justified by performing a power expansion, as indicated below).

It should be emphasized that the function $\theta(t, t_0, \lambda_0)$ can in many cases be explicitly computed solving the differential Equation (3.21); for an example see Subsection 5.1. It may then be also possible to compute in explicit form the coefficients $\theta_{nl}(t, t_0)$ of the expansion[1]

$$[\theta(t, t_0; \lambda_0)]^{2n+1} = \sum_{l=0}^{\infty} \theta_{nl}(t, t_0)\, \lambda_0^l\,; \tag{3.24}$$

and using these coefficients (3.20) can be rewritten formally in the form

$$C^{(n)}(t) = \sum_{l=0}^{\infty} \theta_{nl}(t, t_0)\, C^{(l+n)}(t_0)\,. \tag{3.25}$$

Equation (3.20), as well as this last equation (when applicable) can be interpreted in two ways. Firstly, they display explicitly the time evolution of each $C^{(n)}(t)$, once the (initial) values of all the quantities $C^{(l+n)}(t_0)$, $l \geqq 0$, are given; this is particularly useful if the t-dependence of the (explicit) coefficients $\theta_{nl}(t, t_0)$ is periodic. A second interpretation of (3.20) or (3.25) obtains from the remark that these formulae remain valid if the roles of t and t_0 are exchanged, so that instead of (3.20) one can write

$$C^{(n)}(t_0) = \int_{-\infty}^{+\infty} dx [\theta(t_0, t; L)]^{2n+1} L^n [x u_x(x, t) + 2u(x, t)],\ n = 0, 1, 2, \ldots \tag{3.26}$$

(where of course the operator L is now defined by (2.3)), and instead of (3.25)

$$C^{(n)}(t_0) = \sum_{l=0}^{\infty} \theta_{nl}(t_0, t)\, C^{(n+l)}(t) \tag{3.27}$$

(the function θ, and the coefficients θ_{nl}, being always defined by (3.21–22) and (3.24)). The last two formulae define now quantities that are time-independent for the (generalized) class (2.1) of nonlinear evolution equations; however, since generally the λ_0-dependence of $\theta(t_0, t; \lambda_0)$ is not polynomial, the sum in (3.27) does extend to infinity, so that even when the coefficients $\theta_{nl}(t_0, t)$ are explicitly known and have a simple time-dependence (see the example of subsection 5.1), the theoretical significance of these "constants of the motion" is not transparent and their practical usefulness is moot.

These results imply of course that if the function $\theta(t, t_0; \lambda_0)$ defined by (3.21) and (3.22) is, for all values of λ_0, periodic in t with period T, all the quantities $C^{(n)}(t)$ are also periodic, $C^{(n)}(t+T) = C^{(n)}(t)$. Necessary conditions for this have been mentioned above. Note that they imply that the phenomenon of periodicity of the quantities $C^{(n)}(t)$ does not occur for the simpler equations of the class (2.1). As we will see in the following Section (and in Subsection 5.3) the situation is different for the class (2.22) of nonlinear evolution equations solvable via the spectral transform associated to the generalized Zakharov-Shabat spectral problem.

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1 Clearly this expansion holds only if the function $\theta(t, t_0; \lambda_0)$ is holomorphic in λ_0 in the neighborhood of $\lambda_0 = 0$. A sufficient condition for this is that $\beta(z, t)$ be itself holomorphic in z in the neighborhood of $z = 0$

Let us emphasize that periodicity of all the quantities $C^{(n)}$ does not imply that the generic solution of (2.1) is also periodic.

We end this Section noting that, in the special case $\beta(z,t)=\beta_0(t)$, the system (3.16) decouples and its explicit solution is then given by the formula

$$C^{(n)}(t)=\exp\left[(2n+1)\int_{t_0}^{t} dt'\beta_0(t')\right]C^{(n)}(t_0). \tag{3.28}$$

This is, however, also the case in which the novel nonlinear evolution Equation (2.1) is reducible by a change of variables [30] to the old form (with $\beta=0$).

