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# Phase Transitions in Anisotropic Lattice Spin Systems

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**Abstract.** A general method for proving the existence of phase transitions is presented and applied to six nearest neighbor models, both classical and quantum mechanical, on the two dimensional square lattice. Included are some two dimensional Heisenberg models. All models are anisotropic in the sense that the groundstate is only finitely degenerate. Using our method which combines a Peierls argument with reflection positivity, i.e. chessboard estimates, and the principle of exponential localization we show that five of them have long range order at sufficiently low temperature. A possible exception is the quantum mechanical, anisotropic Heisenberg ferromagnet for which reflection positivity is *not* proved, but for which the rest of the proof is valid.

## I. Summary of Results and General Strategy of Proofs

One of the main purposes of this paper is to explain a general method for proving the existence of phase transitions, in the sense of long range order at sufficiently low temperatures, in classical and quantum lattice systems. In principle, our method can be applied to arbitrary lattice systems satisfying *reflection positivity* (a condition closely related to the existence of a self-adjoint positive definite transfer matrix), the groundstates of which are essentially *finitely degenerate* (e.g. the space of groundstates decomposes into finitely many subspaces labelled by a discrete order parameter, sometimes related to a broken discrete symmetry group).

Our method is inspired by recent work of Glimm, Jaffe and Spencer concerning phase transitions in the $(\lambda\phi^4)_2$ quantum field model, [16]. In this paper their ideas are extended in two ways:

1. We systematize the use of *reflection positivity* and *chessboard estimates*

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in obtaining upper bounds on the statistical weight of contours arising in a Peierls argument and we show how to apply these methods to *quantum* lattice systems. This reduces the proof of long range order to estimating a ratio between a *constrained* partition function and the usual partition function. (This is basically a *thermodynamic* estimate).

2. We introduce the *principle of exponential localization* in order to derive upper bounds on constrained partition functions. This principle is particularly useful in the analysis of quantum lattice systems.

Reflection positivity, originally inspired by work of Osterwalder and Schrader [18], and the principle of exponential localization are useful tools in contexts other than the theory of phase transitions.

In Section I.A we introduce six different classical and quantum mechanical models on the two dimensional square lattice in terms of which we develop and illustrate our general method. A summary of our main results concludes that section.

In Section I.B we recall the connections between phase transitions and the occurrence of various forms of long range order (LRO) at sufficiently low temperatures.

In Section I.C, D and E we present the main ideas behind our general method; (Section I.C contains a convenient variant of the Peierls argument, essentially identical to the one of [16]; see also [10]).

In Section II we establish, (or review) reflection positivity for five of our six models, the exception being the quantum Heisenberg ferromagnet. We prove a generalization of the Hölder inequality for traces which, when combined with reflection positivity, yields the *chessboard estimates.* They extend constructive field theory estimates of [19].

In Section III we introduce the *principle of exponential localization* and apply it to our models for the purpose of estimating constrained partition functions. This is an expansion of the idea used in [22].

In Section IV the proofs of our main results are completed by combining the estimates of Sections I.C, II and III. Sections II and III contain results which are of some interest in their own right: Theorems 2.1, 2.2, 3.1 and Corollary 3.2. The reader can understand their statements and proofs without being familiar with the rest of this paper.

Next, we describe the models studied in this paper in general terms and recall some typical aspects of two dimensional lattice systems.

Two facts are well established about two-dimensional (quantum or classical) lattice spin systems with short range interactions:

(i) The Ising model has a first order phase transition (i.e. long range order for large $\beta = (kT)^{-1}$); for all values $S = 1/2, 1, \ldots$ of the spin.

(ii) Models with *continuous* symmetry (e.g. the isotropic Heisenberg models) have *no* such ordering. The proof of this is due to Mermin-Wagner [1], Mermin [2] and Hohenberg [3], (MWH). Thus, a natural question is whether the anisotropic models have LRO for all values of the anisotropy parameter, $\alpha$, with $0 < \alpha < 1$. For the *classical* Heisenberg (H) model this was proved recently by Malyshev [4]. Kunz, Pfister and Vuillermot [5] later gave a simplified proof for the planar rotator. Ginibre [6] and Robinson [7] proved LRO for the quantum Heisenberg *ferromagnet* for very small $\alpha$.

In [8] we announced proofs of LRO for a variety of anisotropic models: in particular for the quantum ferromagnetic H model, for all $\alpha < 1$. Subsequently we became aware of a flaw in one of the lemmas for the ferromagnetic model. This is basically the same flaw as in the announced Dyson, Lieb and Simon [9] proof of LRO for the *three* dimensional H model. The other results stated in [8] are correct. Here we will present the details of the proofs, including the part of the proof for the ferromagnetic H model that is correct. It is hoped that before very long the missing piece of the puzzle will be filled in.

An obvious remark has to be made: All the models we consider have *no* LRO at high temperature, a fact which can be proved by high temperature expansions, for example. Since LRO implies the existence of a spontaneously polarized state, our proof of LRO at low temperature implies the existence of a phase transition.

The models discussed here are all two-dimensional but, as in the usual Peierls argument [23], all our results and methods can be extended to higher dimensions. They can also be extended to some other lattices, e.g. the honeycomb lattice; see [12].

Some of our results were reviewed in [10] and [11]. Additional applications of the ideas presented here are to be found [10], [12] and [13].

### *I.A. Description of the Models and Main Results*

All models are on the square lattice $\mathbb{Z}^2$ and have only nearest neighbor interactions. Thus $\sum_{\langle i,j\rangle}$ means a sum over nearest neighbors, each term being included once; $H$ is the Hamiltonian.

(1) Classical $N$-Vector Model ($N > 1$)

$$H = -\sum_{\langle i,j\rangle}\left\{S_i^1 S_j^1 + \alpha\sum_{k=2}^{N} S_i^k S_j^k\right\}. \tag{1.1}$$

Each $S_i$ is a unit vector in $\mathbb{R}^N$, uniformly distributed on the sphere. (More general rotationally symmetric spin distributions could be accomodated by our methods.) Note that in this classical case, the ferromagnet (minus sign in (1.1)) is equivalent to the antiferromagnet (plus sign) by reversing the spins on the odd sublattice. This is *not* true in quantum models.

Our result in this case is that for every $\alpha < 1$ there is LRO at low $T$. In other words for every $\alpha < 1$ there is a $\beta_c(\alpha)$ such that there is LRO for $\beta > \beta_c(\alpha)$. Our estimate on $\beta_c(\alpha)$ is

$$\beta_c(\alpha) = O((1-\alpha)^{-1}). \tag{1.2}$$

The MWH result is that $\beta_c(\alpha = 1) = \infty$. Our proof is simpler than Malyshev's [4].

(2) Classical Anharmonic Crystal (AC Model)

The Hamiltonian of this model is given by

$$H = \sum_{\langle i,j \rangle} \phi(x_i, x_j),$$

where $x_i$ is the coordinate of an $N$-vector classical oscillator bound to site $i$, with apriori distribution $d^N x$;$\phi$ is some *continuous*, anisotropic interaction potential,

$$\phi(x,y) = \phi_1(x) + \phi_1(y) + \phi_2(x,y),$$

where $\phi_1 \geqq 0$ is a one body potential, and $\phi_2$ is a two body potential.

In other words we are considering some sort of anharmonic, anisotropic classical crystal (resp. a Euclidean lattice field theory). We will prove LRO at high $\beta$ under the following assumptions on $\phi$:

$$\min_{x,y} \phi(x,y) = \varepsilon_0$$

occurs for $x$ and $y$ in the *same* direction, (Typically at $x = y = x_0$, for some $x_0 \neq 0$). But if $x^1$ and $y^1$ (the 1-components of $x$, resp. $y$) have *opposite* signs

$$\phi(x,y) \geqq \varepsilon_0 + \alpha + \lambda(\phi_1(x) + \phi_1(y)),$$

for some $\alpha > 0$ and some $\lambda > 0$ with the property that for sufficiently large $\beta$

$$\int e^{-\beta\lambda\phi_1(x)} d^N x < \infty .$$

Examples of such potentials are:

1. $\phi_1(x)$ e.g. $gx^4 - \frac{1}{4}(x^1)^2 + (64g)^{-1}$, for some $g > 0, \phi_2(x,y) = V(x-y)$, where $V(x)$ is some strictly convex function with minimum at $x = 0$.
2. $\phi_1(x)$ e.g. $\gamma x^2, \gamma \ll 1, \phi_1(x,y) = V(x-y)$, with $V(x)$ e.g. $gx^4 - \frac{1}{4}(x^1)^2$, for some $g > 0$, (or $V$ an arbitrary continuous function with two sharp minima at $x = \pm(x^{01}, 0, \ldots, 0)$).
3. $\phi_1(x) = \gamma \log(|x| + 1)$, $\phi_2$ as in example 2.

Examples 2 and 3 (of anti-ferromagnetic type) are *not* of the general form of model AC, but can be brought into this form by replacing $x_i$ by $-x_i$ on one of the sublattices.

*Remarks.* It is of interest to consider also the case where $\phi_1(x)$ is replaced by const. $\beta^{-1}\phi_1(x)$. Then these models certainly do *not* have LRO for large $\beta$, as can be shown by a high temperature expansion.

The symmetry $\phi(x,y) = \phi(-x,-y)$ is *not* crucial for our arguments; see also [10, 12]. The main point of the study of model AC is that $\exp[-\beta\phi_2(x,y)]$ is *not* required to be of positive type. Next nearest neighbor interactions (coupling $x_{(m,n)}$ with $x_{(m\pm1,n\pm1)}$) could be included.

Physically more interesting models of an anharmonic crystal would be obtained by setting $\phi_1(x) = 0$ and assuming that $\phi_2$ is translation invariant. Our methods do *not* apply to such models.

(3) Quantum Antiferromagnetic Heisenberg Model

$$H = H^a = S^{-2}[H^z + \alpha H^{xy}] \tag{1.3}$$

$$H^z = \sum_{\langle i,j\rangle} S_i^z S_j^z$$

$$H^{xy} = \sum_{\langle i,j\rangle} \{S_i^x S_j^x + S_i^y S_j^y\}\,; \tag{1.4}$$

$S = 1/2, 1, 3/2, \ldots$ is the total spin at each site. We will prove that there is LRO at sufficiently large $\beta$ and small $\alpha$: For each $S$ there is an $\alpha(S)$ and $\beta_c(\alpha)$ such that for $\alpha < \alpha(S)$ and $\beta > \beta_c(\alpha)$ there is LRO. As $S \to \infty, \alpha(S) \to 1$. We do not know if there is LRO for all $\alpha < 1$ when $S$ is finite. This is an *open problem.* Because the $S \to \infty$ limit is the same as the classical model [14], we have here a generalization of the Malyshev result.

There is an equivalent form for (1.4) which is more convenient for our purposes, namely

$$H^z = -\sum_{\langle i,j\rangle} S_i^z S_j^z$$

$$H^{xy} = -\sum_{\langle i,j\rangle} \{S_i^x S_j^x + (iS_i^y)(iS_j^y)\}\,. \tag{1.4a}$$

This is obtained by making a rotation of $\pi$ about the $y$-axis for the spins on one of the two sublattices; for such spin operators $S^z \to -S^z, S^x \to -S^x, S^y \to +S^y$. In this representation *all* the terms in (1.3) are then of the form—(real matrix at $i$) (real matrix at $j$). Then reflection positivity, as discussed in Section II.A, holds. See [9] for more details.

(4) Quantum Ferromagnetic Heisenberg Model

$$H = H^f = -H^a. \tag{1.5}$$

The announced result [8] was that there is LRO for all $\alpha < 1$ when $\beta$ is large enough (uniformly in $S$). Unfortunately we cannot prove this because the proof of reflection positivity (Section II) is missing, but the second stage of the proof is correct and is given in Section III.

(5) The Two Quantum Models Can Be Modified as Follows

$$H^z = \sum_{\langle i,j\rangle} (S_i^z - S_j^z)^2$$

$$H^{xy} = \sum_{\langle i,j\rangle} \{(S_i^x - S_j^x)^2 + (S_i^y - S_j^y)^2\}\,.$$

$$H = S^{-2}[H^z + \alpha H^{xy}]\,. \tag{1.6}$$

This was mentioned in [8]. We will not give the details of the proofs here which are straight forward variants of the ones for (3), (4). This model is, however, interesting for the following reasons:

First, consider this model classically. When $\alpha = 0$ there is no LRO for any

$\beta$ by the Brascamp-Lieb argument [15]. Refined statements about exponential clustering were proved in [21]. When $\alpha = 1$ there is no LRO by MWH. We expect that there is no LRO for any $0 < \alpha < 1$ and any $\beta$.

