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The φ_2^4 Quantum Field as a Limit of Sine-Gordon Fields

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Abstract. We exhibit the $\lambda\varphi_2^4$ quantum field theory as the limit of Sine-Gordon fields as suggested by the identity

$$\varphi^4/4! = \lim_{\varepsilon\to 0} (\varepsilon^{-4}\cos\varepsilon\varphi - \varepsilon^{-4} + \tfrac{1}{2}\varepsilon^{-2}\varphi^2).$$

The proofs of finite volume stability for the two models, due to Nelson and Fröhlich respectively, are unrelated. We find a generalized stability argument that incorporates ideas from both of the simpler cases. The above limit, for the Schwinger functions, then proceeds uniformly in ε.

As a by-product, let $(\varphi, d\mu)$ be a Gaussian random field, φ_κ $(1 \leqq \kappa < \infty)$ a regularization of φ, and V a function satisfying:

(i) $V(\varphi_\kappa) \geqq -a\kappa^\alpha$

(ii) $\| V(\varphi) - V(\varphi_\kappa)\|_p \leqq bp^\beta \kappa^{-\gamma}, \qquad 2 \leqq p < \infty.$

Then $e^{-V(\varphi)} \in L^1(d\mu)$ provided $\alpha(\beta-1) < \gamma$.

I. Introduction and Results

In this paper we show how to obtain the $\lambda\varphi_2^4$ quantum field theory as a uniform limit of Sine-Gordon ($\lambda_s \cos\varepsilon\varphi$) quantum fields. Formally one might expect such a relationship as a consequence of the identity

$$\lambda\varphi^4/4! = \lim_{\varepsilon\to 0} \lambda(\varepsilon^{-4}\cos\varepsilon\varphi - \varepsilon^{-4} + \tfrac{1}{2}\varepsilon^{-2}\varphi^2), \tag{1.1}$$

which suggests convergence of the $\lambda_s \cos\varepsilon\varphi$ model to $\lambda\varphi^4$ as $\varepsilon \to 0$, provided we perform infinite vacuum energy, mass and coupling constant renormalizations.

There are serious technical problems to be overcome before this idea may be extended to quantum field theory. To prove convergence of the corresponding Schwinger functions, some uniformity in ε will be needed, such as a uniform bound for $\langle e^{-V_\varepsilon}\rangle$ where V_ε is the finite-volume action. However the proofs that e^{-V} is integrable for the φ_2^4 and Sine-Gordon theory, due respectively to Nelson [1]

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and Fröhlich [2], are unrelated in structure. Thus a uniform bound on $\langle e^{-V_\varepsilon}\rangle$ cannot be obtained directly. We will generalize Nelson's proof (see Theorem 3) so as to allow interactions of Sine-Gordon type which have a Wick lower bound $V(\phi_\kappa) \geqq -\kappa^{\varepsilon^2/4\pi}$ as compared to $-(\ln\kappa)^2$ for φ_2^4. One of the basic estimates required for Theorem 3, a bound on $\|V(\varphi)-V(\varphi_\kappa)\|_p$, will require Fröhlich's methods as well as φ^4 estimates. We have therefore combined Nelson's and Fröhlich's results in a more general framework. A smooth transition from Sine-Gordon to φ^4 is possible in this setting, and in fact we prove uniformity in ε—see Theorems 1 and 2. Since the essential difficulties are ultraviolet effects, we will consider only the finite-volume interactions in this paper, but extension to the infinite-volume limit will not be difficult.

The Schwinger functions for volume Δ, which we take to be a unit square, for the models in (1.1) are defined by:

$$S_\varepsilon(f_1,\dots,f_n)=\int d\mu_0 e^{-\lambda V_\varepsilon}\varphi(f_1)\dots\varphi(f_n)/\int d\mu_0 e^{-\lambda V_\varepsilon}, \tag{1.2}$$

$$S(f_1,\dots,f_n)=\int d\mu_0 e^{-\lambda V}\varphi(f_1)\dots\varphi(f_n)/\int d\mu_0 e^{-\lambda V}. \tag{1.3}$$

Here $\varepsilon^2<4\pi, \lambda\geqq 0, f\in\mathscr{S}(R^2), d\mu_0$ denotes the measure for the free euclidean field of mass $m_0>0$ with covariance $C=(-\Delta+m_0^2)^{-1}$,

$$V\equiv\int_\Delta d^2x:\varphi^4:(x),\qquad V_\varepsilon\equiv\int_\Delta d^2x\varepsilon^{-4}:\cos\varepsilon\varphi-1+\tfrac{1}{2}\varepsilon^2\varphi^2:(x),$$

and Wick ordering will always be with respect to $d\mu_0$. We define $\|f\|\equiv\|C^{1/2}f\|_2$. Our principal result is:

Theorem 1. *$S(f_1,\dots,f_n)=\lim_{\varepsilon\to 0}S_\varepsilon(f_1,\dots,f_n)$. For $|\varepsilon|<1/2$ there are constants c_1,c_2 independent of ε, $\{f_i\}$, c_2 independent of λ, with*

$$S_\varepsilon(f_1,\dots,f_n)\leqq c_1c_2^n n!^{1/2}\prod_{i=1}^n\|f_i\|,$$

$$|S(f_1,\dots,f_n)-S_\varepsilon(f_1,\dots,f_n)|\leqq\varepsilon^2c_1c_2^n n!^{1/2}\prod_{i=1}^n\|f_i\|. \tag{1.4}$$