4. Conserved Quantities for the Class of Nonlinear Evolution Equations Solvable by the Spectral Transform Associated to the (Generalized) Zakharov-Shabat Linear Problem

We proceed now in close parallelism to the treatment of the preceeding Section, and therefore our presentation is terse. The use in some cases (below and above) of the same notation in the Schrödinger and Zakharov-Shabat contexts should cause no confusion, since the two cases are always separate.

It is now convenient to associate a function $\varphi(k,t)$ to $\beta^{(\pm)}(k,t)$ through the definition

$$\beta^{(+)}(k,t)=\exp[2i\varphi(k,t)]\,\beta^{(-)}(k,t). \tag{4.1}$$

It is then immedately seen that $\varphi(k,t)$ satisfies the equation

$$\varphi_t(k,t)+\frac{i}{2}\mu(k,t)\,\varphi_k(k,t)$$

$$=-\frac{i}{2}\int_{-\infty}^{+\infty} dx(v(x,t),(\sigma_+-\sigma_-)\,\tilde{\mu}(k,L;t)\,xv(x,t)). \tag{4.2}$$

Insertion in this equation of the asymptotic expansion (that might even converge as $N\to\infty$)

$$\varphi(k,t)=\sum_{n=1}^{N}\varphi^{(n)}(t)\,(2ik)^{-n}+0(k^{-N-1}), \tag{4.3}$$

yields again two formulas, resulting from the identification of the coefficients of the positive respectively negative powers of k under the assumption that $\mu(z,t)$ is expressed by the power expansion formula

$$\mu(z,t)=\sum_m \mu_m(t)\,(2iz)^m \tag{4.4}$$

(this is the analog of Equation (3.4)).

The first formula reads

$$\varphi^{(n)}(t)=iC^{(n)}(t)/n\,, \tag{4.5}$$

where we define

$$C^{(n)}(t)=\int_{-\infty}^{+\infty} dx(v(x,t),(\sigma_+-\sigma_-)\,(2iL)^n xv(x,t)),\; n=1,2,\ldots \tag{4.6}$$

Thus it expresses a property of the spectral problem rather than of the time evolution. For instance, for the single soliton case

$$r(x) = -i\varrho^{(+)}\exp(-p_-\xi - p_+x)/\cosh[p_-(x-\xi)], \tag{4.7a}$$

$$q(x) = i\varrho^{(-)}\exp(-p_-\xi + p_+x)/\cosh[p_-(x-\xi)], \tag{4.7b}$$

where $p_\pm = -i(k^{(+)} \pm k^{(-)})$ and $\varrho^{(+)}\varrho^{(-)} = -2p_-^2 \exp(2p_-\xi)$ (see Eq. (4.2.21) of I), one easily finds that

$$\beta^{(\pm)}(k) = [k - k^{(\mp)}]/[k - k^{(\pm)}], \tag{4.8}$$

so that in this case

$$C^{(n)} = [2ik^{(-)}]^n - [2ik^{(+)}]^n; \tag{4.9}$$

and one can verify by explicit computation, at least for the first few values of n, the consistency of this last formula with (4.6) and (4.7).

The second formula, rewritten using (4.5), reads

$$C_t^{(n)}(t) = -n \sum_m \mu_m(t)\, C^{(n+m-1)}(t), n = 1, 2, 3, \ldots; \tag{4.10}$$

it characterizes therefore the time evolution of the quantities $C^{(n)}(t)$. Clearly it implies that these quantities are constants of motion for the "old" class of nonlinear evolution equations, i.e. (2.22) with $\mu \equiv 0$ (including the mKdV, NLS and SG equations, corresponding respectively to $\gamma(z,t) = -(2iz)^3$ and $q = r$, $\gamma(z,t) = i(2iz)^2$ and $q = -r^*$, $\gamma(z,t) = -(2iz)^{-1}$ and $q = -r$, with an additional change of dependent variable in this latter case). It can indeed be easily verified that the compact expression (4.6) (with (2.27)) reproduces the well-known constants, for instance

$$C^{(1)}(t) = -\int_{-\infty}^{+\infty} dx\, q(x,t)\, r(x,t), \tag{4.11a}$$

$$C^{(2)}(t) = \int_{-\infty}^{+\infty} dx [q_x(x,t)\, r(x,t) - q(x,t)\, r_x(x,t)], \tag{4.11b}$$

$$C^{(3)}(t) = 3\int_{-\infty}^{+\infty} dx \{[q(x,t)\, r(x,t)]^2 + q_x(x,t)\, r_x(x,t)\}. \tag{4.11c}$$