However, the quantum model has a phase transition. In view of the foregoing remark, it is not surprising that our method yields the following in the ferromagnetic case (assuming reflection positivity): For $\alpha < 1$ there is a $\beta_c(\alpha, S)$, with LRO when $\beta > \beta_c(\alpha, S)$. However, $\beta_c(\alpha, S) \to \infty$ as $S \to \infty$ or $\alpha \to 1$.

(6) Quantum and Classical $xy$ Model

For convenience we take this model in the form

$$H = -S^{-2} \sum_{\langle i,j \rangle} \{S_i^z S_j^z + \alpha S_i^x S_j^x\}. \tag{1.7}$$

This is the *ferromagnet*. However by making a notation by $\pi$ about the $y$-axis for all spins on one sublattice (as in model (3)), we see that the antiferromagnet (defined with a $+$ sign in (1.7)) is *equivalent* to the ferromagnet. See [9] for further details. For this model, as given by (1.7), reflection positivity does hold: (see Section II.A and use the standard representation in which $S^z$ and $S^x$ are real matrices).

Since the results and proofs for this model are the same as for the antiferromagnet (model (3)), resp. for model (1), we will not give further details.

## *I.B. Remarks about Long Range Order*

Let $\langle - \rangle_\Lambda$ be the Gibbs state of a system in a bounded rectangle $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions, at inverse temperature $\beta$. The system in the thermodynamic limit, $\Lambda \uparrow \mathbb{Z}^2$, is said to have LRO if

$$\sigma(\beta) \equiv \lim_{\Lambda \uparrow \mathbb{Z}^2} \langle m_\Lambda^2 \rangle_\Lambda > 0, \tag{1.8}$$

where $m_\Lambda = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} m_i$ is the magnetization, and $m_i$ is defined, in the different models, by

(1) $m_i = S_i^1$
(2) $m_i = x_i^1$
(3) $m_i = S^{-1}(-1)^{i_1 + i_2} S_i^z$
(this is the staggered magnetisation)
(4), (5), (6) $m_i = S^{-1} S_i^z$.

The inequality $\sigma(\beta) > 0$ implies that there is spontaneous magnetization; see e.g. [9]. It is well known that $\sigma(\beta) \geqq M^2 > 0$ is implied by

$$\langle m_0 m_j \rangle_\Lambda \geqq M^2 > 0, \tag{1.9}$$

uniformly in $\Lambda$ and $j$. We will establish (1.9) at small temperature.

For this purpose, define $P_i^{\pm\delta}$ to be the projection operator onto all configurations satisfying $m_1 \geqq \delta$, resp. $m_i \leqq -\delta$. Moreover

$$P_i^{<\delta} = 1 - P_i^{+\delta} - P_i^{-\delta} \tag{1.10}$$

is the projection onto all configurations for which $|m_i| < \delta$. Finally, $P_i(\lambda) = P_i^{-\lambda} + P_i^{<\lambda}$ is the projection onto all configurations for which $m_i < \lambda$. For all models, except the AC model, $|m_i| \leqq 1$. Then

$$\begin{aligned}\langle m_0 m_j \rangle_\Lambda &= \int \lambda\lambda' \langle dP_0(\lambda) dP_j(\lambda') \rangle_\Lambda \\ &\geqq \delta^2 \{\langle P_0^{+\delta} P_j^{+\delta} \rangle_\Lambda + \langle P_0^{-\delta} P_j^{-\delta} \rangle_\Lambda\} \\ &\quad - \{\langle P_0^{+\delta} P_j^{-\delta} \rangle_\Lambda + \langle P_0^{-\delta} P_j^{+\delta} \rangle_\Lambda\} \\ &\quad - \delta^2 \langle P_0^{<\delta} P_j^{<\delta} \rangle_\Lambda . \end{aligned} \tag{1.11}$$

The three terms on the right side of (1.11) are labelled I, II, III.

First we discuss II. Since, in all models, $m_0$ and $m_j$ commute, for all $j$, $P_0^{+\delta} P_j^{-\delta} \leqq P_0^+ P_j^-$, for $\delta > 0$, with

$$\begin{aligned} P_j^+ &= P_j^{+(\delta=0)} = 1 - P_j(0), \\ P_j^- &= P_j(0). \end{aligned} \tag{1.12}$$

Therefore

$$\langle P_0^{+\delta} P_j^{-\delta} \rangle_\Lambda \leqq \langle P_0^+ P_j^- \rangle_\Lambda . \tag{1.13}$$

The right side of (1.13) will be estimated by means of a new version of the Peierls argument inspired by work of Glimm, Jaffe and Spencer [16], and will be shown to be small, for large $\beta$, in the following sense which depends on the model: For some $\varepsilon > 0$ and $\beta$ large enough

$$\langle P_0^+ P_j^- \rangle_\Lambda < \frac{\varepsilon}{2}, \tag{1.14}$$

uniformly in $\Lambda$ and $j$. Thus,

$$\mathrm{II} > -\varepsilon . \tag{1.15}$$

Next we discuss term III on the right side of (1.11). By the Schwarz inequality for the state $\langle - \rangle_\Lambda$

$$\langle P_0^{<\delta} P_j^{<\delta} \rangle_\Lambda \leqq \langle P_0^{<\delta} \rangle_\Lambda , \tag{1.16}$$

and we have used

$$(P_j^{<\delta})^2 = P_j^{<\delta}$$

and

$$\langle P_j^{<\delta} \rangle_\Lambda = \langle P_0^{<\delta} \rangle_\Lambda ,$$

which follows from the translation invariance of $\langle - \rangle_\Lambda$. We will prove by purely thermodynamic considerations that for some $\varepsilon > 0$ and sufficiently large $\beta$ (depending on the model)

$$\langle P_0^{<\delta} \rangle_\Lambda < \varepsilon . \tag{1.17}$$

Therefore

$$\mathrm{III} > -\varepsilon\delta^2 > -\varepsilon . \tag{1.18}$$

Finally we discuss term I on the right side of (1.11).

$$\begin{aligned}\langle P_0^{+\delta}P_j^{+\delta}\rangle_\Lambda &= \langle P_0^{+\delta}(1-P_j^{-\delta}-P_j^{<\delta})\rangle_\Lambda \\ &= \langle P_0^{+\delta}\rangle_\Lambda - \langle P_0^{+\delta}P_j^{<\delta}\rangle_\Lambda - \langle P_0^{+\delta}P_j^{-\delta}\rangle_\Lambda \\ &\geqq \langle P_0^{+\delta}\rangle_\Lambda - \langle P_0^{<\delta}\rangle_\Lambda - \langle P_0^{+\delta}P_j^{-\delta}\rangle_\Lambda .\end{aligned} \tag{1.19}$$

In all the models considered in this paper there is a symmetry taking $m_j$ to $-m_j$, for all $j\in\Lambda$. Therefore

$$\langle P_0^{-\delta}\rangle_\Lambda = \langle P_0^{+\delta}\rangle_\Lambda$$

so that

$$\langle P_0^{+\delta}\rangle_\Lambda = \tfrac{1}{2} - \tfrac{1}{2}\langle P_0^{<\delta}\rangle_\Lambda . \tag{1.20}$$

Combination of (1.14), (1.17), (1.19) and (1.20) yields

$$\langle P_0^{+\delta}P_j^{+\delta}\rangle_\Lambda > \tfrac{1}{2} - 2\varepsilon, \tag{1.21}$$

uniformly in $\Lambda$ and $j$, (provided $\beta$ is large enough, depending on the model). Therefore

$$\mathrm{I} \geqq \delta^2 - 4\delta^2\varepsilon.^1 \tag{1.22}$$

Insertion of (1.15), (1.18) and (1.22) into inequality (1.11) gives

$$\langle m_0 m_j\rangle_\Lambda \geqq \delta^2 - 4\delta^2\varepsilon - 2\varepsilon > \delta^2 - 6\varepsilon, \tag{1.23}$$

uniformly in $\Lambda$ and $j$. Therefore

$$\sigma(\beta) > \delta^2 - 6\varepsilon. \tag{1.24}$$

In each model we will choose $\delta$ and $\varepsilon$ to depend on $\beta$ in such a way that, for sufficiently large $\beta$, $\delta^2 - 6\varepsilon > 0$.

The most difficult inequality to prove is (1.14). The strategy will be explained in three steps, *C, D* and *E* below. The inequality (1.17) is relatively simple and will be given in Section IV.

### *I.C. The Peierls Argument*

In this section we describe a general form of the Peierls argument. We consider a finite classical or quantum lattice system in a square $\Lambda \subset \mathbb{Z}^2$. For convenience we wrap $\Lambda$ on a torus, but this is inessential for this part of the argument. At each site $i\in\Lambda$ we are given two orthogonal projection operators, $P_j^{\pm}$, with

$$P_j^+ + P_j^- = 1, \text{ for all } j. \tag{1.25}$$

In the following $\langle - \rangle \equiv \langle - \rangle_\Lambda$. We propose to derive an upper bound on $\langle P_m^+ P_n^- \rangle$, where $m$ and $n$ are arbitrary, fixed sites in $\Lambda$, and $m \neq n$. The first step is the trivial identity

$$\langle P_m^+ P_n^- \rangle = \left\langle P_m^+ P_n^- \prod_{\substack{j\in\Lambda \\ m\neq j\neq n}} (P_j^+ + P_j^-)\right\rangle, \tag{1.26}$$

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[1] This also holds when $\langle P_0^{+\delta}\rangle_\Lambda \neq \langle P_0^{-\delta}\rangle_\Lambda$

an immediate consequence of (1.25). We now expand the product on the right side of (1.26).

*Definition 1.* A *configuration* $c$ is a function on $\Lambda$ with values in $\{+,-\}$, and $c(m)=+, c(n)=-$. A *contour* $\gamma \subset \Lambda$ is a family of *nearest neighbor pairs* $\{\langle i_1, j_1\rangle, \ldots, \langle i_l, j_l\rangle : l=4,6,\ldots\}$ which decomposes $\Lambda$ into *precisely two* disjoint subsets

$$\Lambda_m = \Lambda_m(\gamma) \supset \{i_1, \ldots, i_l, m\}, \text{ and}$$
$$\Lambda_n = \Lambda_n(\gamma) \supset \{j_1, \ldots, j_l, n\} \text{ with}$$
$$\Lambda_m \cup \Lambda_n = \Lambda.$$

Given a configuration $c$, we let $\Gamma(c)$ denote that class of all contours $\gamma = \{\langle i_1, j_1\rangle, \ldots, \langle i_l, j_l\rangle\}$ with $c(i_k)=+, i_k \in \Lambda_m(\gamma), c(j_k)=-, j_k \in \Lambda_n(\gamma), k=1,\ldots,l$. Since, for any configuration $c, c(m)=+, c(n)=-$, we conclude that, given an arbitrary $c$, there exists a contour $\gamma(c) \in \Gamma(c)$ with the property that there exists a connected set $\Lambda_c \subset \Lambda_m(\gamma(c))$ such that $m \in \Lambda_c, c(i)=+$, for all $i \in \Lambda_c, \{i_1, \ldots, i_l\} \subseteq \Lambda_c$. (A set $X$ is connected if any two sites $i,j$ in $X$ belong to a chain $\{i=i_0, i_1, \ldots, i_k, i_{k+1}=j\} \subseteq X$ such that $i_l$ and $i_{l+1}$ are nearest neighbors, $l=0,\ldots,k$). Using Definition 1 we get from (1.26) by expanding

$$\begin{aligned}\langle P_m^+ P_n^- \rangle &= \sum_c \Big\langle \prod_{j\in\Lambda} P_j^{c(j)} \Big\rangle \\ &= \sum_\gamma \sum_{\{c,\gamma(c)=\gamma\}} \Big\langle \prod_{j\in\Lambda} P_j^{c(j)} \Big\rangle. \end{aligned} \tag{1.27}$$

Next, we note that

$$0 \leqq P_i^{\pm} \leqq 1, [P_i^{c(i)}, P_j^{c(j)}] = 0,$$

for $i \neq j$, arbitrary $c$. Hence, for $Y \subseteq X$,

$$0 \leqq \prod_{j\in X} P_j^{c(j)} \leqq \prod_{j\in Y} P_j^{c(j)}, \text{ so that}$$

$$0 \leqq \Big\langle \prod_{j\in X} P_j^{c(j)} \Big\rangle \leqq \Big\langle \prod_{j\in Y} P_j^{c(j)} \Big\rangle \leqq 1, \tag{1.28}$$

for all $c$. Therefore

$$\begin{aligned} &\sum_{\{c:\gamma(c)=\gamma\}} \Big\langle \prod_{j\in\Lambda} P_j^{c(j)} \Big\rangle \\ &\quad \leqq \sum_{\{c,\Gamma(c)\ni\gamma\}} \Big\langle \prod_{j\in\Lambda} P_j^{c(j)} \Big\rangle \\ &\quad = \Big\langle P_m^+ P_n^- \prod_{\langle i,j\rangle \in\gamma} P_i^+ P_j^- \Big\rangle \\ &\quad < \Big\langle \prod_{\langle i,j\rangle\in\gamma} P_i^+ P_j^- \Big\rangle. \end{aligned} \tag{1.29}$$

Therefore we have the inequality

$$\langle P_m^+ P_n^- \rangle \leqq \sum_\gamma \left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle. \tag{1.30}$$

Let $|\gamma|$ denote the number of nearest neighbor pairs in $\gamma$, (the "length" of $\gamma$).