Proof. We reduce the bounds (1.4) and the convergence to the corresponding results for $e^{-\lambda V_\varepsilon}$. First, note that by Jensen's inequality the denominators in (1.2), (1.3) are bounded below by 1. Consequently it suffices to prove the bounds (1.4) only for the numerators in (1.2), (1.3). The Schwartz inequality yields

$$\begin{aligned}\int d\mu_0 e^{-\lambda V_\varepsilon}\varphi(f_1)\dots\varphi(f_n)&\leqq\|e^{-\lambda V_\varepsilon}\|_2\|\varphi(f_1)\dots\varphi(f_n)\|_2\\&\leqq\|e^{-2\lambda V_\varepsilon}\|_1^{1/2}2^{n/2}n!^{1/2}\prod_i\|f_i\|\end{aligned}$$

and similarly

$$\begin{aligned}&\left|\int d\mu_0(e^{-\lambda V}-e^{-\lambda V_\varepsilon})\varphi(f_1)\dots\varphi(f_n)\right|\leqq\|e^{-\lambda V}-e^{-\lambda V_\varepsilon}\|_{3/2}\|\varphi(f_1)\dots\varphi(f_n)\|_3\\&\leqq\lambda\|V-V_\varepsilon\|_2(\|e^{-\lambda V}\|_6+\|e^{-\lambda V_\varepsilon}\|_6)\|\varphi(f_1)\dots\varphi(f_n)\|_2 2^{n/2}\\&\leqq\lambda 2^n(n!)^{1/2}\prod_{i=1}^n\|f_i\|\,\|V-V_\varepsilon\|_2(\|e^{-6\lambda V}\|_1+\|e^{-6\lambda V_\varepsilon}\|_1+2),\end{aligned}$$

where we have used Holder's inequality and the hypercontractivity of the free field. Thus Theorem 1 is proven if we can establish the following uniform bounds which are the technical core of this paper.

Theorem 2. (*a*) *Let* $|\varepsilon| < 1/2$. *Then* $\int d\mu_0 e^{-\lambda V_\varepsilon}$ *is bounded uniformly in* ε, λ *for* λ *in bounded subsets of* $[0, \infty)$.

(*b*) *Let* $\alpha \equiv \varepsilon^2/4\pi < 1/e$. *Then* $\| V - V_\varepsilon \|_2 \leqq m_0^{-1}(1 - \alpha e)\alpha$.

Theorem 2(a) will be proved by combining a generalization of Nelson's stability proof for φ_2^4, [1, 3], with Fröhlich's proof for Sine-Gordon theory [2]. Theorem 3 gives the required generalization, while the estimates required to apply it are proved in Sections 2, 3. Theorem 2(b) is obtained by an explicit computation which we defer to the end of this section along with the proof of Theorem 2(a).

Theorem 3 provides two conditions on any interaction $V(\varphi)$ which ensure that $\int d\mu_0 e^{-\lambda V}$ is integrable. In the following, φ_κ, $1 \leqq \kappa < \infty$, will denote a momentum cutoff field, for example $\varphi_\kappa = \varphi * h_\kappa$ with $h_\kappa \in L_2(R^2)$, and we will write V, V_κ for $V(\varphi), V(\varphi_\kappa)$.

Theorem 3. *Let* $V(\varphi)$ *be a function such that for constants* $a, b, \alpha, \beta, \gamma$:

(i) $V_\kappa \geqq -a\kappa^\alpha$,

(ii) $\| V - V_\kappa \|_p \leqq b p^\beta \kappa^{-\gamma}, \qquad p \in [2, \infty)$.

Then $e^{-\lambda V} \in L^1(d\mu_0)$, $\lambda \in [0, \infty)$, *provided* $\alpha(\beta - 1) < \gamma$.

Proof. Let κ_j be an increasing sequence of cutoffs with $\kappa_1 = 1, \kappa_j \to \infty$. We will use the identity, valid *a.e.* with respect to $d\mu_0$:

$$e^{-\lambda V} = \sum_{r=0}^{\infty} (-)^r \prod_{j=1}^{r} \lambda(V - V_{\kappa_j}) \int^< d^r s e^{-\lambda[(1-s_1)V_{\kappa_1} + (s_1 - s_2)V_{\kappa_2} + \ldots + (s_{r-1} - s_r)V_{\kappa_r} + s_r V_{\kappa_{r+1}}]},$$

where $\int^< d^r s$ denotes integration over the domain $1 \geqq s_1 \geqq \ldots \geqq s_r \geqq 0$. For $0 < \nu < 1/2$ let $c(\nu) \equiv \sum_1^\infty j^{-1-\nu} > 2$. Since the quantity in the exponent is a convex combination of V_{κ_i}, we obtain from estimate (i) and (ii):

$$\begin{aligned}\int d\mu_0 e^{-\lambda V} &\leqq \sum_{r=0}^{\infty} \left\| \prod_{j=1}^{r} \lambda(V - V_{\kappa_j}) \right\|_1 e^{\lambda a \kappa_{r+1}^\alpha} \int^< d^r s \\ &\leqq \sum_{r=0}^{\infty} \lambda^r \prod_{j=1}^{r} \| V - V_{\kappa_j} \|_{j^{1+\nu} c(\nu)} e^{\lambda a \kappa_{r+1}^\alpha} (r!)^{-1} \\ &\leqq \sum_{r=0}^{\infty} (\lambda b)^r (r!)^{\beta(1+\nu)-1} c(\nu)^{\beta r} \prod_{j=1}^{r} \kappa_j^{-\gamma} e^{\lambda a \kappa_{r+1}^\alpha}\end{aligned}$$