Actually the system (4.7) can formally be solved also in the case with $\mu \neq 0$:

$$C^{(n)}(t) = \int_{-\infty}^{+\infty} dx (v(x,t_0), (\sigma_+ - \sigma_-) \cdot [2i\chi(t,t_0;L_0)]^n x v(x,t_0)), n = 1, 2, 3, \ldots. \tag{4.12}$$

Here L_0 is the operator L of Equation (2.27), but with $v(x,t)$ replaced by $v(x,t_0)$, and χ is essentially the function already defined in the second part of Section 2, being characterized by the differential equation

$$\chi_t(t,t_0;k_0) = \tfrac{1}{2} i\mu[\chi(t,t_0;k_0), t] \tag{4.13}$$

and the initial condition

$$\chi(t_0,t_0;k_0) = k_0. \tag{4.14}$$

The function χ can in many cases be explicitly evaluated (see Section 5 below). It may then be also possible to evaluate in explicit form the coefficients of the expansion

$$[\chi(t,t_0;k_0)]^n = \sum_{l=0}^{\infty} \chi_{nl}(t,t_0)\,k_0^l, \tag{4.15}$$

and using these coefficients to express the quantities $C^{(n)}(t)$ as linear combinations of the quantities $C^{(n)}(t_0)$:

$$C^{(n)}(t) = \sum_{l=1}^{\infty} \chi_{nl}(t,t_0)\,C^{(l)}(t_0),\, n=1,2,3,\ldots. \tag{4.16}$$

Note that (4.14) implies that the coefficients χ_{nl} in (4.15) (and therefore also in (4.16)) vanish identically for $l<n$ if $\mu(z,t)/z$ is analytic in $z=0$.

This last equation, as well as (4.12), displays the time-evolution of each $C^{(n)}(t)$ whenever all the quantities $C^{(l)}(t_0)$ are given; and in some cases (for instance, if χ is periodic in t) it provides important information on the time-evolution of each $C^{(n)}(t)$ even if the initial values of the quantities $C^{(n)}(t_0)$ are unknown. Examples are discussed in the following Section. The two Equations (4.12) and (4.16) also provide explicit, if hardly useful, expressions of time-independent quantities, after the rôles of t and t_0 have been exchanged.

In the special case $\mu(z,t)=2i\mu_1(t)z$ the system (4.10) decouples, and its explicit solution reads simply

$$C^{(n)}(t) = \exp\left[-n\int_{t_0}^{t} dt'\,\mu_1(t')\right] C^{(n)}(t_0),\, n=1,2,3,\ldots; \tag{4.17}$$

but this is also the case in which the novel nonlinear evolution Equation (2.22) is reducible by a change of variables to the old form (with $\mu=0$; see the second paper of [29]).

Note that, in order that the system (4.10) become triangular, it is required either that all μ_m vanish for $m\leqq 0$, or that they all vanish for $m\geqq 2$. In these cases, and if moreover μ is time-independent, periodic solutions obtain iff μ_1 is pure imaginary; and in the second case, they have a very simple structure and can be explicitly evaluated. On the other hand we emphasize that in the case of this section, in contrast to the preceeding section, the periodic behavior can emerge even when $\mu(z,t)$ is real, time-independent and polynomial in z, i.e. for the simpler nonlinear equations of the class (2.22). An explicit example is discussed in Subsection 5.3.

We end this Section reporting the special form that take the results of this Section for the two subclasses of nonlinear evolution equations described under case i) and ii) at the end of Section 2.

In case i), all the quantities $C^{(n)}$ vanish if n is even, while those with odd index can be conveniently rewritten using the operator M of Equation (2.51):

$$C^{(2n+1)}(t) = -2\varepsilon\int_{-\infty}^{+\infty} dx\,q(x,t)\,M^n[xq_x(x,t)+q(x,t)],\, n=0,1,2,\ldots. \tag{4.18}$$

In the special case of the single soliton solution, they take the values

$$C^{(2n+1)} = 2p^{2n+1},\, n=0,1,2,\ldots, \tag{4.19}$$

where $p=p_-$ is real (see (4.7); note that in this case p_+ vanishes). This last formula applies only for $\varepsilon=+$, because only for this value of ε solitons exist.

