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Unbounded Derivations of Commutative C^*-Algebras

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Abstract. It is shown that an unbounded *-derivation δ of a unital commutative C^*-algebra A is quasi well-behaved if and only if there is a dense open subset U of the spectrum of A such that, for any f in the domain of δ, $\delta(f)$ vanishes at any point of U where f attains its norm. An example is given to show that even if δ is closed it need not be quasi well-behaved. This answers negatively a question posed by Sakai for arbitrary C^*-algebras.

It is also shown that there are no-zero closed derivations on A if the spectrum of A contains a dense open totally disconnected subset.

1. Introduction

Unbounded derivations have recently become one of the most important branches of the theory of C^*-algebras, since they include the infinitesimal generators of the one-parameter *-automorphism groups representing time-evolution of quantum dynamical systems. Several authors have shown how results from Banach space theory take on special forms for C^*-algebras (see e.g. [2, 3]). In his recent survey of the theory of unbounded derivations [6], Sakai raised several questions concerning closed *-derivations. In this paper, we obtain negative answers to two of these questions.

Sakai proved that a sufficient condition for the commutative C^*-algebra $C(\Omega)$ of continuous complex-valued functions on a compact Hausdorff space Ω to have no non-zero closed *-derivations is that Ω should be totally disconnected, and asked whether this condition is also necessary [6, Problem 1.1]. We show here that it is not by proving that another weaker sufficient condition is that Ω should contain a dense open totally disconnected subset. This result has also been obtained independently by B. E. Johnson.

Let δ be a *-derivation of any C^*-algebra A [δ is assumed to have dense domain $\mathscr{D}(\delta)$]. An element x of the self-adjoint part $\mathscr{D}(\delta)^s$ of $\mathscr{D}(\delta)$ is said to be *well-behaved* if there is a state ϕ of A with $|\phi(x)|=\|x\|$ and $\phi(\delta(x))=0$, and to be *strongly well-behaved* if $\phi(\delta(x))=0$ for all self-adjoint linear functionals ϕ in A^* with

$\phi(x)=\|\phi\|\,\|x\|$. The set of all well-behaved elements will be denoted by $W(\delta)$. Then δ is said to be *quasi well-behaved* if the interior $\operatorname{Int} W(\delta)$ of $W(\delta)$ in $\mathscr{D}(\delta)^s$ is dense in $\mathscr{D}(\delta)^s$. Sakai [6, Theorem 2.8] showed that a quasi well-behaved *-derivation δ is closable, and that if δ is closed then $W(\delta)$ is dense, and he asked whether all closed *-derivations are quasi well-behaved. In [1] it was shown that the elements of $\operatorname{Int} W(\delta)$ are strongly well-behaved and that $\mathscr{D}(\delta)^s$ contains a dense set of strongly well-behaved elements, irrespective of whether δ is closed. Here we construct a commutative C^*-algebra with a closed *-derivation which is not quasi well-behaved, and give a general condition, slightly weaker than the property of being quasi well-behaved, for a *-derivation to be closable.

2. Existence of Closed Derivations

Let δ be a closed *-derivation of a commutative C^*-algebra A, and let p be a projection in A. It may be proved that p belongs to $\mathscr{D}(\delta)$ by using either the Silov idempotent theorem as in [6, Proposition 1.11], or [2, Theorem 3] and the fact that any two distinct projections in A are distance 1 apart, or [2, Theorem 17] and a decomposition $p=f(x)$ where $x\in\mathscr{D}(\delta)^s$, $\|x-p\|<1/3$ and f is a continuously differentiable function with $f(s)=0$ $(s\leqq 1/3)$ and $f(t)=1$ $(t\geqq 2/3)$. Then $\delta(p)=2p\delta(p)$, so $\delta(p)=0$.

Theorem 1. *Let Ω be a compact Hausdorff space containing a dense open totally disconnected subset Ω_1. Then there are no non-zero closed *-derivations on $C(\Omega)$.*

Proof. Let δ be a closed *-derivation on $C(\Omega)$. Each point in Ω_1 has a compact totally disconnected neighbourhood, and therefore has a neighbourhood basis consisting of sets which are both open and closed [4, Chapter II, §4]. Thus the projections in $C(\Omega)$ separate the points of Ω_1 from each other and from $\Omega\backslash\Omega_1$. If f is any function in $C(\Omega)$ vanishing on $\Omega\backslash\Omega_1$, then by the Stone-Weierstrass theorem, f is uniformly approximable by linear combinations of prejections, so $f\in\mathscr{D}(\delta)$ and $\delta(f)=0$, since δ is closed.