where in the second to last step we have used Holder's inequality since $\sum_{j=1}^{r} j^{-1-\nu} \cdot c(\nu)^{-1} < 1$. In particular, if we choose the cutoffs to be $\kappa_r = r^\mu, \mu > 0$, we obtain the bound:

$$\int d\mu_0 e^{-\lambda V} \leqq \sum_{r=0}^{\infty} (\lambda bc(v)^{\beta})^r (r!)^{\beta(1+v)-1-\mu\gamma} e^{\lambda a(r+1)^{\mu\alpha}}.$$

Since $r! \sim e^{r \ln r}$, the above series converges provided that $\mu\alpha \leqq 1$ and $\beta(1+v) < 1+\mu\gamma$, or equivalently if

$$\{\beta(1+v)-1\}\gamma^{-1} < \mu \leqq \alpha^{-1}.$$

Because $v > 0$ may be chosen arbitrarily small we can always find such a μ provided $\alpha(\beta-1) < \gamma$. We remark that our bound on $\int d\mu_0 e^{-\lambda V}$ leads to a bound uniform in λ on bounded sets of $[0, \infty)$. □

Proof of Theorem 2. (a) In Sections II, III Corollary 2.2, Theorem 3.1, we prove that for any $\alpha = \varepsilon^2/4\pi < 1$, $\delta > 0$, and a sharp momentum cutoff,

(i) $V_{\varepsilon,\kappa} \geqq -a(\delta)\kappa^{\alpha+\delta}$

(ii) $\| V_\varepsilon - V_{\varepsilon,\kappa} \|_p \leqq b(\alpha) p^{18} \kappa^{-\gamma(\alpha)}$, $\gamma(\alpha) = 3(1-\alpha)/8(1+\alpha)$,

with $b(\alpha)$ uniformly bounded on closed subsets of $[0, 1)$. The condition of Theorem 3: $(\alpha+\delta)(18-1) \leqq \gamma(\alpha)$ is satisfied if we choose $\alpha \leqq 1/50$ $\delta = 10^{-3}$, leading to the required bound on $\int d\mu_0 e^{-\lambda V_\varepsilon}$, uniformly in ε since $b(\alpha)$ and $\gamma(\alpha)$ are uniformly bounded on $[0, 1/50]$.

We prove Theorem 2(b) by expanding $:\cos \varepsilon\varphi:$ in a power series:

$$\| V_\varepsilon - V \|_2 \leqq \sum_{n=3}^{\infty} \varepsilon^{2n-4} (2n)!^{-1} \| \int_\Delta dx :\varphi^{2n}:(x) \|_2 .$$

We show below that

$$\| \int_\Delta dx :\varphi^{2n}:(x) \|_2 \leqq m_0^{-1} (e/4\pi)^n (2n)!, \tag{1.5}$$

and consequently

$$\| V_\varepsilon - V \|_2 \leqq \sum_{n=3}^{\infty} \varepsilon^{2n-4} (2n)!^{-1} m_0^{-1} (e/4\pi)^n (2n)! = e^3 (16\pi^2 m_0)^{-1} (1-\alpha e)\alpha.$$

To prove (1.5) we note that for $p \in [2, \infty)$, p integral:

$$\begin{aligned} \| \int_\Delta dx :\varphi^p:(x) \|_2^2 &= p! \int_\Delta dx dy C(x-y)^p \\ &\leqq p! \| C \|_p^p \\ &\leqq p! \| \tilde{C} \|_q^p, \frac{1}{p} + \frac{1}{q} = 1, \text{ by Hausdorf-Young,} \\ &\leqq p! \{ \int d^2k (4\pi^2(k^2 + m_0^2))^{-q} \}^{p/q} \\ &= p! m_0^{-2} \pi^{p-1} (4\pi^2)^{-p} (p-1)^{p-1} \\ &\leqq m_0^{-2} (e/4\pi)^p p!^2 \quad \text{since} \quad p! > p^p e^{-p}. \end{aligned}$$

The bound (1.5) follows immediately. □

Remark. One might wonder whether similar estimates yield the bounds on $\| V_\varepsilon - V_{\varepsilon,\kappa} \|_p$ required by Theorem 3. Since by hypercontractivity

$$\Big\| \int_\Delta dx : \varphi^{2n} :(x) \Big\|_p \leqq (p-1)^n \Big\| \int_\Delta dx : \varphi^{2n} :(x) \Big\|_2$$
$$\leqq m_0^{-1}((p-1)e/4\pi)^n (2n)!,$$

the corresponding bound for $\| V_\varepsilon - V_{\varepsilon,\kappa} \|_p$ converges only if $\alpha < ((p-1)e)^{-1}$, which is useless since we require all $p \in [2, \infty)$ for each α.

II. Uniform Lower Bounds on $V_\varepsilon(\varphi_\kappa)$

Theorem. *Let* $\alpha \equiv \varepsilon^2/4\pi$ *and* $c_\kappa \equiv 2\pi \langle \varphi_\kappa^2 \rangle$. *Then* $V_\varepsilon(\varphi_\kappa) \geqq -(3/4\pi)^2 c_\kappa^2 e^{\alpha c_\kappa}$.