In case ii), the quantities $C^{(n)}$ turn out to be real or imaginary depending on the parity of n; again they are more conveniently written using the operator S introduced at the end of Section 2, Equation (2.54). The formula reads

$$C^{(n)}(t)=(2i)^{n+1}\varepsilon \int_{-\infty}^{+\infty} dx\, \mathrm{Im}[q^*(x,t)\, S^n xq(x,t)], n=1,2,3,\ldots. \tag{4.20}$$

In the special case of the single soliton (see (4.7))

$$C^{(n)}=-(2i)^{n+1}\,\mathrm{Im}[k^{(+)n}], n=1,2,3,\ldots. \tag{4.21}$$

This last formula applies only for $\varepsilon=-$, because only for this value of ε solitons exist.

The simplifications that these reductions imply on the formulas given in this Section for the general class of nonlinear evolution equations (2.22) are too trivial to require explicit display.

5. Examples

In this Section we apply the results of the two preceeding Sections to the generalized KdV, mKdV and NLS equations [30–32]. Our treatment is terse, since we draw on the analysis of these equations [30–32] without reporting here all the corresponding results.

5.1. *Generalized* KdV *Equation* [30]

This equation reads

$$u_t=\alpha_0 u_x+\alpha_1(u_{xxx}-6u_x u)+\beta_0(xu_x+2u)+\beta_1 \cdot\left[xu_{xxx}+4u_{xx}-6xu_x u-8u^2+2u_x \int_x^{+\infty} dx' u(x',t)\right], \tag{5.1.1}$$

where α_0, α_1, β_0 and β_1 are constants, and of course $u\equiv u(x,t)$ (except where explicitly indicated otherwise).

We consider firstly the case with $\beta_0 \neq 0$, $\beta_1 \neq 0$. Then (from (3.21) and (3.22); or see Equation (8) of [30])

$$\theta(t,t_0;\lambda_0)=\theta(t-t_0;\lambda_0) =\exp[\beta_0(t-t_0)]\,[1+\mu\lambda_0\{1-\exp[2\beta_0(t-t_0)]\}]^{-1/2} \tag{5.1.2}$$

with

$$\mu=\beta_1/\beta_0, \tag{5.1.3}$$

so that (see (3.24)).

$$\theta_{nl}(t,t_0)=\theta_{nl}(t-t_0)=\{[2(n+l)-1]!!/[l!(2n-1)!!]\}\cdot \exp[(2n+l+1)\beta_0(t-t_0)]\,\mu^l \sinh^l[\beta_0(t-t_0)]. \tag{5.1.4}$$

Insertion of these expressions in (3.20) and (3.25), or in (3.26) and (3.27), yields therefore the formulae appropriate to this case. Note that the periodicity of all the quantities $C^{(n)}(t)$ is now confirmed, iff β_0 is imaginary. Also note that, if instead β_0 is real, a sufficient condition for the absolute convergence of the sum (3.25) in this case is that the initial data obey (for some $\varepsilon>0$) the limitation

$$\lim_{n\to\infty} [C^{(n)}(t_0)\exp(\varepsilon n^{1+\varepsilon})]=0; \tag{5.1.5}$$

if instead, for some (positive) λ and for some N

$$|C^{(n)}(t_0)|<\lambda^n \quad \text{for} \quad n>N, \tag{5.1.6}$$

the series (3.24) converges only for values of t such that

$$\tfrac{1}{2}\ln[1-|\mu\lambda|^{-1}]<\beta_0(t-t_0)<\tfrac{1}{2}\ln[1+|\mu\lambda|^{-1}], \tag{5.1.7}$$

the lower-limit restriction being operative only if $|\lambda\mu|>1$.