Now consider g in $\mathscr{D}(\delta)$ and ω in Ω_1. There is a function f in $C(\Omega)$ vanishing on $\Omega\backslash\Omega_1$ but not at ω. By the above, $\delta(fg)=\delta(f)=0$, so $0=\delta(fg)(\omega)=f(\omega)\delta(g)(\omega)$. Hence $\delta(g)(\omega)=0$ for all ω in Ω_1, and, since Ω_1 is dense, $\delta=0$.

3. Well-Behaved Points and Derivations

Let Ω be a compact Hausdorff space and δ be any *-derivation of $C(\Omega)$. In order to study properties of δ it will be convenient to convert the definition of well-behaved elements of $\mathscr{D}(\delta)^s$ into a corresponding notion for points of Ω. Thus ω in Ω is said to be *well-behaved* if $\delta(f)(\omega)=0$ whenever $f\in\mathscr{D}(\delta)^s$ and $|f(\omega)|=\|f\|$. We shall denote the set of well-behaved points of Ω by Ω_δ. It follows from [1, Proposition 7] that $\Omega_\delta=\Omega$ if and only if $W(\delta)=\mathscr{D}(\delta)^s$.

Proposition 2. *Let f be a real-valued function in $C(\Omega)$ and let*

$$\alpha_1=\sup\{|f(\omega)|:\omega\in\operatorname{Int}\Omega_\delta\}$$

$$\alpha_2=\sup\{|f(\omega)|:\omega\in\Omega\backslash\Omega_\delta\}$$

(where the supremum of the empty set is taken to be $-\infty$*). Then for any* $\varepsilon>0$ *and* $\beta<\frac{1}{2}(\alpha_1+\varepsilon+\mathrm{Min}(\varepsilon-\alpha_2,0))$, *there is a function* g *in* $\mathscr{D}(\delta)^s$ *with* $\|f-g\|<\varepsilon$ *such that* $W(\delta)$ *contains the closed ball in* $\mathscr{D}(\delta)^s$ *with centre* g *and radius* β.

Proof. If either $\mathrm{Int}\,\Omega_\delta$ or $\Omega\backslash\Omega_\delta$ is empty, the result is trivial, so we may assume that α_1 and α_2 are finite. It suffices also to assume that $f\in\mathscr{D}(\delta)$. Choose real numbers ε_j $(1\leqq j\leqq 6)$ such that

$$0<\varepsilon_j<1$$
$$\alpha_1\varepsilon_1+(1+\varepsilon)\varepsilon_3+(\alpha_1+\alpha_2)\varepsilon_4+\varepsilon_5+\alpha_2\varepsilon_6<\alpha_1+\varepsilon+\mathrm{Min}(\varepsilon-\alpha_2,0)-2\beta$$
$$\|f\|\varepsilon_4<\varepsilon_5<\varepsilon$$
$$\varepsilon_2+\varepsilon\varepsilon_3<\alpha_2\varepsilon_6<\varepsilon\,.$$

Put $\varepsilon'=\mathrm{Min}(\varepsilon\alpha_2^{-1},1)-\varepsilon_6$. There exists ω_0 in $\mathrm{Int}\,\Omega_\delta$ such that $|f(\omega_0)|\geqq\alpha_1(1-\varepsilon_1)$. We may suppose that $f(\omega_0)\geqq 0$. There exists an open set V in Ω containing $\Omega\backslash\mathrm{Int}\,\Omega_\delta$, but which does not have ω_0 as a limit point, and which satisfies $|f(\omega)|<\alpha_2+\varepsilon_2$ for all ω in V. Then there are functions g_1 and g_2 in $C(\Omega)$ with $0\leqq g_j\leqq 1$, $g_1(\omega_0)=1$, $g_1=0$ on V, $g_2=1$ on $\Omega\backslash\Omega_\delta$ and $g_2=0$ on $\Omega\backslash V$. Since $\mathscr{D}(\delta)$ is a dense *-subalgebra of $C(\Omega)$, there exist g_3 and g_4 in $\mathscr{D}(\delta)$ with $0\leqq g_3\leqq 1$, $\|g_3-g_1\|<\varepsilon_3$, $0\leqq g_4\leqq 1$ and $\|g_4-g_2\|<\varepsilon_4$. Let $g=f(1-\varepsilon' g_4)+(\varepsilon-\varepsilon_5)g_3$. Then $g\in\mathscr{D}(\delta)$ and for ω in $\Omega\backslash V$,