Proof. $V_\varepsilon(\varphi_\kappa) = \int_\Delta dx \varepsilon^{-4} : \cos \varepsilon\varphi_\kappa - 1 + \frac{1}{2}\varepsilon^2 \varphi_\kappa^2 : (x)$

$$= (4\pi\alpha)^{-2} \int_\Delta dx \{ e^{\alpha c_\kappa} \cos \varepsilon\varphi_\kappa + \tfrac{1}{2}(\varepsilon\varphi_\kappa)^2 - 1 - \alpha c_\kappa \}. \tag{2.1}$$

Consider the function on $(-\infty, \infty)$

$$f(x) = a \cos x + \tfrac{1}{2}x^2, \; a > 1,$$

which takes its absolute minimum on $[-\pi, \pi]$. For $x \in [-\pi, \pi]$:

$$\cos x \geqq 1 - x^2/2 + x^4/4! - x^6/6!$$
$$\geqq 1 - x^2/2 + x^4/4! - \pi^2 x^4/6!$$
$$\geqq 1 - x^2/2 + x^4/36.$$

Thus on $[-\pi, \pi]$:

$$f(x) = a \cos x + \tfrac{1}{2}x^2 \geqq a - (a-1)x^2/2 + ax^4/36$$
$$\geqq a - 9(a-1)^2/4a \tag{2.2}$$

where the last quantity is the minimum of the fourth order polynomial. Inserting (2.2) into the expression (2.1) for $V_\varepsilon(\varphi_\kappa)$ yields:

$$V_\varepsilon(\varphi_\kappa) \geqq (4\pi\alpha)^{-2} \{ e^{\alpha c_\kappa} - 1 - \alpha c_\kappa - (9/4)(e^{\alpha c_\kappa} - 1)^2 e^{-\alpha c_\kappa} \}$$
$$\geqq -(3/4\pi)^2 [\sinh(\tfrac{1}{2}\alpha c_\kappa)/\alpha]^2$$
$$= -(3/4\pi)^2 [\sinh(\tfrac{1}{2}\alpha c_\kappa)/\alpha c_\kappa]^2 e^{-\alpha c_\kappa} \cdot c_\kappa^2 e^{\alpha c_\kappa}$$
$$\geqq -(3/4\pi)^2 c_\kappa^2 e^{\alpha c_\kappa}, \tag{2.3}$$

since $g(x) \equiv x^{-2} e^{-x} \sinh^2(x/2)$ satisfies $0 \leqq g(x) \leqq 1$ on $[0, \infty)$. □

Corollary 2.2. *Let* $\varphi_\kappa = \varphi * h_\kappa$ *where* $\tilde{h}_\kappa(k) = \chi\{|k| \leqq \kappa/m_0\}$. *Then for any* $\delta > 0$ *there is a constant* $a(\delta)$ *independent of* ε, κ *with*

$$V_\varepsilon(\varphi_\kappa) \geqq -a\kappa^{\alpha+\delta}. \tag{2.4}$$

Proof. By explicit computation, $c_\kappa = \frac{1}{2}\ln(1+\kappa^2) \leqq 1 + \ln \kappa$. Since $x^2 \leqq \delta^{-2} e^{\delta x}$ and since $\alpha < 1$, (2.4) follows from (2.3) with $a = \delta^{-2} e^{1+\delta}$. □

III. $\|V_\varepsilon - V_{\varepsilon,\kappa}\|_p \leqq bp^\beta\kappa^{-\gamma}$

In this section we choose for convenience a sharp momentum cutoff: $\varphi_\kappa = \varphi * h_\kappa$, $\tilde{h}_\kappa(k) = \chi(|k| \leqq \kappa m_0^{-1})$, $\kappa \geqq 1$.

Theorem 3.1. *Let $\alpha = \varepsilon^2/4\pi < 1$ and $p \geqq 2$. There is a constant $b(\alpha)$, bounded uniformly on closed subsets of $[0, 1)$, such that with $\gamma(\alpha) \equiv 3(1-\alpha)/(1+\alpha)$,*

$$\|V_\varepsilon - V_{\varepsilon,\kappa}\|_p \leqq b(\alpha) p^{18} \kappa^{-\gamma(\alpha)}.$$

Proof. We introduce an interpolating field $\varphi_{\kappa(s)} = s\varphi + (1-s)\varphi_\kappa$. Then from Taylor's formula with remainder

$$V_\varepsilon - V_{\varepsilon,\kappa} = \varepsilon^{-4} \int_0^\varepsilon d\varepsilon_1 \int_0^{\varepsilon_1} d\varepsilon_2 \int_0^{\varepsilon_2} d\varepsilon_3 \int_0^{\varepsilon_3} d\varepsilon_4 \int_0^1 ds \frac{\partial^4}{\partial \varepsilon_4^4} \frac{\partial}{\partial s} \int_\Delta dx :\cos \varepsilon_4 \varphi_{\kappa(s)}:(x).$$