**Theorem 1.1.** (*Peierls Argument*). *Suppose that* (*for large enough* $|\Lambda|$)

$$\left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle \leqq e^{-K|\gamma|}, \tag{1.31}$$

*for some constant* $K > \ln 3$ (*independent of* $\Lambda$). *Then*

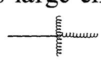
$$\langle P_m^+ P_n^- \rangle \leqq \sum_{l=2}^{\infty} 2l 3^{2l-2} e^{-2lK} < \infty, \tag{1.32}$$

*for arbitrary m and n in* $\Lambda$ (*and all sufficiently large squares* $\Lambda$).

*Remark.* The assertions of Theorem 1.1 do not depend on the size of $\Lambda$ and extend without change to the infinite system $\Lambda = \mathbb{Z}^2$; (see the subsequent proof and [10, Section 3]).

*Proof.* By Definition 1 the length of a contour is always *even*. The smallest contours are $\{\langle m,j_1\rangle, \langle m,j_2\rangle, \langle m,j_3\rangle, \langle m,j_4\rangle\}$ and $\{\langle i_1,n\rangle, \dots, \langle i_4,n\rangle\}$, i.e. have length 4. Hence, by (1.30),

$$\langle P_m^+ P_n^- \rangle \leqq \sum_{l=2}^{\infty} \sum_{\{\gamma : |\gamma| = 2l\}} \left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle. \tag{1.33}$$

(When $\Lambda$ is finite these sums are finite). Given some fixed length $2l$, well known combinatorics shows that there are no more than $2(l-1)3^{2l-2}$ contours of length $2l$, provided $\Lambda$ is large enough, depending on $m$ and $n$. (The factor $3^{2l-2}$ comes from a standard  -argument and the fact that all contours consist of one or two *closed* pieces. The factor $2(l-2)$ comes from the fact that each contour must separate $m$ from $n$). Theorem 1.1 now follows from (1.33) and the inequality $K > \ln 3$ which guarantees that the series $\sum_{l=2}^{\infty} 2(l-1)3^{2l-2}e^{-2lK}$ *converges*.

Q.E.D.

Theorem 1.1 has the following

**Corollary 1.2.** *Given* $\varepsilon > 0$, *there exists some finite* $K(\varepsilon)$ *such that, for all* $K \geqq K(\varepsilon)$,

$$\langle P_m^+ P_n^- \rangle < \frac{\varepsilon}{2}.$$

*Remarks.* 1. The relevance of Corollary 1.2 for the proof of long range order has been explained in Section 1.B.

2. Theorem 1.1 and Corollary 1.2 can easily be generalized to the case of more than two positive operators (e.g. projections) $P_i^1, \dots, P_i^M$ say, with

$$\sum_{l=1}^{M} P_i^l = 1.$$

We could apply this more refined Peierls argument to the quantum ferromagnet, model (4), with $P_i^1 = P_i^{+\delta}$, $P_i^2 = P_i^{<\delta}$, $P_i^3 = P_i^{-\delta}$. This extension is important in models with more complicated phase diagrams involving at least $M > 2$ pure phases; (see e.g. [10], Sections 3 and 8).

3. Clearly these techniques extend to arbitrary dimensions $\geqq 2$ and other than simple, cubic lattices. See also [12].

## *I.D. Reflection Positivity and Chessboard Estimate*

In Section I.C we have seen that in order to prove

$$\langle P_m^+ P_n^- \rangle < \varepsilon,$$

uniformly in $m$ and $n$, it is sufficient to show

$$\left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle \leqq e^{-K|\gamma|} \tag{1.34}$$

for some constant $K = K(\varepsilon) \gg \ln 3$. Here we want to sketch how (1.34) can be reduced to a purely thermodynamic estimate.

Let $\Lambda$ be a square with sides of length $N = 4M, M = 1, 2, 3, \ldots$. We define a "universal projection"

$$P_\Lambda = \prod_{m=0}^{M-1} \left[ \prod_{n=0}^{N-1} P^+_{(4m,n)} P^-_{(4m+1,n)} P^-_{(4m+2,n)} P^+_{(4m+3,n)} \right]. \tag{1.35}$$

The following self explanatory figure illustrates Equation (1.35).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |
| + | − | − | + | + | − | − | + |

$\Lambda$

$N = 8, M = 2$

**Fig. 1**

One of the *key estimates* in our approach to proving LRO is the inequality

$$\left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle \leqq \langle P_\Lambda \rangle^{|\gamma|/2|\Lambda|} \tag{1.36}$$

which we shall prove for models (1), (3), (5) (antiferromagnetic case) and (6), i.e. all models except the quantum ferromagnets and the anharmonic classical crystal, model (2). For the former, we believe that (1.36) holds but we have no proof; (1.36) will be assumed to hold in the sequel.

For the AC model, the definition of the universal projection has to be modified: Let $\Lambda$ be a square with sides of even length $N = 2M$. We define

$$P_\Lambda^{AC} = \prod_{m=0}^{M-1} \left[ \prod_{n=0}^{M-1} P^+_{(2n,2m)} P^-_{(2n+1,2m)} \right]. \tag{1.37}$$

The following figure explains the definition of $P_\Lambda^{AC}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| + | − | + | − | + | − | + | − |
| | | | | | | | |
| + | − | + | − | + | − | + | − |
| | | | | | | | |
| + | − | + | − | + | − | + | − |
| | | | | | | | |
| + | − | + | − | + | − | + | − |
| | | | | | | | |

$N = 8, M = 4$

Fig. 2

In the case of the anharmonic classical crystal we prove

$$\left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle \leqq \langle P_\Lambda^{AC} \rangle^{|\gamma|/2|\Lambda|}. \tag{1.38}$$

(This inequality *also* holds for the classical $N$-vector model, model (1)). Our proofs of inequalities (1.36) and (1.38) are based on the notion of *reflection positivity* (or $O-S$ positivity) which we now explain: We choose a pair of lines $l$ parallel to one of the coordinate axes, cutting $\Lambda$ into two congruent pieces $\Lambda_+$ and $\Lambda_-$ (Note: $l$ is a pair of lines, because $\Lambda$ is wrapped on a torus). In models (1), (3), (4) and (5) the lines $l$ are *between* two lattice lines, so that $\Lambda_+ \cap \Lambda_- = \emptyset$, whereas in model (2) $l$ consists of two *lattice lines*, and $\Lambda_+ \cap \Lambda_- = l$. Let $\theta_l$ be the reflection at $l$. Let $F = F(m)_{\Lambda_+}$ be a complex-valued function of all the $m_i$'s (see Section 1.B), with $i \in \Lambda_+$. We define $\theta_l F = \theta_l F(m)_{\Lambda_-}$ to be the function obtained from $F$ by substituting $m_{\theta_l j}$ for $m_j$. *Reflection positivity* is the inequality

$$\langle F(\theta_l \bar{F}) \rangle \geqq 0, \tag{1.39}$$

where $\bar{F}$ is the *complex conjugate* of $F$. A somewhat more general inequality (also called reflection positivity) is discussed in Section II; as an example we mention that in the $N$-vector model, (1.39) is true for arbitrary, complex-valued functions $F$ of $\{\mathbf{S}_i : i \in \Lambda_+\}$ and both choices of $l$ (between two lattice lines or coinciding with a lattice line).

Reflection positivity (1.39) yields the following Schwarz inequality: If $F$ and $G$ are functions of $(m)_{\Lambda_+}$ then

$$|\langle F(\theta_l \bar{G})\rangle|^2 \leqq \langle F(\theta_l \bar{F})\rangle \langle G(\theta_l \bar{G})\rangle. \tag{1.40}$$

Next, we indicate how (1.36), resp. (1.38) follow from (1.40). Let $\gamma_h$ be all pairs $\langle i,j\rangle$ of the contour $\gamma$ for which $j-i$ points in the 1-direction. Such pairs are called "horizontal". Furthermore $\gamma_V = \gamma \backslash \gamma_h$ denotes all "vertical" pairs in $\gamma$. For $\langle i,j\rangle \in \gamma_h$, let $i \wedge j$ denote the site with smaller 1-coordinate; for $\langle i,j\rangle \in \gamma_V$, $i \wedge j$ is the site with smaller 2-coordinate. Suppose that reflection positivity (1.39) holds for reflections $\theta_l$ at lines $l$ *between* two lattice lines. Then we define

$$\gamma_{h,e} = \{\langle i,j\rangle \in \gamma_h : i \wedge j \text{ even}\}$$

$$\gamma_{h,0} = \{\langle i,j\rangle \in \gamma_h : i \wedge j \text{ odd}\} = \gamma_h \backslash \gamma_{h,e}.$$

Similarly $\gamma_{V,e}$ and $\gamma_{V,0}$ are defined. By the standard Schwarz inequality for $\langle - \rangle$ we have

$$\begin{aligned}\left\langle \prod_{\langle i,j\rangle \in \gamma} P_i^+ P_j^- \right\rangle &\leqq \left\langle \prod_{\langle i,j\rangle \in \gamma_h} P_i^+ P_j^- \right\rangle^{1/2} \left\langle \prod_{\langle i,j\rangle \in \gamma_V} P_i^+ P_j^- \right\rangle^{1/2} \\ &\leqq \prod_{\substack{\alpha = h,V \\ \beta = e,0}} \left\langle \prod_{\langle i,j\rangle \in \gamma_{\alpha,\beta}} P_i^+ P_j^- \right\rangle^{1/4}\end{aligned} \tag{1.41}$$

To each factor on the right side we now apply reflection positivity (1:39) and inequality (1.40) repeatedly, for many different choices of $l$. This yields

$$\left\langle \prod_{\langle i,j\rangle \in \gamma_{\alpha,\beta}} P_i^+ P_j^- \right\rangle \leqq \langle P_\Lambda \rangle^{2|\gamma_{\alpha,\beta}|/|\Lambda|}. \tag{1.42}$$

This inequality is a special case of a general corollary of reflection positivity, called *chessboard estimate* [19], which we prove in Section 2. Clearly, inequalities (1.41) and (1.42) yield the *key inequality* (1.36).