$$\begin{aligned}|(g-f)(\omega)|&<\varepsilon'\|f\|\varepsilon_4+\varepsilon-\varepsilon_5\leqq\varepsilon-(\varepsilon_5-\|f\|\varepsilon_4)\\&<\varepsilon\end{aligned}$$

while for ω in V

$$\begin{aligned}|(g-f)(\omega)|&<(\alpha_2+\varepsilon_2)\varepsilon'+(\varepsilon-\varepsilon_5)\varepsilon_3\\&<\varepsilon-\alpha_2\varepsilon_6+\varepsilon_2+\varepsilon\varepsilon_3\\&<\varepsilon\,.\end{aligned}$$

Thus $\|g-f\|<\varepsilon$. Also, for ω in $\Omega\backslash\Omega_\delta$,

$$\begin{aligned}|g(\omega)|+\beta&<\alpha_2(1-\varepsilon'(1-\varepsilon_4))+(\varepsilon-\varepsilon_5)\varepsilon_3+\beta\\&<\alpha_2\varepsilon_6-\mathrm{Min}(\varepsilon-\alpha_2,0)+\alpha_2\varepsilon_4+\varepsilon\varepsilon_3+\beta\\&<\alpha_1(1-\varepsilon_1)-\alpha_1\varepsilon_4+\varepsilon-\varepsilon_3-\varepsilon_5-\beta\\&<\alpha_1(1-\varepsilon_1)(1-\varepsilon'\varepsilon_4)+(\varepsilon-\varepsilon_5)(1-\varepsilon_3)-\beta\\&<g(\omega_0)-\beta\,.\end{aligned}$$

Hence if $h\in\mathscr{D}(\delta)^s$ and $\|g-h\|\leqq\beta$, then $|h(\omega_1)|=\|h\|$ for some ω_1 in Ω_δ, so $\delta(h)(\omega_1)=0$ and $h\in W(\delta)$.

Proposition 3. *Let* f *be a function in* $\mathscr{D}(\delta)^s$, ε *be a positive real number and suppose that* $W(\delta)$ *contains the open ball in* $\mathscr{D}(\delta)^s$ *with centre* f *and radius* ε. *Then* Ω_δ *contains all points* ω *of* Ω *with* $|f(\omega)|>\|f\|-2\varepsilon$.

Proof. Replacing f by $-f$ if necessary, we may assume that $f(\omega)\geqq 0$. Adjusting f by a small function in $\mathscr{D}(\delta)^s$ non-zero at ω, we may assume that $f(\omega)>0$. There is a real polynomial p with $p(0)=0$, $p(f(\omega))=\|p(f)\|$ and $\|p(f)-f\|<\varepsilon$. Then

$p(f) \in \operatorname{Int} W(\delta)$, so $p(f)$ is strongly well-behaved. Hence $\delta(p(f))(\omega)=0$. Now suppose that $g(\omega)=\|g\|$ for some g in $\mathscr{D}(\delta)^s$. Then $\|f-p(f)-\lambda g\|<\varepsilon$ for small $\lambda>0$, so $p(f)+\lambda g$ is strongly well-behaved. But $(p(f)+\lambda g)(\omega)=\|p(f)+\lambda g\|$, so $\delta(p(f)+\lambda g)(\omega)=0$. Hence $\delta(g)(\omega)=0$.

Theorem 4. *A *-derivation δ of $C(\Omega)$ is quasi well-behaved if and only if* $\operatorname{Int}\Omega_\delta$ *is dense in Ω.*

Proof. Suppose δ is quasi well-behaved, but $\operatorname{Int}\Omega_\delta$ is not dense. Then there is a function g in $\operatorname{Int} W(\delta)$ with $|g(\omega)| \leqq \frac{1}{2}$ for ω in $\operatorname{Int}\Omega_\delta$, but $g(\omega_0)=\|g\|=1$ for some ω_0 in Ω. By Proposition 3, $\omega_0 \in \operatorname{Int}\Omega_\delta$. But this is a contradiction.