In the special case $\beta_0=0$ (with $\beta_1\neq0$), (5.1.2) is replaced by

$$\theta(t,t_0;\lambda_0)=\theta(t-t_0;\lambda_0)=[1-2\beta_1\lambda_0(t-t_0)]^{-1/2} \tag{5.1.8}$$

and (5.1.4) by

$$\begin{aligned}\theta_{nl}(t,t_0)&=\theta_{nl}(t-t_0)\\ &=\{[2(n+l)-1]!!/[l!(2n-1)!!]\}\,[\beta_1(t-t_0)]^l.\end{aligned} \tag{5.1.9}$$

Also in this case (5.1.5) guarantees the absolute convergence of (3.25), while (5.1.6) implies convergence only for values of t such that

$$|t-t_0|<|2\beta_1\lambda|^{-1}. \tag{5.1.10}$$

Finally if instead $\beta_1=0$ (with $\beta_0\neq0$),

$$\theta(t-t_0;\lambda_0)=\theta(t-t_0)=\exp[\beta_0(t-t_0)], \tag{5.1.11}$$

so that in this case the time-dependence of the quantities $C^{(n)}$ is given by the completely explicit formula

$$C^{(n)}(t)=\exp[(2n+1)\beta_0(t-t_0)]\,C^{(n)}(t_0). \tag{5.1.12}$$

This case is however not really new, being reducible to the usual KdV equation by a change of variables [30] (this equation had been previously investigated by Hirota and Satsuma [33]; and its reducibility to the KdV equation was first pointed out by Ablowitz and Wadati [private communication]).

For the single-soliton solution the quantities $C^{(n)}(t)$ are of course given by Equation (3.11), with p replaced by the (time-dependent) function $p(t)$ given in explicit form in [30]; it is instructive to verify that this same result obtains from the formalism given above, starting from the initial values $C^{(n)}(0)$ given again by Equation (3.11), but now with p replaced by $p_0=p(0)$.

5.2. *Generalized* mKdV *Equation* [31][2]

The formulae for this case are presented below without any comment, in view of their similarity to those for the generalized KdV equation discussed in the preceeding subsection; but we use here the notation of Section 4 (and of the second half of Section 2), while in the preceeding Subsection the appropriate notation was of course that of Section 3 (and of the first half of Section 2).

The equation reads

$$q_t+\gamma_1 q_x+\gamma_3(q_{xxx}+6q^2q_x)+\mu_1(q+xq_x)+\mu_3 \cdot\left[xq_{xxx}+3q_{xx}+4q^3+6xq^2q_x-2q_x\int_x^{+\infty}dx'q^2(x',t)\right]=0, \tag{5.2.1}$$

with γ_1, γ_3, μ_1 and μ_3 real constants, and $q\equiv q(x,t)$ (except where explicitly indicated otherwise).

For $\mu_1 \neq 0, \mu_3 \neq 0$,

$$\chi(t,t_0;k_0)=\chi(t-t_0;k_0) =k_0[\exp[2\mu_1(t-t_0)]+4\beta k_0^2\{1-\exp[2\mu_1(t-t_0)]\}]^{-1/2}, \tag{5.2.2}$$

with

$$\beta=\mu_3/\mu_1, \tag{5.2.3}$$

so that $\chi_{nl}(t,t_0)=\chi_{nl}(t-t_0)$ vanishes if $l<n$ or if l and n have different parities while

$$\chi_{2m+1,2p+2m+1}(\tau)=(4\beta)^p(2m+2p-1)!!/[p!(2m-1)!!]\cdot \exp[-(2m+p+1)\mu_1\tau]\sinh^p(\mu_1\tau). \tag{5.2.4}$$

The coefficients χ_{nl} with even n could be easily computed, but they are not relevant in view of the fact that all $C^{(n)}$'s with even n in this case vanish (see the last part of Section 4).

Finally if $\mu_3=0$ (with $\mu_1\neq 0$),

$$\chi(t,t_0;k_0)=\chi(t-t_0;k_0)=k_0\exp[-\mu_1(t-t_0)], \tag{5.2.5}$$

implying in this case

$$C^{(n)}(t)=\exp[-n\mu_1(t-t_0)]\,C^{(n)}(t_0), \tag{5.2.6}$$

which merely confirms Equation (4.17).