In the classical, anharmonic crystal, model (2), we first decompose $\gamma$ into two pieces, $\gamma_h$ consisting of horizontal and $\gamma_V$ consisting of vertical pairs. For $\langle i,j\rangle \in \gamma_h$, let $(ij)_2$ denote the 2-coordinate of both $i$ and $j$, for $\langle i,j\rangle \in \gamma_V$, let $(ij)_1$ be the 1-coordinate of $i$ and $j$. We define

$$\gamma_{h,e} = \{\langle i,j\rangle \in \gamma_h : (ij)_2 \text{ even}\}$$

$$\gamma_{h,0} = \{\langle i,j\rangle \in \gamma_h : (ij)_2 \text{ odd}\}$$

$$\gamma_{V,e} = \{\langle i,j\rangle \in \gamma_V : (ij)_1 \text{ even}\}$$

$$\gamma_{V,0} = \{\langle i,j\rangle \in \gamma_V : (ij)_1 \text{ odd}\}.$$

Applying again the standard Schwarz inequality for $\langle - \rangle$, we obtain

$$\left\langle \prod_{\langle i,j\rangle\in\gamma} P_i^+ P_j^- \right\rangle \leqq \prod_{\substack{\alpha=h,V\\ \beta=e,0}} \left\langle \prod_{\langle i,j\rangle\in\gamma_{\alpha,\beta}} P_i^+ P_j^- \right\rangle^{1/4}. \tag{1.43}$$

To each term on the r.s. of (1.43) we apply the Schwarz inequality (1.40) repeatedly, for all allowed choices of reflections $\theta_l$ at *lattice lines* $l$. This yields (see Section II)

$$\left\langle \prod_{\langle i,j\rangle\in\gamma_{\alpha,\beta}} P_i^+ P_j^- \right\rangle \leqq \langle P_\Lambda^{AC} \rangle^{2|\gamma_{\alpha,\beta}|/|\Lambda|}. \tag{1.44}$$

The *key inequality* (1.38) follows from inequalities (1.43) and (1.44). Further details concerning reflection positivity and the chessboard estimates (1.42) and (1.44) are given in Section II.

### *I.E. Estimate of $\langle P_\Lambda \rangle$ and Exponential Localization*

In this section we sketch the main ideas of how to estimate

$R_\Lambda(\beta) \equiv \langle P_\Lambda \rangle$ and $R_\Lambda^{AC}(\beta) \equiv \langle P_\Lambda^{AC} \rangle$.

By definition of $\langle - \rangle$,

$$R_\Lambda^{(AC)}(\beta) \equiv \frac{\mathrm{Tr}(\exp[-\beta H_\Lambda] P_\Lambda^{(AC)})}{\mathrm{Tr}(\exp[-\beta H_\Lambda])}. \tag{1.45}$$

where $R_\Lambda^{(AC)}$ means either $R_\Lambda$ or $R_\Lambda^{AC}$. Here $H_\Lambda$ is the Hamiltonian of the model under consideration, and Tr is the usual trace in the quantum mechanical models, and, in the classical models, an integral with measure the product of the single spin distributions over all sites in $\Lambda$. Let $E_\Lambda(de)$ denote the spectral measure of the Hamiltonian $H_\Lambda$. By the spectral theorem

$$\mathrm{Tr}(\exp[-\beta H_\Lambda] C) = \int_{e_0}^{\infty} e^{-\beta e}\, \mathrm{Tr}(E_\Lambda(de) C), \tag{1.46}$$

where $e_0 \equiv e_0(\Lambda) = \mathrm{infspec}\, H_\Lambda$ is the groundstate energy, and $C$ is an arbitrary operator, resp. function. We will choose some positive number $\Delta = \Delta(\beta)$, depending on the model under consideration, and decompose $R_\Lambda^{(AC)}(\beta)$ into two pieces

$$R_-^{(AC)}(\beta, \Delta) = Z_\Lambda(\beta)^{-1} \int_{e_0}^{e_0+\Delta|\Lambda|} e^{-\beta e}\, \mathrm{Tr}(E_\Lambda(de) P_\Lambda^{(AC)})$$

$$R_+^{(AC)}(\beta, \Delta) = Z_\Lambda(\beta)^{-1} \int_{e_0+\Delta|\Lambda|}^{\infty} e^{-\beta e}\, \mathrm{Tr}(E_\Lambda(de) P_\Lambda^{(AC)}), \tag{1.47}$$

where

$$Z_\Lambda(\beta) = \mathrm{Tr}(\exp[-\beta H_\Lambda]) = \int_{e_0}^{\infty} e^{-\beta e}\, \mathrm{Tr}(E_\Lambda(de)) \tag{1.48}$$

is the partition function. We estimate $R_+^{(AC)}(\beta, \Delta)$ by

$$R_+^{(AC)}(\beta,\Delta) \leqq Z_\Lambda(\beta)^{-1} \exp\{-\beta[e_0+\Delta|\Lambda|]\} \int_{e_0+\Delta|\Lambda|}^{\infty} \mathrm{Tr}(E_\Lambda(de))$$
$$\leqq \exp\{-\beta[e_0+\Delta|\Lambda|]\}\{\mathrm{Tr}(1)Z_\Lambda(\beta)^{-1}\}. \tag{1.49}$$

The Peierls-Bogoliubov inequality will be shown to give

$$\mathrm{Tr}(1)/Z_\Lambda(\beta) \leqq \exp \beta[e_0 + \tfrac{1}{2}\Delta|\Lambda|], \tag{1.50}$$

for $\beta$ sufficiently large. Thus

$$R_+^{(AC)}(\beta,\Delta) \leqq \exp\{-\beta\Delta|\Lambda|/2\}. \tag{1.51}$$

Next we consider $R_-^{(AC)}(\beta,\Delta)$. In the classical cases this will vanish for the following reason: $\Delta$ will be chosen sufficiently small so that $P_\Lambda^{(AC)}$ will be a projection onto configurations with energy greater than $e_0+\Delta|\Lambda|$. Thus the integral for $R_-^{(AC)}(\beta)$ will vanish identically.

In the quantum cases, the situation is more complicated. Although $P_\Lambda$ will be a projection onto states whose *average energy* exceeds $e_0+\Delta|\Lambda|$, the integral does *not* vanish, because $P_\Lambda$ will have nonvanishing matrix elements in eigenstates of $H_\Lambda$ with energy $< e_0+\Delta|\Lambda|$. To be explicit, let $e_0 \leqq e_1 \leqq$ be the eigenvalues of $H_\Lambda$ with eigenvectors $\phi_0, \phi_1, \ldots$. Then

$$R_-(\beta,\Delta) = Z_\Lambda(\beta)^{-1} \sum_{i\Delta}{}' C_i \exp[-\beta e_i] \tag{1.52}$$

where $\sum_\Delta'$ means a sum over $i$ such that $e_i \leqq e_0 + \Delta|\Lambda|$, and

$$C_i \equiv (\phi_i, P_\Lambda \phi_i). \tag{1.53}$$

Now $C_i$ is independent of $\beta$, and terefore $R_-(\beta,\Delta)$ does not necessarily vanish as $\exp[-\beta(\text{const.})]$. What we have to show is that $C_i \to 0$ sufficiently fast as $i \to 0$. Then we can hope that $R_-(\beta,\Delta)$ goes to zero sufficiently fast as $\beta \to \infty$, for a suitable choice of $\Delta$.

The estimate on $C_i$, carried out in Section III, comes about in the following way: We write $H_\Lambda = A_\Lambda + B_\Lambda$, where $B_\Lambda$ is suitably small compared to $A_\Lambda$, and such that $P_\Lambda$ is an eigenprojection for $A_\Lambda$ onto $A_\Lambda$ eigenvectors having energy $> e_0 + n\Delta|\Lambda|$ for some $n$. In model (3), for example, $A = S^{-2}H^z$. If $B_\Lambda$ were zero then $C_i = 0$ for $e_i \leqq e_0 + \Delta|\Lambda|$. The *principle of exponential localization* will tell us that $A_\Lambda$ eigenvectors of $A_\Lambda$ energy greater than $e_0 + n\Delta|\Lambda|, (n \geqq 2)$ when expanded in the $\phi_i$, are strongly (indeed, *exponentially well* in $|\Lambda|$) localized around $\phi_i$ with $e_i > e_0 + \Delta|\Lambda|$. This, in turn, will lead to $C_i$ being *small* for $e_i \approx e_0$.

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## II. Reflection Positivity and Chessboard Estimates

### *II.A. Reflection Positivity*

In this section we recall the proofs of reflection positivity, inequality (1.39), for the models studied in this paper. For the classical $N$-vector models, reflection positivity is shown in [20]. In terms of a transfer matrix formalism it is used in [10]. For the quantum anti-ferromagnet and the quantum mechanical $xy$ model (models (3) and (6)) reflection positivity was discovered in [9]. The proof given there also applies to the classical $N$-vector models.

First we consider the classical, anharmonic crystal, model (2), for which (1.39) is new. We choose a pair of lattice lines $l$ cutting $\Lambda$ into two congruent pieces, $\Lambda_+$ and $\Lambda_-$, with $\Lambda_+ \cap \Lambda_- = l$. Let $\tilde{\Lambda}_\pm = \Lambda_\pm \backslash l$. The $N$-vector oscillators attached to sites in $\tilde{\Lambda}_\pm$ have coordinates $(y)_\pm \equiv \{y_j \in \mathbb{R}^N : j \in \tilde{\Lambda}_\pm\}$. The coordinates of the $N$-vector oscillators attached to sites in $l$ are denoted by

$$(z) \equiv \{z_j \in \mathbb{R}^N : j \in l\}.$$

Given a function $F$ of $(y)_+, (z)$, we define $\theta_l F$ to be the function of $(y)_-, (z)$ obtained by substituting $y_{\theta_l j}$ for $y_j$, for all $j \in \tilde{\Lambda}_\pm$, i.e. $\theta_l F$ is the reflection of $F$ in the lines $l$. The Hamilton function $H_\Lambda$ of the AC model is given by

$$\begin{aligned} H_\Lambda &= \sum_{\langle i,j\rangle \subset \Lambda} \phi(x_i, x_j) \\ &= \sum_{\langle i,j\rangle \subset \Lambda_+} \phi(x_i, x_j) + \sum_{\langle i,j\rangle \subset \Lambda_-} \phi(x_i, x_j) \\ &= \sum_{\langle i,j\rangle \subset \Lambda_+} \{\phi(x_i, x_j) + \phi(x_{\theta_l i}, x_{\theta_l j})\} \\ &\equiv B((y)_+, (z)) + (\theta_l B)((y)_-, (z)). \end{aligned}$$

Let $dx$ be the a priori distribution of a single oscillator, and set

$$\begin{aligned} d(y)_\pm &= \prod_{j \in \tilde{\Lambda}_\pm} d^N y_j \\ d(z) &= \prod_{j \in l} d^N z_j. \end{aligned}$$

Let $F = F((y)_+, (z))$ be an arbitrary function localized on $\Lambda_+$. Then

$$\begin{aligned} \langle F(\overline{\theta_l F})\rangle &= Z_\Lambda(\beta)^{-1} \int d(z) d(y)_+ d(y)_- e^{-\beta H_\Lambda} F((y)_+, (z)) \overline{\theta_l F((y)_-, (z))} \\ &= Z_\Lambda(\beta)^{-1} \int d(z) \{\int d(y)_+ e^{-\beta B((y)_+, (z))} F((y)_+, (z))\} \\ &\quad \times \{\int d(y)_- e^{-\beta(\theta_l B)((y)_-, (z))} \overline{(\theta_l F)((y)_-, (z))}\} \\ &= Z_\Lambda(\beta)^{-1} \int d(z) \left| \int d(y)_+ \right. \\ &\quad \times \left. e^{-\beta B((y)_+, (z))} F((y)_+, (z)) \right|^2, \end{aligned} \tag{2.1}$$

i.e.

$$\langle F(\overline{\theta_l F})\rangle \geqq 0, \tag{2.2}$$

which is reflection positivity. Clearly, this form of reflection positivity also holds for the classical $N$-vector models. Next, we consider the quantum mechanical models and the classical $N$-vector models. We let $l$ be a pair of lines between lattice lines cutting $\Lambda$ into two *disjoint*, congruent pieces, $\Lambda_+$ and $\Lambda_-$. Let $\mathfrak{A}_j$ denote the family of all bounded functions of the spin $\mathbf{S}_j$ ("algebra of observables" at site $j$). We define

$$\mathfrak{A}_\pm = \bigotimes_{j \in \Lambda_\pm} \mathfrak{A}_j,$$

and

$$\mathfrak{A} = \mathfrak{A}_+ \otimes \mathfrak{A}_- .$$

Given some $B \in \mathfrak{A}$, we define $\theta B \equiv \theta_l B$ by

$$(\theta B)((\mathbf{S})_\Lambda) = B((\theta \mathbf{S})_\Lambda), \tag{2.3}$$

where

$$(\mathbf{S})_\Lambda = \{\mathbf{S}_j : j \in \Lambda\}, \text{ and}$$

$$(\theta \mathbf{S})_\Lambda = \{\mathbf{S}_{\theta_l j} : j \in \Lambda\},$$

i.e. $\theta B$ is obtained from $B$ by substituting $\mathbf{S}_{\theta_l j}$ for $\mathbf{S}_j$, all $j \in \Lambda$. Clearly $\theta$ defines an isomorphism from $\mathfrak{A}_+$ onto $\mathfrak{A}_-$ (and conversely). Furthermore we define

$$\bar{B} = (B^T)^* \tag{2.4}$$

to be the *complex conjugate* (*not* the adjoint) of $B$, for arbitrary $B \in \mathfrak{A}$. Following [9] we study Hamiltonians of the following general form:

$$H = B + \theta(\bar{B}) - \sum_i C_i \theta(\bar{C}_i), \tag{2.5}$$

where $B, C_1, \ldots, C_k, \ldots$ are in $\mathfrak{A}_+$, (and $B = B^*, C_i = \pm C_i^*$, for all $i$, so that $H$ is selfadjoint). The following result is a slight variation of Theorem E.1 of [9].