Now suppose that $\operatorname{Int}\Omega_\delta$ is dense and f is a real function in $C(\Omega)$. Then in the notation of Proposition 2, $\alpha_1=\|f\| \geqq \alpha_2$, so that proposition shows that for $0<\beta<\varepsilon<\|f\|$, there exists g in $\mathscr{D}(\delta)^s$ such that $\|g-f\|<\varepsilon$ and $W(\delta)$ contains the closed ball in $\mathscr{D}(\delta)^s$ with centre g and radius β.

In commutative C^*-algebras, Theorem 4 often gives a convenient method of determining whether a *-derivation is quasi well-behaved. For example if Ω is a compact subset of the real line $\mathbb{R}$ with no isolated points and δ is the derivation of $C(\Omega)$ defined by

$$\delta(f)(t)=\lim_{\substack{s\to t\\ s\in\Omega}} \frac{f(s)-f(t)}{s-t}$$

whenever this defines a continuous function, then Ω_δ consists of those points t in Ω for which

$$\sup\{s\in\Omega : s<t\}=t=\inf\{s\in\Omega : s>t\}\ .$$

It is easy to see that $\operatorname{Int}\Omega_\delta$ is the interior of Ω in $\mathbb{R}$. Hence (as may also be seen directly) δ is quasi well-behaved if and only if Ω is the closure of an open subset of $\mathbb{R}$. Similar considerations in the plane are involved in the construction of the following example of a closed *-derivation which is not quasi well-behaved.

Example 5. For $n \geqq 1$ and $k \geqq 1$, let $\alpha_{kn}=(k-1)2^{-(n-1)}$ and $\beta_{kn}=(2k-1)2^{-n}$. For $1 \leqq k \leqq 2^{n-1}$ and $1 \leqq l \leqq 2^{n-1}$, let E_{kln} and F_{kln} be the following subsets of $\mathbb{R}^2$:

$$E_{kln}=(\alpha_{kn},\beta_{kn})\times\{\beta_{ln}\}$$
$$F_{kln}=[\beta_{kn},\alpha_{k+1,n}]\times[\beta_{ln}-2^{-(n+2)},\beta_{ln}+2^{-(n+2)}]\ .$$

Note that the sets E_{kln} are disjoint, and if E_{kln} intersects F_{ijm}, then $m<n$ and $E_{kln}\subset F_{ijm}$. Let

$$E=\left(\bigcup_{n=1}^{\infty}\bigcup_{k,l=1}^{2^{n-1}}E_{kln}\right)\Big\backslash\left(\bigcup_{n=1}^{\infty}\bigcup_{k,l=1}^{2^{n-1}}F_{kln}\right)$$
$$=\bigcup_{n=1}^{\infty}\bigcup_{k,l=1}^{2n-1}\left(E_{kln}\Big\backslash\bigcup_{m=1}^{n-1}\bigcup_{i,j=1}^{2^{m-1}}F_{ijm}\right).$$

Then E is a union of sets of the form E_{kln}. Let Ω be the closure of E, and Ω_0 be the set of points of the form (β_{kn},β_{ln}) in Ω.

Suppose k. l, and n are such that E_{kln} is contained in E, and that $p \geqq 2$. Then inspection shows that $E_{k'l'n'}$ is contained in E, where $(k-1)2^p < k' \leqq (k-1)2^p + 2^{p-1}$, $l' = (2l-1)2^{p-1}$ and $n' = n+p$. Now let (ξ, β_{ln}) be a typical point of E_{kln}, and put $n_p = n+p$, $l_p = (2l-1)2^{p-1}$ and choose k_p so that $|\xi - \beta_{k_p n_p}|$ is minimised. Then $(k-1)2^p < k_p \leqq (k-1)2^p + 2^{p-1}$, so $E_{k_p l_p n_p}$ is contained in E, and $(\beta_{k_p n_p}, \beta_{l_p n_p}) \in \Omega_0$. But this sequence converges to (ξ, β_{ln}). Thus Ω_0 is dense in Ω.

Now define δ by $\delta(f) = g$ where g is a (necessarily unique) function in $C(\Omega)$ such that for (ξ, η) in E

$$g(\xi, \eta) = \lim_{\zeta \to 0} \zeta^{-1}(f(\xi + \zeta, \eta) - f(\xi, \eta)) .$$

Then δ is a densely-defined closed *-derivation. Consider the point (β_{kn}, β_{ln}) of Ω_0 and put

$$f(\xi, \eta) = [1 + (\xi - \beta_{kn} - 2^{-(n+2)})^2 + (\eta - \beta_{ln})^2]^{-1} .$$

Then f is well-defined on Ω and attains its maximum at the point (β_{kn}, β_{ln}) since Ω contains no point of the interior of F_{kln}. However $\delta(f)(\beta_{kn}, \beta_{ln}) > 0$. Thus (β_{kn}, β_{ln}) does not belong to Ω_δ.