5.3. *Generalized* NLS *Equation* [32]

We refer again to the notation of Section 4 (and the second half of Section 2), and present the results tersely.

The equation to be analyzed reads

$$iq_t+(\gamma_0+i\mu_1+\mu_0 x)q+i(\gamma_1+\mu_1 x)q_x +(\gamma_2+\mu_2 x)(q_{xx}+2|q|^2q) +2\mu_2\left(q_x-q\int_x^{+\infty}dx'|q(x',t)|^2\right)=0, \tag{5.3.1}$$

[2] Let us recall that in this case there is one quantity, namely the integral of the field over all space, that is conserved (not only for the mKdV, but for the generalized mKdV as well; see [31]); this quantity, however, does not belong to the sequence of $C^{(n)}$ considered here

with γ_j and μ_j, $j=0,1,2$, real constants and $q\equiv q(x,t)$ (except where explicitly indicated otherwise).

For the generic case with $\mu_j \neq 0$ ($j=0,1,2$),

$$\chi(t,t_0;k_0)=[k_0 c(t-t_0)-(k_0+\mu_0/\mu_1)s(t-t_0)]/ \cdot [c(t-t_0)+(1-4k_0\mu_2/\mu_1)s(t-t_0)], \tag{5.3.2}$$

where

$$c(t)=\cosh(\mu t/2), \tag{5.3.3a}$$

$$s(t)=(\mu_1/\mu)\sinh(\mu t/2), \tag{5.3.3b}$$

$$\mu=(4\mu_0\mu_2+\mu_1^2)^{1/2}. \tag{5.3.4}$$

Note that c and s, Equations (5.3.3), are well defined, and real, for all real and imaginary values of μ. The corresponding expression of the expansion coefficients reads

$$\chi_{nl}(t,t_0)=\chi_{nl}(t-t_0)=(4\mu_2/\mu_1)^{l-n}[1-c(t-t_0)/s(t-t_0)]^n \cdot [1+c(t-t_0)/s(t-t_0)]^{-l} f_{nl}([c^2(t-t_0)-s^2(t-t_0)]^{-1}), \tag{5.3.5}$$

with

$$f_{nl}(x)=x d^l/dx^l[x^{l-1}(1-x)^n]/l!, l=1,2,3,\dots. \tag{5.3.6}$$

The special expressions that these results take if some of the quantities μ_j, or the quantity μ of Equation (5.3.4), vanish, are easily obtained from the formulae given above, and will not be reported here. In some cases these expressions simplify considerably; for instance if μ_0 vanishes, the argument of the functions f_{nl} in the r.h.s. of (5.3.5) becomes simply unity (see (5.3.4) and (5.3.3)), implying that χ_{nl} vanishes if $l<n$ (see (5.3.6); and note the consistency of this result with (5.3.2) and (4.15)). Of course the same simplification occurs if $\mu_2=0$; but in this case the overall simplification is more drastic, implying

$$\chi_{nl}(\tau)=(-)^{n+l}\binom{n}{l}(\mu_0/\mu_1)^{n-l}\exp[-(n+l)\mu_1 t/2]\sinh^{n-l}(\mu_1 t/2), \quad 0\leqq l\leqq n, \tag{5.3.7a}$$

$$\chi_{nl}(\tau)=0, l>n. \tag{5.3.7b}$$

Thus in this last case the time-dependence of all the quantities $C^{(n)}(t)$ is given by the finite sum

$$C^{(n)}(t)=\sum_{l=0}^{n}\chi_{nl}(t-t_0)C^{(l)}(t_0); \tag{5.3.8}$$

this is, however, not a very interesting instance, since it is just the case in which the generalized NLS equation (5.3.1) can be reduced by appropriate changes of variable to the standard NLS (see [32]).

We end this Subsection noting that these results imply that now the necessary and sufficient condition for the periodicity (with the period T defined below) of *all* the quantities $C^{(n)}(t)$ is that the quantity μ of Equation (5.3.4) be pure imaginary,

$$\mu = i\omega = 2\pi i/T. \tag{5.3.9}$$

Of course this condition may well hold even when the 3 constants μ_j are all real (as it is assumed in this Subsection). Note that if μ is imaginary, periodic features also appear in the behavior of the solitons [32]; we emphasize however that, in this case, the generic solution of (5.3.1) (or, for that matter, even the soliton solution) is of course not periodic.