**Theorem 2.1.** (*Reflection Positivity*). *Let $F \in \mathfrak{A}_+$. Then*

$$\langle F(\overline{\theta F}) \rangle \equiv \frac{\mathrm{Tr}(e^{-\beta H} F(\overline{\theta F}))}{\mathrm{Tr}(e^{-\beta H})} \geqq 0,$$

*where* "Tr" *means the usual trace in the quantum case and an integral in the classical case.*

*Proof.* It clearly suffices to prove that $\mathrm{Tr}(e^{-\beta H} F(\overline{\theta F})) \geqq 0$. By the Trotter product formula,

$$e^{-\beta H} = \lim_{n \to \infty} G_n, \qquad \text{where}$$

$$G_n = \left( e^{-(\beta/n)B} e^{-(\beta/n)\theta \bar{B}} \left[ 1 + \frac{\beta}{n} \sum_i C_i \theta \bar{C}_i \right] \right)^n. \tag{2.6}$$

Thus, Theorem 2.1 is proved if

$$\mathrm{Tr}(G_n F(\overline{\theta F})) \geqq 0, \text{ for all } n. \tag{2.7}$$

To prove (2.7), note that all elements in $\mathfrak{A}_+$ commute with all elements in $\mathfrak{A}_-$. In (2.7) all elements with a $\theta$ (which are in $\mathfrak{A}_-$) can therefore be moved to the right of all elements without a $\theta$ (which are in $\mathfrak{A}_+$). This shows that $\mathrm{Tr}(G_n F(\overline{\theta F}))$ is a sum of terms of the form

$$\begin{aligned}\mathrm{Tr}(D_1 \ldots D_m F \theta \bar{D}_1 \ldots \theta \bar{D}_m \theta \bar{F}) &= \mathrm{Tr}(D_1 \ldots D_m F \theta(\bar{D}_1 \ldots \bar{D}_m \bar{F})) \\ &= \mathrm{Tr}(D_1 \ldots D_m F)\, \mathrm{Tr}(\bar{D}_1 \ldots \bar{D}_m \bar{F}),\end{aligned}$$

with $D_1, \ldots, D_m$ in $\mathfrak{A}_+$. Here we have used the obvious facts that $\mathrm{Tr}(AB) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$, for $A \in \mathfrak{A}_+, B \in \mathfrak{A}_-$, and $\mathrm{Tr}(\theta A) = \mathrm{Tr}(A)$, for all $A \in \mathfrak{A}_+$. Finally

$$\mathrm{Tr}(D_1 \ldots D_m F)\,\mathrm{Tr}(\bar{D}_1 \ldots \bar{D}_m \bar{F}) = |\mathrm{Tr}(D_1 \ldots D_m F)|^2 \geqq 0,$$

by definition of complex conjugation ($B \mapsto \bar{B}$). Q.E.D.

We leave it to the reader to check that the Hamiltonians $H_\Lambda$ of models (1), (3) and (6) are of the form (2.5). See also [9]. Hence Theorem 2.1 proves *reflection positivity*, inequality (1.39), for these models. However, for the quantum ferromagnet, models (4) and (5) (ferromagnetic case), $H_\Lambda$ is *not* of the form (2.2) (because of the $S_i^y S_j^y$ terms), and the proof of Theorem 2.1 breaks down. At present, *no* useful form of reflection positivity is known for these models. In the sequel, we will assume that inequality (1.36), which follows from reflection positivity (as shown in Section II.B) *does* hold for the ferromagnetic models, even though we have no proof of it.

### *II.B. Chessboard Estimate*

Our goal in this subsection is to use reflection positivity to prove inequalities (1.42) and (1.44) (chessboard estimate). We prove a general theorem that includes (1.42) as a special case.

**Theorem 2.2.** (*Generalized Hölder Inequality*). *Let $\mathfrak{A}$ be a vector space with anti-linear involution $J$ (to be thought of as complex conjugation). Let $\omega$ be a multilinear functional on $\mathfrak{A}^{\times 2M}$, for some integer $M > 0$, with the properties*

*(C) $\omega(A_1, \ldots, A_{2M}) = \omega(A_2, \ldots, A_{2M}, A_1)$ (cyclicity), and*

*($\theta$) The matrix $K$ whose matrix elements $K_{ij}$ are given by*

$$K_{ij} = \omega(JA_{i,1}, \ldots, JA_{i,M}, A_{j,1}, \ldots, A_{j,M}),$$

*with $A_{l,m}$ an arbitrary vector in $\mathfrak{A}$, for all $l = 1, \ldots, n, m = 1, \ldots M$, is a positive semi-definite $n \times n$ matrix, for all $n = 1, 2, \ldots$; (Reflection Positivity). Then*

$$(1)\ |\omega(A_1, \ldots, A_{2M})| \leqq \prod_{j=1}^{2M} \omega(JA_j, A_j, \ldots, JA_j, A_j)^{1/2M},$$

*(chessboard estimate)*

*and*

$$(2)\ \|A\|_{2M} \equiv \omega(JA, A, \ldots, JA, A)^{1/2M}$$

*is a semi-norm on $\mathfrak{A}$.*

*Proof.* 1. *A Schwarz Inequality*. Let $\mathscr{L}(\mathfrak{A}^{\times M})$ be the vector space over the complex numbers spanned by all elements in $\mathfrak{A}^{\times M}$. Hypothesis ($\theta$) tells us precisely that $\omega$ defines an *inner product* on $\mathscr{L}(\mathfrak{A}^{\times M})$. As a special consequence of the *Schwarz inequality* for this inner product we have

$$|\omega(A_1, \ldots, A_{2M})| \leqq \omega(A_1, \ldots, A_M, JA_M, \ldots, JA_1)^{1/2} \cdot \omega(JA_{2M}, \ldots, JA_{M+1}, A_{M+1}, \ldots, A_{2M})^{1/2}. \tag{2.8}$$

2. *Proof of Theorem* 2.2 *for* $M = 2$. This serves to exhibit the main ideas behind the proof of the general case. By (2.8) and hypothesis ($C$),

$$\begin{aligned}|\omega(A,B,C,D)| &\leqq \omega(A,B,JB,JA)^{1/2}\omega(JD,JC,C,D)^{1/2}\\ &= \omega(B,JB,JA,A)^{1/2}\omega(JC,C,D,JD)^{1/2}\\ &\leqq \omega(B,JB,B,JB)^{1/4}\omega(JA,A,JA,A)^{1/4}\\ &\quad\cdot\omega(JC,C,JC,C)^{1/4}\omega(D,JD,D,JD)^{1/4}\\ &= \omega(JA,A,JA,A)^{1/4}\omega(JB,B,JB,B)^{1/4}\\ &\quad\cdot\omega(JC,C,JC,C)^{1/4}\omega(JD,D,JD,D)^{1/4}\end{aligned}$$

which is (1); (2) follows from the multilinearity of $\omega$ and (1).

3. *The General Case.* Since $\omega$ is multi-linear and

$$\omega(JA_j, A_j, \ldots, JA_j, A_j) = \omega(A_j, JA_j, \ldots, A_j, JA_j),$$

by hypothesis (C), we may assume that

$$\omega(JA_j, A_j, \ldots, JA_j, A_j) = 1, \tag{2.9}$$

for all $j = 1, \ldots, 2M$; (if not, replace $A_j$ by $\omega(JA_j, A_j, \ldots, JA_j, A_j)^{-1/2M} \cdot A_j$). We set $JA_j \equiv A_{j+2M}, j = 1, \ldots, 2M$. A *configuration* $c$ is a function on $\{1, \ldots, 2M\}$ with values in $\{1, \ldots, 4M\}$. Let $z \equiv \max\limits_c |\omega(A_{c(1)}, A_{c(2)}, \ldots, A_{c(2M)})|$, i.e.

$$z \geqq |\omega(A_{c(1)}, \ldots, A_{c(2M)})|, \text{ for all } c. \tag{2.10}$$

**Lemma.** $z = 1$.

*Proof.* For $c$ defined by

$$c(2m-1) = j + 2M, c(2m) = j,$$

$m = 1, \ldots, M$,

$$\omega(A_{c(1)}, \ldots, A_{c(2M)}) = 1,$$

by (2.9). Hence $z \geqq 1$. Thus, it suffices to show $z \leqq 1$. Let $\tilde{c}$ be a configuration for which

$$|\omega(A_{\tilde{c}(1)}, \ldots, A_{\tilde{c}(2M)})| = z.$$

Let $c(M+1) \equiv j$. Then, by the Schwarz inequality (2.8),

$$\begin{aligned}z &= |\omega(A_{\tilde{c}(1)}, \ldots, A_{\tilde{c}(2M)}|\\ &\leqq \omega(A_{\tilde{c}(1)}, \ldots, A_{\tilde{c}(M)}, JA_{\tilde{c}(M)}, \ldots, JA_{\tilde{c}(1)})^{1/2}\\ &\quad\cdot\omega(JA_{\tilde{c}(2M)}, \ldots, JA_j, A_j, \ldots, A_{\tilde{c}(2M)})^{1/2}\\ &\leqq z^{1/2}\omega(JA_{\tilde{c}(2M)}, \ldots, JA_j, A_j, \ldots, A_{\tilde{c}(2M)})^{1/2}, \text{ by (2.10)}\\ &= z^{1/2}\omega(JA_{\tilde{c}(2M-1)}, \ldots, JA_j, A_j, \ldots, A_{\tilde{c}(2M)}, JA_{\tilde{c}(2M)})^{1/2}, \text{ by hypothesis (C)}\\ &\leqq z^{3/4}\omega(JA_{\tilde{c}(2M-1)}, \ldots, JA_j, A_j, JA_j, A_j, \ldots, A_{\tilde{c}(2M-1)})^{1/4}, \text{ by (2.8) and (2.10)}\end{aligned}$$

$$\begin{aligned}
&\leqq \dots \\
&\leqq z^{1-2^{-(m-1)}} \omega(\underbrace{JA_j, A_j, \dots, JA_j, A_j}_{M}, \underbrace{*, *, \dots}_{M})^{2^{-(m-1)}} \\
&\leqq z^{1-2^{-m}} \omega(JA_j, A_j, \dots, JA_j, A_j)^{2^{-m}}, \text{ by (2.8) and (2.10)} \\
&= z^{1-2^{-m}}, \text{ by (2.9),}
\end{aligned}$$

for some $m$ with $2^{m-1} \geqq M > 2^{m-2}$. Hence $z^{2^{-m}} \leqq 1$, i.e. $z \leqq 1$.

Q.E.D.

To prove Theorem 2.1, (1), let $c$ be given by $c(j) = j, j = 1, \dots, 2M$. By (2.10) and the Lemma,

$$|\omega(A_{c(1)}, \dots, A_{c(2M)})| = |\omega(A_1, \dots, A_{2M})| \leqq z = 1. \tag{2.11}$$

The multilinearity of $\omega$ and (2.11) completes the proof of (1). Theorem 2.2, (2) follows from the multilinearity of $\omega$ and hypothesis ($\theta$) (which imply $\|A\|_{2M} \geqq 0$ and $\|\lambda A\|_{2M} = |\lambda| \, \|A\|_{2M}$) and from (1) (which implies that $\|A+B\|_{2M} \leqq \|A\|_{2M} + \|B\|_{2M}$). Q.E.D.