Consequently with $|\varepsilon| \leqq \varepsilon_0 < \sqrt{4\pi}$ and $p = 2n$, an even integer,

$$\begin{aligned}
\|V_\varepsilon - V_{\varepsilon,\kappa}\|_{2n}^{2n} &\leqq \sup_{|\varepsilon| \leqq \varepsilon_0} \sup_{s \in [0,1]} \int d\mu_0 \left(\int_\Delta dx \frac{\partial^4}{\partial \varepsilon^4} \frac{\partial}{\partial s} :\cos \varepsilon \varphi_{\kappa(s)}:(x) \right)^{2n} \\
&= \sup_{|\varepsilon| \leqq \varepsilon_0} \sup_{s \in [0,1]} \left[\prod_{i=1}^{2n} \left(\int_\Delta dx_i \frac{\partial^4}{\partial \varepsilon_i^4} \frac{\partial}{\partial s_i} \right) \int d\mu_0 \prod_{i=1}^{n} :\cos \varepsilon_i \varphi_{\kappa(s_i)}:(x_i) \right]_{\substack{s_i = s \\ \varepsilon_i = \varepsilon}} \\
&= \sup_{|\varepsilon| \leqq \varepsilon_0} \sup_{s \in [0,1]} \left[\prod_{i=1}^{2n} \left(\int_\Delta dx_i \sum_{\delta_i = \pm 1} \frac{\partial^4}{\partial \varepsilon_i^4} \frac{\partial}{\partial s_i} \right) e^{-(1/2) \sum_{i \neq j} \delta_i \delta_j \varepsilon_i \varepsilon_j C_\kappa(s_i s_j; x_i - x_j)} \right]_{\substack{s_i = s \\ \varepsilon_i = \varepsilon}}
\end{aligned} \tag{3.1}$$

where

$$C_\kappa(st; x) = \int d\mu_0 \varphi_{\kappa(s)}(x) \varphi_{\kappa(t)}(0) = C_\kappa(x) + st\delta C_\kappa(x). \tag{3.2}$$

In Lemma 3.2 below we show that the quantity in square brackets in (3.1) is bounded by:

$$\begin{aligned}
&\sum_{t=n}^{2n} \sum_{r=4n-t}^{8n} \sum_{\{i_l, \alpha_l\}} \sum_{\{j_m, \beta_m\}} \frac{(r+2t)!}{(2r+2t-8n)!} \frac{t!}{(2t-2n)!} \varepsilon^{2r+8t-8n} s^{2t-2n} \\
&\prod_{i=1}^{2n} \left(\int_\Delta d^2 x_i \sum_{\delta_i = \pm} \right) \prod_{l=1}^{r} C_\kappa(s^2; x_{i_l} - x_{\alpha_l}) \prod_{m=1}^{t} |\delta C_\kappa(x_{j_m} - x_{\beta_m})| e^{-(1/2) \sum_{i \neq j} \delta_i \delta_j \varepsilon^2 C_\kappa(s^2; x_i - x_j)},
\end{aligned} \tag{3.3}$$

where the integers $i_l, \alpha_l, 1 \leqq l \leqq r$, and $j_m, \beta_m, 1 \leqq m \leqq t$, satisfy

$$i_l \leqq i_{l+1}, i_l \neq i_{l+4}, i_l \neq \alpha_l; j_m < j_{m+1}, j_m \neq \beta_m.$$

The Holder inequality is now applied to give the bound

$$\begin{aligned}
&\prod_{i=1}^{2n} \int_\Delta d^2 x_i \prod_{l=1}^{r} C_\kappa(s^2; x_{i_l} - x_{\alpha_l}) \prod_{m=1}^{t} |\delta C_\kappa(x_{j_m} - x_{\beta_m})| e^{-(1/2) \sum_{i \neq j} \delta_i \delta_j \varepsilon^2 C_\kappa(s^2; x_i - x_j)} \\
&\leqq \left\| \prod_{l=1}^{r} C_\kappa(s^2; x_{i_l} - x_{\alpha_l}) \right\|_{p_1} \left\| \prod_{m=1}^{t} \delta C_\kappa(x_{j_m} - x_{\beta_m}) \right\|_{p_2} \left\| e^{-(1/2)\varepsilon^2 \sum_{i \neq j} \delta_i \delta_j C_\kappa(s^2; x_i - x_j)} \right\|_{p_3}
\end{aligned} \tag{3.4}$$

where $p_1^{-1}+p_2^{-1}+p_3^{-1}=1$ and $\| \ \|_p$ denotes the norm for $L_p(\Delta^{2n})$. In Lemmas 3.3, 3.4, 3.5 below we bound the terms appearing in (3.4):

$$\begin{aligned}&\left\| \prod_{l=1}^{r} C_\kappa(s^2;x_{i_l}-x_{\alpha_l}) \right\|_p \leqq c_1(p)^r \\ &\left\| \prod_{m=1}^{t} \delta C_\kappa(x_{j_m}-x_{\beta_m}) \right\|_p \leqq (c_2(p)\kappa^{-1/p})^t \qquad (3.5)\\ &\left\| e^{-(1/2)\varepsilon^2 \sum\limits_{i\neq j} \delta_i\delta_j C_\kappa(s^2;x_i-x_j)} \right\|_p \leqq (c_3(\alpha p))^{2n}(2n)!, \text{ if } \alpha p<1.\end{aligned}$$