6. Concluding Remarks

This paper has investigated the time evolution of the quantities $C^{(n)}(t)$ defined by (3.6) and (2.3), when the scalar field $u(x,t)$ evolves in time according to (2.1); and the time evolution of the quantities $C^{(n)}(t)$ defined by (4.6), (2.23) and (2.27), when the 2-component vector field $v(x,t)$ of Equation (2.24) evolves in time according to (2.22). One conclusion of this analysis has been that the quantities $C^{(n)}$ are constants of the motion if the explicitly x-dependent part of the evolution equations is missing (i.e., $\beta \equiv 0$ in (2.1), $\mu \equiv 0$ in (2.22); in these cases the well-known classes of nonlinear evolution equations extensively studied in the last few years are recovered). While the existence of these constants of the motion in these cases was of course well-known, their compact expressions, Equations (3.6) and (4.6), appears to be new. It is remarkable that the variable x enters explicitly in these formulae, although it never appears in the final expressions of the quantities $C^{(n)}$; like some ingredients in certain cooking recipes, that have to be put in at the beginning but are filtered out before the dish is served.

The way these quantities $C^{(n)}$ have been introduced above, although convenient for the purposes of this paper, is probably not the most appropriate one to clarify their origin. To that end it would presumably be preferable to analyze the effect of scale transformations on the spectral problems; indeed combinations such as $xu_x(x)+2u(x)$ (in the context of the Schrödinger spectral problem) or $xv(x)$ (in the context of the Zakharov-Shabat spectral problem) are clearly related to the virial theorem. But we defer such an analysis to a separate treatment.

After this paper was completed it has been brought to our attention that an extension of the spectral transform method such as the one described in [29] has been performed by Newell [34, 35]. No discussion of conservation laws in this more general context is contained in [34]; as for [35], while we have been informed of the existence of this paper, we have not yet seen it.

Note Added in Proof

The novel nonlinear evolution equations of the enlarged class solvable by the spectral transform associated to the Schrödinger spectral problem, considered in this paper and in preceeding ones of this series [29, 30], possess a (rather large) class of solutions that remain regular through a finite span of the time evolution (indeed, in some cases, throughout the entire time development): they are the multisoliton solutions, whose explicit shape (as a function of x) and evolution (as a function of t) have been described in detail in some instances [29, 30]. These solutions obtain from initial data such that the corresponding reflection coefficient vanishes; for them, all the results reported in this paper hold without reservation. If instead to the initial data is associated a nonvanishing reflection coefficient, then for a large class of these nonlinear evolution equations (including the simpler, and therefore more interesting, ones) the corresponding solutions, even if they are regular at $t=0$, generally develop, for $t>t_s$ (where $t_s=0$ unless the initial reflection coefficient has compact support), a component that is singular (as a function of x, at $x=0$); in k-space, this is apparent from the fact that the reflection coefficient, for $t>t_s$, generally does not vanish as $k\to\pm\infty$, even though it had this property for $t=0$ (see (2.16) and note that, if $|\beta(z,t)|\to\infty$ as $|z|\to\infty$, $k_0(t,k)$ tends generally to finite values when $k\to\pm\infty$); in x-space, this may be traced to the vanishing at $x=0$ of the coefficient of the x-derivative of highest order in the evolution equation. The validity of all results in these cases becomes therefore questionable, since some of the basic assumptions on which the whole spectral theory is based are violated by the time evolution. It is likely, but nontrivial, that the essence of the results remain applicable within a more general framework, involving distributions rather than functions; but a discussion of this requires a separate paper.

Of course all this applies as well (with obvious changes in terminology) to the novel class of nonlinear evolution equations solvable by the spectral transform associated to the Zakharov-Shabat spectral problem [29, 31, 32].

The fact that these nonlinear evolution equations develop a singular behavior for most, but not all, regular initial data, is an intriguing phenomenon that underscores the mathematical interest of these equations but casts doubts on their phenomenological applicability.

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