To apply Theorem 2.2 to the proof of estimates (1.42), resp. (1.44), one makes the following identifications:

$$\omega(\cdot) \mapsto \langle \cdot \rangle$$

$$A_j \mapsto P_i^+ P_j^-, \text{ with } i,j \text{ nearest neighbors;}$$

Theorem 2.2 must be applied twice, once in the *vertical direction* and once in the *horizontal direction*. This gives (1.42), resp. (1.44). We now must check that $\omega(\cdot) = \langle \cdot \rangle$ satisfies the hypothesis of Theorem 2.2: Clearly $\left\langle \prod_{j \in \Lambda} B_j \right\rangle$ is linear in each $B_j$, yielding multi-linearity of $\omega$.

Since we have wrapped $\Lambda$ on a torus (periodic boundary conditions),

$$\left\langle \prod_{j \in \Lambda} B_j \right\rangle = \left\langle \prod_{j \in \Lambda} B_{j+a} \right\rangle,$$

for arbitrary $a \in \Lambda$. This shows that $\omega$ satisfies hypothesis (C) in both, the vertical and the horizontal directions. Finally, hypothesis ($\theta$) of Theorem 2.2 in both, the vertical and the horizontal directions, is an immediate consequence of *reflection positivity* (inequality (2.1), resp. Theorem 2.1). A more direct proof of inequalities (1.42) and (1.44) proceeds as follows; (we sketch the argument leading to (1.42); the case of the anharmonic crystal is treated similarly). Let $B_{h,e}$ denote all pairs of horizontal nearest neighbors $\langle i,j \rangle$ (directed, "horizontal bonds") with $i \wedge j$ even. Let $\mathcal{O}$ be an arbitrary, *non empty* subset of $B_{h,e}$. Let $|\mathcal{O}|$ denote the number of horizontal bonds in $\mathcal{O}$. We consider the family

$$\left\{ \left\langle \prod_{\langle i,j \rangle} P_i^+ P_j^- \right\rangle^{1/2|\mathcal{O}|} : \mathcal{O} \subseteq B_{h,e} \right\}.$$

Let

$$z = \max_{\mathcal{O}} \left\{ \left\langle \prod_{\langle i,j \rangle \in \mathcal{O}} P_i^+ P_j^- \right\rangle^{1/2|\mathcal{O}|} \right\},$$

and let $\tilde{\mathcal{O}}$ be some subset of directed, horizontal bonds on which the maximum $z$ is taken. Using translation invariance of $\langle - \rangle$ (corresponding to hypothesis (C) of Theorem 2.1) and reflection positivity of $\langle - \rangle$ (corresponding to ($\theta$)) and applying the Schwarz inequality (corresponding to (2.8)) repeatedly, as in inequality (2.11), in the horizontal and vertical direction, we obtain

$$z \leqq \langle P_\Lambda \rangle^{1/k|\Lambda|} z^{1-1/k},$$

for some integer $k > 0$. Hence $z \leqq \langle P_\Lambda \rangle^{1/|\Lambda|}$ from which we obtain (1.42). Finally we remark that Theorem 2.2 can be used to give alternate proofs of the general chessboard estimates of the last reference in [19] (Theorem 2.3, periodic boundary conditions) and of [10] (Lemma 4.5). Furthermore Theorem 2.2 implies the *Hölder inequality* for general traces and the *Peierls-Bogolubov* and *Golden-Thompson* inequalities.

## III. Exponential Localization

In this section we explain the difficult part in the required estimate of $R_\Lambda(\beta) = \langle P_\Lambda \rangle$, defined in (1.45), for the quantum mechanical models. We recall that in Section I.E. we have split $R_\Lambda(\beta)$ into two pieces

$$R_\Lambda(\beta) = R_-(\beta, \Delta) + R_+(\beta, \Delta), \tag{3.1}$$

where

$$R_-(\beta, \Delta) = Z_\Lambda(\beta)^{-1} \sum_{i\Delta}' C_i \exp[-\beta e_i]; \tag{3.2}$$

here $\sum_{i\Delta}'$ means a sum over all $i$ such that $e_i \leqq e_0 + \Delta |\Lambda|$, and

$$C_i \equiv (\phi_i, P_\Lambda \phi_i). \tag{3.3}$$

The easy estimate of $R_+(\beta, \Delta)$ is postponed to Section IV. In this section we prove upper bounds on $R_-(\beta, \Delta)$ for models (1)–(6). We claim that, for the classical models (1), (2) and (6) (classical case),

$$R_-(\beta, \Delta) = 0 \tag{3.4}$$

for sufficiently small $\Delta$. To show this we first estimate the minimum $\mathcal{E}(P_\Lambda^{(AC)})$ of the Hamilton function $H_\Lambda$ restricted to the configurations

$\{\mathbf{S} : \mathbf{S} \in P_\Lambda\}$ (models (1), (6)),

resp. $\{x : x \in P_\Lambda^{AC}\}$ (model (2)).

For models (1) and (6)

$$\mathcal{E}(P_\Lambda) \geqq -\tfrac{3}{2}|\Lambda| - \frac{\alpha}{2}|\Lambda|. \tag{3.5}$$

For model (2)

$$\mathscr{E}(P_\Lambda^{AC}) \geqq 2\varepsilon_0|\Lambda| + \frac{\alpha}{2}|\Lambda|. \tag{3.6}$$

Therefore

$$\text{if } \Delta < (1-\alpha)/2 \text{ (models (1), (6))} \tag{3.7}$$

$$\text{resp. } \Delta < \alpha/2 \quad \text{(model (2))} \tag{3.8}$$

then $R_-(\beta, \Delta) = 0$

which proves our contention.

As already noted in Section I.E, (3.4) is *false* for the quantum mechanical models, and we have to work much harder in order to obtain a good upper bound on $R_-(\beta, \Delta)$. The idea is to show that $C_i = (\phi_i, P_\Lambda \phi_i)$ is very small for eigenvalues $e_i$ of $H_\Lambda$ close to the ground state energy $e_0$. Although $P_\Lambda$ is a projection onto states of relatively high $H^z$-energy, $(\phi_i, P_\Lambda \phi_i)$ does *not* vanish, even for $e_i$ very close to $e_0$, as it does in the classical case.

### *III.A. Principle of Exponential Localization*

The following general result will be crucial for our analysis.

**Theorem 3.1.** (*Exponential Localization of Eigenvectors*). *Let $A$ and $B$ be selfadjoint operators (typically finite, hermitean matrices) on a Hilbert space $\mathscr{H}$ such that*

(*i*) $A \geqq 0$

(*ii*) $\pm B \leqq \varepsilon A$,

*with $0 \leqq \varepsilon < 1$. Suppose that*

$$(A+B)\psi = \lambda\psi, \ \|\psi\| = 1.$$

*Choose some $\rho > \lambda \geqq 0$ such that*

$$\sigma \equiv \varepsilon\rho(\rho - \lambda)^{-1} < 1. \tag{3.9}$$

*Let $P_\rho$ be the spectral projection of $A$ corresponding to $[\rho, \infty)$, and $M_\rho \equiv P_\rho \mathscr{H}$, (all "eigenvectors" of $A$ corresponding to eigenvalues $\geqq \rho$). Note that $(A - \lambda)$ restricted to $M_\rho > 0$. Finally, let $\phi \in M_\rho$ be a unit vector with the property*

(*iii*) $\{B(A-\lambda)^{-1}\}^j \phi \in M_\rho$,

*for $j = 0, 1, \ldots, d-1$, with $d \geqq 1$.*

*Then* $$|(\phi, \psi)| \leqq \sigma^d. \tag{3.10}$$

*Remarks.* 1. Since $B \geqq -\varepsilon A$, by (ii),

$$A + B \geqq (1-\varepsilon)A \geqq 0, \tag{3.11}$$

so that all eigenvalues $\lambda$ of $A+B$ are nonnegative.

2. Clearly the condition $|B| \leqq \varepsilon A$ implies (ii), but the converse is *false*, as the example

$$A = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |B| = 1, \varepsilon = 1$$

shows. Hypothesis (ii) is all we need to prove (3.10).

*Proof.* By hypothesis,

$$(A+B)\psi = \lambda\psi, \text{ i.e. } (A-\lambda)\psi = B\psi .$$

Thus, for some $\delta \geqq 0$,

$$\psi = (A - \lambda + i\delta)^{-1}(B\psi + i\delta\psi),$$

so that

$$\begin{aligned} |(\phi,\psi)| &= |(\phi,(A-\lambda+i\delta)^{-1}(B\psi+i\delta\psi))| \\ &= |((A-\lambda-i\delta)^{-1}\phi, B\psi + i\delta\psi)|. \end{aligned}$$

Since $\phi \in M_\rho$ and $\lambda < \rho$, by hypothesis,

$$\lim_{\delta \downarrow 0}(A-\lambda-i\delta)^{-1}\phi = (A-\lambda)^{-1}\phi,$$

hence

$$|(\phi,\psi)| = |(B(A-\lambda)^{-1}\phi,\psi)|. \tag{3.12}$$

By hypothesis (iii), $\{B(A-\lambda)^{-1}\}^j\phi \in M_\rho$, for $j = 0, 1, \dots, d-1$. Therefore, for $d > 1$,

$$|(\phi,\psi)| = |(P_\rho B(A-\lambda)^{-1}\phi,\psi)|, \tag{3.12'}$$

and we can iterate (3.12′) $d-1$ times and then apply (3.12). This yields

$$\begin{aligned} |(\phi,\psi)| &= |(B(A-\lambda)^{-1}\{P_\rho B(A-\lambda)^{-1}P_\rho\}^{d-1}\phi,\psi)| \\ &= |(A^{-1/2}B(A-\lambda)^{-1}\{P_\rho B(A-\lambda)^{-1}P_\rho\}^{d-1}\phi, A^{1/2}\psi)| \\ &\leqq \|A^{-1/2}B(A-\lambda)^{-1/2}P_\rho\| \, \|P_\rho(A-\lambda)^{-1/2}B(A-\lambda)^{-1/2}P_\rho\|^{d-1} \\ &\quad \cdot \|(A-\lambda)^{-1/2}\phi\| \, \|A^{1/2}\psi\|, \end{aligned} \tag{3.13}$$

where we have used that $[A, P_\rho] = 0$. Now

$$\begin{aligned} \|A^{-1/2}B(A-\lambda)^{-1/2}P_\rho\| &\leqq \|A^{-1/2}BA^{-1/2}\| \, \|A^{1/2}(A-\lambda)^{-1/2}P_\rho\| \\ &\leqq \varepsilon\rho^{1/2}(\rho-\lambda)^{-1/2} = (\varepsilon\sigma)^{1/2}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \|P_\rho(A-\lambda)^{-1/2}B(A-\lambda)^{-1/2}P_\rho\| &\leqq \|P_\rho(A-\lambda)^{-1/2}A^{1/2}\|^2 \, \|A^{-1/2}BA^{-1/2}\| \\ &\leqq \rho(\rho-\lambda)^{-1}\varepsilon = \sigma, \end{aligned} \tag{3.15}$$

and we have used the definition of $P_\rho$ and hypothesis (ii). Finally

$$\|(A-\lambda)^{-1/2}\phi\| = \|(A-\lambda)^{-1/2}P_\rho\phi\| \leqq (\rho-\lambda)^{-1/2}\|\phi\| \tag{3.16}$$

and

$$\begin{aligned}\|A^{1/2}\psi\| &= [(\psi, A\psi)]^{1/2}\\ &\leqq (1-\varepsilon)^{-1/2}[(\psi,(A+B)\psi)]^{1/2}, \text{ by (3.11)}\\ &= [(1-\varepsilon)^{-1}\lambda]^{1/2} < \rho^{1/2},\end{aligned} \tag{3.17}$$

since $\sigma \equiv \varepsilon\rho(\rho-\lambda)^{-1} < 1$, i.e. $\lambda < \rho(1-\varepsilon)$. If we combine (3.13)–(3.17) we find $|(\phi,\psi)| \leqq \sigma^{d-1/2}\varepsilon^{1/2}(\rho-\lambda)^{-1/2}\rho^{1/2} = \sigma^d$. Q.E.D.