The constants are independent of $\kappa, s, \varepsilon, \delta_i$ and of the sequences $i_l, j_m, \alpha_l, \beta_m$ and $c_3(x)$ is bounded uniformly on closed subsets of $[0,1)$. Inserting (3.5) into (3.4), (3.3), yields for $\alpha p_3<1$:

$$\| V_\varepsilon - V_{\varepsilon,\kappa} \|_{2n}^{2n} \leqq 2^{2n} c_4^{2n} \sum_{t=n}^{2n} \sum_{r=4n-t}^{8n} \sum_{\{i_l,\alpha_l\}} \sum_{\{j_m,\beta_m\}} \frac{(r+2t)!}{(2r+2t-8n)!} \frac{t!}{(2t-2n)!}$$

$$c_1(p_1)^r (c_2(p_2)\kappa^{-1/p_2})^t c_3(\alpha p_3)^{2n}(2n)!,$$

where $c_4=(4\pi)^6$. The number of sequences $\{\alpha_l\}$ or $\{i_l\}$ is bounded by $(2n)^r$. The number of sequences $\{\beta_m\}$ or $\{j_m\}$ is bounded by $(2n)^t$; also $r+2t\leqq 12n, n\leqq t\leqq 2n$. Thus for $\kappa \geqq 1$:

$$\| V_\varepsilon - V_{\varepsilon,\kappa} \|_{2n}^{2n} \leqq c_5^{2n} n.6n.(2n)^{8n}(2n)^{8n}(2n)^{4n}(12n)!(2n)!(2n)!\kappa^{-n/p_2}$$

where $c_5=2c_4(1+c_1(p_1))^4(1+c_2(p_2))c_3(\alpha p_3)$, which gives

$$\| V_\varepsilon - V_{\varepsilon,\kappa} \|_{2n} \leqq c_6 (2n)^{18}\kappa^{-1/(2p_2)}, c_6=6^7 e c_5. \qquad (3.6)$$

Note that since for any $p\geqq 2$ there is an even integer $2n$ with $p\leqq 2n<2p$, the bound (3.6) remains valid with $2n$ replaced by p provided we change c_6 to $c_7=2^{18}c_6$. Finally we make a choice of p_1, p_2, p_3 above. The only restriction is that $\alpha p_3<1$. Thus we choose $p_1=4(1+\alpha)/(1-\alpha)$, $p_2=4(1+\alpha)/3(1-\alpha)$, $p_3=(1+\alpha)/2\alpha$. The bound of Theorem 3.1 follows immediately, with $b(\alpha)=c_7$. □

Lemma 3.2. *$L \equiv \prod_{i=1}^{2n}\left(\int_\Delta dx_i \sum_{\delta_i=\pm} \frac{\partial^4}{\partial\varepsilon_i^4}\frac{\partial}{\partial s_i}\right) e^{-(1/2)\sum\limits_{i\neq j}\delta_i\delta_j\varepsilon_i\varepsilon_j C_\kappa(s_is_j;x_i-x_j)}\Big|_{\substack{s_i=s\\ \varepsilon_i=\varepsilon}} \leqq R$ where R denotes the expression* (3.3).

Proof. We will use the notation $b_{ij}=\delta_i\delta_j\delta C_\kappa(x_i-x_j)$, $c_{ij}=\delta_i\delta_j C_\kappa(s_is_j;x_i-x_j)$. Noting that the exponent in L is linear in each ε_i, s_i, we apply the identity $\frac{\partial}{\partial x}e^{cx}f(x)=e^{cx}\left(\frac{\partial}{\partial x}+c\right)f(x)$ successively in each of the variables $s_1,\dots,s_{2n}$, $\varepsilon_1,\dots,\varepsilon_{2n}$. Thus

$$L=\prod_{i=1}^{2n}\left(\int_\Delta dx_i \sum_{\delta_i=\pm}\right) e^{-(1/2)\sum\limits_{i\neq j}\delta_i\delta_j\varepsilon^2 C_\kappa(s^2;x_i-x_j)} \prod_{i=1}^{2n}\left(\frac{\partial}{\partial\varepsilon_i}+\sum_{\alpha\neq i}\varepsilon_\alpha c_{i\alpha}\right)^4$$

$$\cdot\prod_{j=1}^{2n}\left(\frac{\partial}{\partial s_j}+\sum_{\beta\neq j}s_\beta\varepsilon_j\varepsilon_\beta b_{j\beta}\right)\cdot 1\Big|_{\substack{s_i=s\\ \varepsilon_i=\varepsilon}}. \qquad (3.7)$$

Expanding out the product P of the differential operators, a given term is characterized by points $1 \leqq i_1 \leqq \ldots \leqq i_r \leqq 2n, 0 \leqq r \leqq 8n$, at most four i_l equal and points $1 \leqq j_1 < \ldots < j_t \leqq 2n, 0 \leqq t \leqq 2n$. With each index i_l or j_m there is an associated index $\alpha_l \neq i_l$ or $\beta_m \neq j_m$. Therefore

$$P = \sum_{r=0}^{8n} \sum_{t=0}^{2n} \sum_{\{\alpha_l, i_l\}} \sum_{\{\beta_m, j_m\}} T \left\{ \prod_{i \notin \{i_l\}} \frac{\partial}{\partial \varepsilon_i} \prod_{l=1}^{r} \varepsilon_{\alpha_l} c_{i_l, \alpha_l} \prod_{j \notin \{j_m\}} \frac{\partial}{\partial s_j} \prod_{m=1}^{t} s_{\beta_m} \varepsilon_{j_m} \varepsilon_{\beta_m} b_{j_m, \beta_m} \right\} \Bigg|_{\substack{s_i = s \\ \varepsilon_i = \varepsilon}} \tag{3.8}$$