Since Ω_δ does not intersect the dense set Ω_0, $\operatorname{Int}\Omega_\delta$ is empty so δ is not quasi well-behaved.

Although the derivation in the above example is not quasi well-behaved, it does have a slightly weaker property. Let δ be a *-derivation of a (non-commutative) C^*-algebra A, and J be a closed ideal of A which is invariant under δ in the sense that $\delta(x) \in J$ whenever $x \in \mathscr{D}(\delta) \cap J$. Then δ induces a densely-defined *-derivation δ_J of A/J defined by

$$\delta_J(\pi_J(x)) = \pi_J(\delta(x)) \quad (x \in \mathscr{D}(\delta)) ,$$

where π_J is the quotient map of A onto A/J. Then δ is *pseudo well-behaved* if there is a family $\mathscr{J}$ of closed invariant ideals J such that δ_J is quasi well-behaved, and $\cap\{J : J \in \mathscr{J}\} = (0)$.

In example 5, for each k, l, and n such that E_{kln} is contained in E, let

$$J_{kln} = \{f \in C(\Omega) : f = 0 \quad \text{on} \quad E_{kln}\} .$$

Then J_{kln} is invariant, $\delta_{J_{kln}}$ is quasi well-behaved and $\cap J_{kln} = (0)$. Thus δ is pseudo well-behaved.

More generally, a *-derivation of a commutative C^*-algebra $C(\Omega)$ is pseudo well-behaved if and only if there exists a family of closed subsets E of Ω with dense union such that for each E in Ω,

$$f \in \mathscr{D}(\delta), \ f|_E = 0 \Rightarrow \delta(f)|_E = 0$$

and the interior of E_δ (in E) is dense in E where

$$E_\delta = \{\omega \in E : \delta(f)(\omega) = 0 \ \text{whenever} \ f \in \mathscr{D}(\delta)^s \ \text{and} \ |f(\omega)| = \|f|_E\|\} .$$

In particular if δ is pseudo well-behaved, then Ω_δ is dense in Ω.

Proposition 6. *A pseudo well-behaved *-derivation on a C^*-algebra A is closable.*

Proof. Let x_n be a sequence in $\mathscr{D}(\delta)$ such that $x_n \to 0$ and $\delta(x_n) \to y$, and let J be an invariant ideal such that δ_J is quasi well-behaved. Then $\pi_J(x_n) \to 0$, $\delta_J(\pi_J(x_n)) \to \pi_J(y)$ and δ_J is closable [6, Theorem 2.8], so $\delta_J(y) = 0$. Since the intersection of such ideals is zero, $y = 0$.

There are analogues of Proposition 6 for Banach spaces. A condition [satisfied by any derivation δ with $W(\delta) = \mathscr{D}(\delta)^s$] which ensures that a densely-defined operator Z on a Banach space is closable was given in [5, Lemma 3.3], and a weaker condition of this type (satisfied by the restriction of a quasi well-behaved derivation δ to $\mathscr{D}(\delta)^s$) appears in the remarks following [1, Theorem 5]. In particular Z is closable if its domain contains a dense open set of weakly dissipative elements (see [1] for definitions). As in Proposition 6 it may be deduced that Z is closable if there exists a family of closed invariant subspaces of X with zero intersection, on whose quotients the operators induced by Z have dense open sets of weakly dissipative elements.

It does not appear easy to construct a closed *-derivation which is not pseudo well-behaved. In particular it is not clear whether every closed *-derivation of a simple C^*-algebra is quasi well-behaved, or whether Ω_δ is dense whenever δ is a closed *-derivation on $C(\Omega)$. However we do have the following example of a *-derivation for which Ω_δ is dense, but which is not closable.

Example 7. Let K be the Cantor set, and define $\delta(f) = g$ where

$$g(t) = \lim_{\substack{s \to t \\ s \in K}} \frac{f(s) - f(t)}{s - t} .$$

Then the discussion before Example 5 shows that δ is a densely-defined *-derivation on $C(K)$ which is not closable, but K_δ is dense in K.

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