**Corollary 3.2.** *Suppose $N \subset M_\rho$ is a subspace of $M_\rho$ such that each $\phi \in N$ satisfies hypothesis (iii) of Theorem 3.1. If $P$ is the projection onto $N$ then (in the notations of Theorem 3.1)*

$$\langle \psi, P\psi \rangle \leqq \sigma^{2d}.$$

The proof is essentially identical to the one of Theorem 3.1. We now apply Corollary 3.2 to estimating the overlap of the universal projection $P_\Lambda$ with the low lying eigenstates of $H_\Lambda$, i.e. the numbers $C_i = (\phi_i, P_\Lambda \phi_i)$, when the eigenvalues $e_i \leqq e_0 + \Delta|\Lambda|$, for models (3) and (4), (quantum mechanical antiferromagnet, resp. ferromagnet. The case of the $xy$ model is similar to the antiferromagnet). For this purpose we identify

$$P = P_\Lambda \tag{3.18}$$

$$A = S^{-2}H^z - e_0(\alpha = 1), \tag{3.19}$$

where $e_0(\alpha = 1)$ is the groundstate energy of the *isotropic* Hamiltonian,

$$B = \alpha S^{-2}H^{xy}. \tag{3.20}$$

In all the models discussed here, the groundstate energy $e_0(\alpha = 1)$ of the Hamiltonian $H = S^{-2}(H^z + H^{xy})$ is bounded above by the groundstate energy $e_0^z$ of $S^{-2}H^z$. Therefore

$$A \geqq 0. \tag{3.21}$$

Furthermore

$$A + \frac{1}{\alpha}B = S^{-2}(H^z + H^{xy}) - e_0(1) \geqq 0. \tag{3.22}$$

If we rotate all spins on one of the sublattices by an angle $\pi$ around the $z$-axis we see that $A + \frac{1}{\alpha}B$ is unitarily equivalent to $A - \frac{1}{\alpha}B$, as $H^{xy}$ is taken into $-\mathrm{H}^{xy}$ under this unitary transformation, but $H^z$ is unchanged. Therefore

$$\pm B \leqq \alpha A, \text{ i.e. } \varepsilon = \alpha. \tag{3.23}$$

Finally we set

$$\rho = e_0^z - e_0(\alpha = 1) + n\Delta|\Lambda|, \tag{3.24}$$

where $\Delta$ and $n$ will be chosen to be dependent on the model. Thus the hypotheses of Theorem 3.1 and Corollary 3.2 are satisfied.

*III.B. Estimates for the Antiferromagnet*

Next we consider the quantum mechanical antiferromagnet, model (3), in detail. We shall estimate the overlap coefficients

$$C_i = (\phi_i, P_\Lambda \phi_i),$$

for all eigenvectors $\phi_i$ of the Hamiltonian $H_\Lambda$ corresponding to eigenvalues $e_i$ with

$$e_i \leqq e_0^z + \Delta|\Lambda|. \tag{3.25}$$

From (3.19) and (3.20) we infer that

$$A + B = H_\Lambda - e_0(\alpha = 1).$$

Therefore the eigenvalue $\lambda$ of $A + B$ introduced in Theorem 3.1, Corollary 3.2 satisfies

$$\lambda \leqq \delta|\Lambda| + \Delta|\Lambda|, \tag{3.26}$$

where

$$\delta \equiv \frac{1}{|\Lambda|}(e_0^z - e_0(\alpha = 1)).$$

It is shown in [17] that

$$e_0(\alpha = 1) \geqq -\left(1 + \frac{1}{4S}\right)2|\Lambda| \tag{3.27}$$

so that

$$\delta \leqq (2S)^{-1}. \tag{3.28}$$

Combining (3.23) and (3.24) with (3.26) and (3.28) we arrive at the following estimate for $\sigma$:

$$\begin{aligned}\sigma \equiv \varepsilon \frac{\rho}{\rho - \lambda} &\leqq \varepsilon \frac{(\delta + n\Delta)|\Lambda|}{(n-1)\Delta|\Lambda|} \\ &\leqq \alpha\left(1 + \frac{1 + 2S\Delta}{2S\Delta(n-1)}\right).\end{aligned} \tag{3.29}$$

Let

$$\mathscr{E}^z(P_\Lambda) \equiv \inf \operatorname{spec}(P_\Lambda H^z P_\Lambda) - e_0(\alpha = 1)$$

be the minimal $A$-energy of any state in $N \equiv P_\Lambda \mathscr{H}$. Recalling definition (1.35) of the universal projection $P_\Lambda$ we see that

$$\mathscr{E}^z(P_\Lambda) \geqq e_0^z - e_0(\alpha = 1) + \tfrac{1}{2}|\Lambda|,$$

so that

$$\mathscr{E}^z(P_\Lambda) - \rho \geqq (\tfrac{1}{2} - n\Delta)|\Lambda|. \tag{3.30}$$

Since $P_\Lambda$ plays the role of the projection $P$ introduced in Corollary 3.2, we have the constraint

$$\tfrac{1}{2} - n\Delta > 0. \tag{3.31}$$

**Lemma 3.3.** (*Estimate on d for Antiferromagnet*). *Let $A \equiv S^{-2}H^z - e_0(\alpha = 1)$*
*(1) Let $\psi$ be a vector of A-energy at least e, i.e. $(1 - P_{p=e})\psi = 0$. Then the A-energy of $H^{xy}\psi$ is at least $e - 8S^{-1}$, i.e. $(1 - P_{e-8S^{-1}})H^{xy}\psi = 0$.*
*(2) $d \geqq [\frac{1}{16}(1 - 2n\Delta)S|\Lambda|]$, for $2n\Delta < 1$, where $[a]$ is the largest integer $\leqq a$.*

*Proof.* In our representation (1.40) of the antiferromagnet

$$H^{xy} = -S^{-2} \sum_{\langle i,j\rangle \subset \Lambda} \{S_i^x S_j^x + (iS_i^y)(iS_j^y)\}$$

$$= -\frac{1}{2S^2} \sum_{\langle i,j\rangle \subset \Lambda} \{S_i^+ S_j^+ + S_i^- S_j^-\}, \tag{3.32}$$

where $S^+, S^-$ are the spin-raising, resp. spin-lowering operators. Using (3.32) we see that one application of $H^{xy}$ to a vector $\psi$ can raise (resp. lower) the $z$-components of the spins of *one* nearest neighbor pair $\langle i,j\rangle \subset \Lambda$ by 1. Clearly, this cannot change the minimal $A$-energy of $\psi$ by more than $S^{-2}\cdot 8S = 8S^{-1}$, as a minute of reflection shows. More precisely,

$$(1 - P_{e-8/S})H^{xy}P_e = 0. \tag{3.33}$$

This completes the proof of (1). The proof of (2) is an immediate consequence of the definition of $d$, of inequality (3.30) and of part (1). Q.E.D.

**Proposition 3.4.** *$R_- \leqq \sigma^{2d}$, where*

$$\sigma = \alpha(1 + (S\eta)^{-1} + 0(\beta^{-\xi})) \text{ and}$$

$$d \geqq \left[\frac{(1-\eta)S}{16}|\Lambda|\right]$$

*for arbitrary $\xi < 1$ and $\eta < 1$.*

*Proof.* We choose

$$\Delta = \beta^{-\xi} \quad \text{and} \quad n = \tfrac{1}{2}\eta\beta^{\xi}. \tag{3.34}$$

Then $\frac{1}{2} - n\Delta = \frac{1}{2}(1-\eta) > 0$, for $\eta < 1$, so that the constraint (3.31) is fulfilled. By Equation (3.2), (3.3),

$$R_- = R_-(\beta, \Delta) = \frac{\sum_{e_i < e_0 + \Delta|\Lambda|} (\phi_i, P_\Lambda \phi_i) e^{-\beta e_i}}{\sum e^{-\beta e_i}}$$

$$\leqq \max_{e_i < e_0 + \Delta|\Lambda|} (\phi_i, P_\Lambda \phi_i). \tag{3.35}$$

Suppose the maximum on the right side of (3.35) occurs for $i = i_0, e_{i_0} < e_0 + \Delta|\Lambda|$. We set $\phi_{i_0} \equiv \psi, P_\Lambda \equiv P$ and apply Corollary 3.2. This gives

$$|(\phi_{i_0}, P_\Lambda \phi_{i_0})| \leqq \sigma^{2d}, \text{ i.e. } R_- \leqq \sigma^{2d}.$$

By (3.34) and (3.29)

$$\begin{aligned}\sigma &\leqq \alpha\left(1+\frac{1+2S\Delta}{2S\Delta(n-1)}\right)\\ &= \alpha\left[1+\left(1-\frac{2}{\eta\beta^{\xi}}\right)^{-1}\left(\frac{1+2S\beta^{-\xi}}{\eta S}\right)\right]\\ &= \alpha(1+(\eta S)^{-1}+0(\beta^{-\xi})).\end{aligned}$$

Furthermore, by Lemma 3.3, (2) and (3.34)

$$\begin{aligned}d &\geqq [\tfrac{1}{16}(1-2n\Delta)S|\Lambda|]\\ &= \left[\frac{1-\eta}{16}S|\Lambda|\right].\end{aligned}$$

Q.E.D.

*Remark.* The dependence of $\sigma$ and $d$ on the total spin $S$ will permit us to show that the critical anisotropy $\alpha_c(S)$, below which a phase transition occurs, tends to 1 as $S \to \infty$.

Our estimate for $d$ is not very good and can be improved; we illustrate how to do so for spin 1/2. We claim $d \geqq |\Lambda|(1-n\Delta)/4$ instead of $|\Lambda|(1-2n\Delta)S/8 = |\Lambda|(1-2n\Delta)/16$; $P_\Lambda$ (Fig. 1) now means a projection onto a definite pattern of up or down spins; $\mathscr{E}^z(P_\Lambda) = -|\Lambda| + \delta$, and we wish to lower it to an $A$-energy of $-2|\Lambda| + n\Delta|\Lambda| + \delta$. Let $e_h$ be the $H^z$-energy of the horizontal bonds. Initially, $e_h = 0$; finally $e_h \leqq -|\Lambda| + n\Delta|\Lambda|$, since the vertical energy $\geqq -|\Lambda|$. Also, $e_h = -|\Lambda| + 2b$ where $b$ is the number of bad (i.e. $+-$ or $-+$) horizontal bonds. At least $k = |\Lambda|(1-n\Delta)/2$ bad horizontal bonds must be removed; $d \geqq d' =$ numbers of steps to do this, while $d' \geqq d''/2$, where $d''$ is the number of *single* spin flips required to do the same thing. Since the initial horizontal pattern in in each row is *bgbgb*... ($g$ = good bond), it is easy to see that $d'' \doteq k$. These arguments give the following improved estimates for $S = 1/2$:

$$\sigma = \alpha\left(1+\frac{2}{\eta}+0(\beta^{-\xi})\right), \tag{3.36}$$

$$d \geqq [\tfrac{1}{4}(1-\eta/2)|\Lambda|]. \tag{3.37}$$

### *III.C. Estimates for the Ferromagnet*

It is well known that in the quantum mechanical ferromagnet (model (4))

$$e_0(\alpha) = e_0^z, \text{ for all } |\alpha| \leqq 1; \tag{3.38}$$

in fact, the groundstates for $|\alpha| < 1$ are identical with the two groundstates of $H^z$. Therefore

$$\rho = n\Delta|\Lambda|, \text{ by (3.24), and}$$

$$A = S^{-2}H^z - e_0^z, B = \alpha S^{-2}H^{xy}.$$

We estimate the overlap coefficients $(\phi_i, P_\Lambda\phi_i)$ for all eigenvectors of the Hamiltonian $H_\Lambda$ corresponding to eigenvalues $e_i$ with $e_i \leqq e_0^z + \Delta|\Lambda|$. Thus the eigenvalue $\lambda$ of $A+B$ introduced in Theorem 3.1 and Corollary 3.2 must satisfy $\lambda \leqq \Delta|\Lambda|$. Therefore

$$\sigma \equiv \varepsilon \frac{\rho}{\rho-\lambda} = \alpha \frac{n}{n-1}. \tag{3.39}$$

As in the antiferromagnet one shows that

$$R_- \leqq \sigma^{2d}; \tag{3.40}$$

see the proof of Proposition 3.4. We are left with estimating the "distance" $d$ on the right side of (3.40).

*Estimate on d.* Let $l$ be an integer such that $|\Lambda|^{1/2}/l$ is an integer. We decompose $\Lambda$ into $|\Lambda|/l^2$ disjoint, congruent squares, $b(=\text{boxes})$, with sides of length $l$. Let $\phi$ be an eigenvector of $\{S_i^z : i \in \Lambda\}$. Clearly $\phi$ is also an eigenvector of $A$. For $\phi$, a *perfect square* is defined to be a square $b = b_\phi$ such that $S_i^z\phi = \sigma_i\phi$ and one of the following two properties holds:

(i) $\sigma_i \geqq (0.9)S$ for all $i \in b_\phi$

(ii) $\sigma_i \leqq -(0.9)S$ for all $i \in b_\phi$.