where T is an ordering operator which places $\frac{\partial}{\partial \varepsilon_i}$ or $\frac{\partial}{\partial s_j}$ in its appropriate position, ordered along with c_{i_l, α_l} or b_{j_m, β_m}. To obtain an upper bound on P, we first replace $b_{j,\beta}, c_{i,\alpha}$ by their absolute values, which is valid since P is a polynomial in these variables with positive coefficients. Next we remove the T operation in (3.8) i.e., we move derivatives in (3.8) to the left until they reach the positions indicated by (3.8) without the symbol T. This increases the bound on P since for μ, ν, λ positive integers

$$x^\lambda \left(\frac{\partial}{\partial x} \right)^\nu x^\mu \leqq \left(\frac{\partial}{\partial x} \right)^\nu x^\lambda x^\mu, x \geqq 0.$$

We can further increase the bound by setting $\varepsilon_i = \varepsilon, s_i = s$ in (3.8) (with T removed) before computing the derivatives rather than afterwards as specified in (3.8). This increases the bound because for μ_i, ν_i positive integers:

$$\left[\prod_{i=1}^{N} \left(\frac{\partial}{\partial x_i} \right)^{\nu_i} \prod_{i=1}^{N} x_i^{\mu_i} \right] \Bigg|_{x_i = x} \leqq \left(\frac{\partial}{\partial x} \right)^{\Sigma \nu_i} x^{\Sigma \mu_i}, \text{ if } x \geqq 0.$$

Thus our final bound on the derivatives P in (3.7) is:

$$P \leqq \sum_{r=0}^{8n} \sum_{t=0}^{2n} \sum_{\{\alpha_l, i_l\}} \sum_{\{\beta_m, j_m\}} \prod_{l=1}^{r} |c_{i_l, \alpha_l}| \prod_{m=1}^{t} |b_{j_m, \beta_m}| \left(\frac{\partial}{\partial \varepsilon} \right)^{8n-r} \varepsilon^{r+2t} \left(\frac{\partial}{\partial s} \right)^{2n-t} s^t$$

and inserting this into (3.7) we obtain the bound $L \leqq R$. □

Lemma 3.3. *For $1 \leqq l \leqq r$, let i_l, α_l be integers in $[1, 2n], i_l \leqq i_{l+1}$, no five i_l equal, $\alpha_l \neq i_l$. Then there is a constant $c_1(p)$, independent of $n, \kappa, s, h, \{i_l, \alpha_l\}$ such that for $1 \leqq p < \infty$:*

$$\left\| \prod_{l=1}^{r} C_\kappa(s^2; x_{i_l} - x_{\alpha_l}) \right\|_{L_p(\Delta^{2n})} \leqq c_1(p)^r. \tag{3.9}$$

Proof. By using the Holder inequality we may reduce the proof to the case where $i_l < i_{l+1}, i_l < \alpha_l$ with p replaced by $8p$. To achieve the reduction we decompose $\{l\} = [1, \ldots, r]$ into four subsets such that in each subset $i_l \neq i_{l+1}$. Each subset may be further decomposed into two subsets characterized by $i_l < \alpha_l, i_l > \alpha_l$ respectively. Holder's inequality is now applied to the product of 8 terms resulting from this decomposition. The cases $i_l > \alpha_l$ and $i_l < \alpha_l$ are handled identically—we discuss the latter.

Suppose now that $i_l < i_{l+1}, i_l < \alpha_l, 1 \leqq l \leqq r$. Introduce new variables by

$$y_i = \begin{cases} x_i & \text{if } i \notin \{i_l\} \\ x_{i_l} - x_{\alpha_l} & \text{if } i = i_l. \end{cases} \tag{3.10}$$

This transformation has upper triangular form because $i_l < \alpha_l$ and has all its diagonal elements equal to 1. Thus the Jacobian is 1 and then

$$\begin{aligned} \left\| \prod_{l=1}^{r} C_\kappa(s^2; x_{i_l} - x_{\alpha_l}) \right\|_{L_p(\Delta^{2n})} &\leqq \left\{ \prod_{i \notin \{i_l\}} \int_\Delta dy_i \prod_{l=1}^{r} \int_\Delta dy_{i_l} C_\kappa(s^2; y_{i_l})^p \right\}^{1/p} \\ &= \| C_\kappa(s^2; \cdot) \|_{L_p(R^2)}^r \\ &\leqq \| \tilde{C}_\kappa(s^2; \cdot) \|_{L_q(R^2)}^r, p^{-1} + q^{-1} = 1, \\ &\quad \text{by Haussdorf-Young } (p \geqq 2), \\ &\leqq (\textstyle\int d^2k (4\pi^2(k^2 + m^2))^{-q})^{r/q} = c'(p)^r. \end{aligned}$$

This proves Lemma 3.3 for the reduced cases, and the general case follows if we choose $c_1(p) = c'(8p)$. □

Lemma 3.4. *For $1 \leqq m \leqq t$ let j_m, β_m be integers in $[1, 2n]$, $j_m < j_{m+1}, \beta_m \neq j_m$. Then there is a constant $c_2(p)$, independent of $n, \kappa, t, \{j_m, \beta_m\}$ such that for $1 \leqq p < \infty$:*

$$\left\| \prod_{m=1}^{t} \delta C_\kappa(x_{j_m} - x_{\beta_m}) \right\|_{L_p(\Delta^{2n})} \leqq (c_2(p)\kappa^{-1/p})^t.$$

Proof. As in Lemma 3.3 we reduce to the case $j_m < \beta_m$ with p replaced by $2p$. With a change of variable similar to (3.10) we have in that case:

$$\begin{aligned} \left\| \prod_{m=1}^{t} \delta C_\kappa(x_{j_m} - x_{\beta_m}) \right\|_{L_p(\Delta^{2n})} &\leqq \| \delta \tilde{C}_\kappa \|_{L_q(R^2)}^t, p^{-1} + q^{-1} = 1, p \geqq 2 \\ &= \left(\int_{|k| \geqq \kappa m_0} d^2k (4\pi^2(k^2 + m_0^2))^{-q} \right)^{t/q} \\ &\leqq (p(m_0\kappa)^{-2/p})^t = (c_2'(p)\kappa^{-2/p})^t. \end{aligned}$$

This completes the proof for the reduced case, the general case follows with the choice $c_2(p) = c_2'(2p), \kappa^{-2/p}$ replaced by $\kappa^{-1/p}$. □

Lemma 3.5. *Provided $\alpha p < 1$, there is a constant $c_3(x)$ independent of $\varepsilon, \kappa, s, p, n$ or $\{\delta_i = \pm 1\}$, bounded uniformly on closed subsets of $[0, 1)$, such that*

$$\left\| e^{-(1/2)\varepsilon^2 \sum_{i \neq j}^{2n} \delta_i \delta_j C_\kappa(s^2; x_i - x_j)} \right\|_{L_p(\Delta^{2n})} \leqq (c_3(\alpha p))^{2n} (2n)!^{1/p}.$$

Proof. We show below that for any choice of $\{\delta_i\}$

$$\begin{aligned} &\left\| e^{-(1/2)\varepsilon^2 \sum_{i \neq j}^{2n} \delta_i \delta_j C_\kappa(s^2; x_i - x_j)} \right\|_{L_p(\Delta^{2n})}^p \\ &\leqq \left\| e^{-(1/2) \sum_{i \neq j}^{2n} \varepsilon_i' \varepsilon_j' C(x_i - x_j)} \right\|_{L_1(\Delta^{2n})} \end{aligned} \tag{3.11}$$

where $\varepsilon_i' = \varepsilon', i \leqq n, \varepsilon_i' = -\varepsilon', i > n, \varepsilon' \equiv p^{1/2}\varepsilon$. Thus the general case (arbitrary

$\delta_i, \kappa, s)$ is bounded by the case $\sum \delta_i = 0, \kappa = \infty, s = 1$. The quantity on the right of (3.11) is recognized as the classical canonical partition function for n positively and n negatively charged particles interacting in volume Δ with Yukawa two-body potential $C(x-y)$. Fröhlich has studied this quantity [2]—in his notation it is $Z_n(C_{m_0}, \chi_\Delta)$. He proves in Theorem 3.7(2) of [2] that if $\alpha' \equiv \varepsilon'^2/4\pi < 1$ there is a constant $K(\alpha')$ with

$$\| e^{-(1/2)\sum_{i \neq j}^{2n} \varepsilon_i' \varepsilon_j' C(x_i - x_j)} \|_{L_1(\Delta^{2n})} \leqq K(\alpha')^n n!^2 \tag{3.12}$$

where it can be checked from his proof that $K(\alpha')$ is bounded uniformly for α' in closed subsets of $[0, 1)$. Lemma 3.5 follows from (3.11), (3.12) if we take $c_3(x) = (1 + K(x))^{1/2}$.

To prove (3.11) we convert the expression on the left to a Gaussian integral. Defining $\varepsilon_i = p^{1/2} \varepsilon \delta_i$, we have

$$\begin{aligned}
&\| e^{-(1/2)\varepsilon^2 \sum_{i \neq j}^{2n} \delta_i \delta_j C_\kappa(s^2; x_i - x_j)} \|_{L_p(\Delta^{2n})}^p = \int_\Delta d^{2n}x e^{-(1/2)\sum_{i \neq j}^{2n} \varepsilon_i \varepsilon_j C_\kappa(s^2; x_i - x_j)} \\
&= \int d\mu_{C_\kappa(s^2, \cdot)} \prod_{i=1}^{2n} \int_\Delta dx : e^{i\varepsilon_i \varphi} :_{C_\kappa(s^2, \cdot)} \\
&\leqq \int d\mu_{C_\kappa(s^2, \cdot)} \prod_{i=1}^{2n} \int_\Delta dx : e^{i\varepsilon_i' \varphi} :_{C_\kappa(s^2, \cdot)} \\
&\leqq \int d\mu_C \prod_{i=1}^{2n} \int_\Delta dx : e^{i\varepsilon_i' \varphi} :_C \\
&= \| e^{-(1/2)\sum_{i \neq j}^{2n} \varepsilon_i' \varepsilon_j' C(x_i - x_j)} \|_{L_1(\Delta^{2n})},
\end{aligned}$$

where we have taken absolute values in going from lines 3 to 4 while in the second last step we have used conditioning, (see for example Simon [4] page 226) since $C_\kappa(s^2; \cdot) \leqq C(\cdot)$. □

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