Suppose now that the $A$-energy of $\phi$ is $\leqq n\Delta|\Lambda|$. We propose to estimate the minimal number, $k$, of perfect squares $b_\phi$ for this $\phi$. For this purpose we assign an $A$-energy to every $b$ square in $\Lambda$ in such a way that the sum of the energies assigned to all squares in $\Lambda$ is $\leqq$ the $A$-energy of $\phi$. The $A$-energy of a perfect square is zero. Therefore, to a square which is *not* perfect, an $A$-energy of at least 2(0.1) must be assigned. There are $(|\Lambda|l^{-2}) - k$ squares which are not perfect. Since the $A$ energy of $\phi$ is $\leqq n\Delta|\Lambda|$, we obtain the inequality

$$(|\Lambda|l^{-2} - k)\,(0.2) \leqq n\Delta|\Lambda|,$$

i.e.

$$k \geqq |\Lambda|(l^{-2} - 5n\Delta). \tag{3.41}$$

Since $l \geqq 2$, we require that $n\Delta < 1/40$. Let $\psi$ be an arbitrary vector in the range of $P_\Lambda$, i.e. $P_\Lambda\psi = \psi$. Define $d$ (see Theorem 3.1) by the condition

$$(1 - P_{n\Delta|\Lambda|})[B(A-\lambda)^{-1}]^d\psi \neq 0,$$

but (3.42)

$$(1 - P_{n\Delta|\Lambda|})[B(A-\lambda)^{-1}]^j\psi = 0,$$

for all $j < d$.

We expand $[B(A-\lambda)^{-1}]^d\psi$ in terms of eigenvectors $\phi_{j,\psi}^z$ of $\{S_i^z : i \in \Lambda\}$. Let $\phi = \phi_{j_0,\psi}^z$ be a vector of $A$-energy $\leqq n\Delta$. By (3.42) such a $\phi \neq 0$ exists. By (3.41) $\phi$ has $k \geqq |\Lambda|(l^{-2} - 5n\Delta)$ perfect squares. In order to obtain a perfect square by repeated application of $B(A-\lambda)^{-1}$ to $\psi$, $B(A-\lambda)^{-1}$ has to be applied to $\psi$

at least $m$ times, where

$$m \geqq (0.9)\frac{l^2}{2}\cdot S\cdot\frac{l}{4}. \tag{3.43}$$

This is so, because $\psi$ is an eigenvector of $P_\Lambda$, so that the $z$-components of $\frac{l^2}{2}$ spins in a box $b$ have to be raised from $S_z \leqq 0$ to $S^z = (0.9)S$, resp. lowered from $S^z \geqq 0$ to $S^z = -(0.9)S$, in order to convert $b$ into a perfect box. (Recall that $P_\Lambda$ is pictorially given by Figure 1, Section I.D).

For the quantum mechanical ferromagnet

$$\begin{aligned} B &= -\sum_{\langle i,j\rangle \subset \Lambda} \{S_i^x S_j^x + S_i^y S_j^y\} \\ &= -\tfrac{1}{2}\sum_{\langle i,j\rangle \subset \Lambda} \{S_i^+ S_j^- + S_i^- S_j^+\}. \end{aligned} \tag{3.44}$$

Equation (3.44) shows that when the $z$-component of a spin at some site is raised (lowered) the $z$-component of a spin at a nearest neighbor site is lowered (raised). Thus, in order to raise the $z$-component of a spin at some site $i \in b$ from $S^z \leqq 0$ to $S^z = (0.9)S$ without lowering the $z$-components of other spins in $b$, $B$ has to be applied

$$(0.9)\,\mathrm{dist}(i, \text{boundary of } b)\cdot S$$

times; hence, on the average, $(0.9)S\cdot\frac{l}{4}$ times. This completes the proof of (3.43). If we combine (3.41) with (3.43) we obtain

$$d \geqq m\cdot k \geqq |\Lambda|(l^{-2} - 5n\Delta)\frac{l^3 S}{8}(0.9).$$

Choosing $l = [(10n\Delta)^{-1/2}]\,(\geqq 2)$ yields

**Proposition 3.5.** *Provided $n\Delta < 1/40$*

$$R_- \leqq \sigma^{2d},$$

*where*

$$\sigma = \alpha\frac{n}{n-1}, \textit{ and}$$

$$d \geqq (0.9)|\Lambda| S[16(10n\Delta)^{1/2}]^{-1}.$$

*Remark.* The estimate on $d$ obtained in Proposition 3.5 for the ferromagnet is vastly superior to the estimates on $d$ obtained for the antiferromagnet (Proposition 3.4 and (3.37)). This will become apparent in the next section where we will allow $n\Delta$ to go to zero as $\beta \to \infty$. Then $d \to \infty$ for the ferromagnet, but *not* for the antiferromagnet. Finally we note that the general methods developed in this section can be applied in other contexts than the one considered here in order to get bounds on expectations of global observables in equilibrium states.

## IV. Estimates on $R_+$ and Completion of the Proof

### *IV.A. Summary of Previous Results*

Recall that our proof of LRO at low temperatures is completed by showing that

$$R_\Lambda(\beta) = \langle P_\Lambda \rangle$$

is "small" for large $\beta$, namely we require that

$$\sum_{n=2}^{\infty} 2n3^{2n-2} R_\Lambda^{n/|\Lambda|} < \tfrac{1}{4}; \tag{4.1}$$

see Section I.C, Theorem 1.1 and Section I.D, inequalities (1.34), (1.36) and (1.38). In Section III we decomposed $R_\Lambda(\beta)$ into two parts,

$$R_\Lambda(\beta) = R_-(\beta, \Delta) + R_+(\beta, \Delta), \tag{4.2}$$

and we have established upper bounds on $R_-(\beta, \Delta)$, namely:

(a) In models (1) and (6) (classical case), i.e. the classical $N$-vector models:

$$R_-(\beta, \Delta) = 0, \text{ for } \Delta \leqq \tfrac{1}{2}(1-\alpha); \tag{4.3}$$

see (3.7).

(b) In model (2), the classical, anharmonic crystal:

$$R_-(\beta, \Delta) = 0, \text{ for } \Delta \leqq \alpha/2; \tag{4.4}$$

see (3.8).

(c) In model (3), the quantum antiferromagnet:

$$R_-(\beta, \Delta) \leqq \sigma^{2d}, \text{ for } \Delta = \beta^{-\xi}, \tag{4.5}$$

where

$$\sigma = \alpha(1 + (S\eta)^{-1} + 0(\beta^{-\xi})),$$

and

$$d \geqq \left[ \frac{(1-\eta)S}{16} |\Lambda| \right],$$

resp.

$$d \geqq [\tfrac{1}{4}(1-\eta/2)|\Lambda|], \text{ for } S = 1/2,$$

with

$$0 < \xi < 1, \; 0 < \eta < 1$$

(to be chosen later). See Propositions 3.4 and (3.37). The estimates for model (6) (quantum $xy$ model) are identical.

(d) In model (4) (quantum ferromagnet)

$$R_-(\beta, \Delta) \leqq \sigma^{2d}, \tag{4.6}$$

where

$$\sigma = \alpha \frac{n}{n-1}, \tag{4.7}$$

and

$$d \geqq (0.9)|\Lambda| S[16(10n\Delta)^{1/2}]^{-1}, \tag{4.8}$$

with $n > 1$ and $\Delta > 0$ to be chosen later. We require $n\Delta < 1/40$.

### *IV.B. The $R_+$ Estimate*

We now estimate $R_+(\beta, \Delta)$ for these models.

($a'$) *Models* (*1*) *and* (*6 Classical*). Let $P_\Lambda^{>\delta}$ be the subset of configurations such that $m_i \equiv S^{-1} S_i^z \geqq (1-\delta)^{1/2}$, for all $i \in \Lambda$. Then (with Tr defined by the usual normalized integral, i.e. $\mathrm{Tr}(1) = 1$)

$$\begin{aligned} Z_\Lambda(\beta) &\geqq \mathrm{Tr}\{P_\Lambda^{>\delta} \exp[-\beta H_\Lambda]\} \\ &\geqq \{\mathrm{Tr}\, P_\Lambda^{>\delta}\} \exp\{-\beta\, \mathrm{Tr}[P_\Lambda^{>\delta} H_\Lambda]/\mathrm{Tr}\, P_\Lambda^{>\delta}\} \end{aligned}$$

by Jensen's inequality. By symmetry, the term proportional to $\alpha$ vanishes in $\mathrm{Tr}\, P_\Lambda^{>\delta} H_\Lambda$. Furthermore, $H_\Lambda \leqq -2|\Lambda|(1-\delta)$ whenever $P_\Lambda^{>\delta} \neq 0$. Moreover,

$$\mathrm{Tr}(P_\Lambda^{>\delta}) = \{\tfrac{1}{2}[1-(1-\delta)^{1/2}]\}^{|\Lambda|}. \tag{4.9}$$

Hence, choosing $\delta = \Delta/4 = (1-\alpha)/8$, we obtain

$$R_+(\beta, \Delta) \leqq \exp[\beta|\Lambda|(2-\Delta)] Z_\Lambda^{-1} \leqq e^{-(\beta/4)(1-\alpha)|\Lambda|} e^{c(\alpha)|\Lambda|}, \tag{4.10}$$

where $c(\alpha) \propto \alpha - \ln(1-\alpha)$ for $\alpha \approx 1$, is *independent* of $\beta$. Thus

$$\langle P_\Lambda \rangle^{1/|\Lambda|} = R_\Lambda(\beta)^{1/|\Lambda|} = R_+(\beta, \Delta)^{1/|\Lambda|} \leqq e^{-(\beta/4)(1-\alpha)+c(\alpha)}, \tag{4.11}$$

which tends to 0, as $\beta \to \infty$. *This completes the proof of* LRO *for models* (1) and (6, classical) *for large enough* $\beta$. An estimate on the spontaneous magnetization $\langle m_0 \rangle = S^{-1}\langle S_0^z \rangle$, resp. $\sigma(\beta)$ (see Section I.B) as a function of $\beta$ is given later.

($b'$) *Model* (*2*). By definition of model (2) (anharmonic crystal, Section I.A),

$$\min \phi(x, y) = \varepsilon_0,$$

occurs when $x$ and $y$ have the same direction. Without loss of generality we may assume that there exist some $x_0 \neq 0$ such that

$$\phi(x_0, x_0) = \varepsilon_0. \tag{4.12}$$

But when $x^1$ and $y^1$ (the 1-components of $x$, resp. $y$) have opposite sign

$$\phi(x, y) \geqq \varepsilon_0 + \alpha + \lambda(\phi_1(x) + \phi_1(y)), \tag{4.13}$$

for some $\lambda > 0$; see Section I.A. We now choose $\delta > 0$ such that $x_0^1 - \delta \geqq 0$ and

$$\phi(x, y) \leqq \varepsilon_0 + \alpha/8,$$

for all $x$ and $y$ in a ball of radius $\delta$ centered at $x_0$. We can do so, since the interaction potential $\phi$ is by assumption *continuous*
Hence

$$Z_\Lambda(\beta) \geqq e^{-2\beta(\varepsilon_0+\alpha/8)|\Lambda|}(v_N(\delta))^{|\Lambda|}, \tag{4.14}$$

where $v_N(\delta)$ is the volume of a ball of radius $\delta$ in $\mathbb{R}^N$. Furthermore, for $\Delta = \alpha/2$ i.e. $R_-^{AC}(\beta, \Delta) = 0$ (see (4.4)),

$$\begin{aligned} R_+^{AC}(\beta) &= R_+^{AC}(\beta, \Delta = \alpha/2) \\ &\leqq \exp[-2\beta(\varepsilon_0 + \alpha/4)|\Lambda|]g(\beta)^{|\Lambda|}Z_\Lambda(\beta)^{-1}, \end{aligned} \tag{4.15}$$

where

$$g(\beta) \equiv \int e^{-4\beta\lambda\phi_1(x)}d^N x.$$

This is an immediate consequence of (4.13); (see also inequality (1.49) of Section I.E). Combination of (4.14) and (4.15) yields

$$R_+^{AC}(\beta, \alpha/2) \leqq \exp[-(\beta\alpha/4)|\Lambda|](g(\beta)/v_N(\delta))^{|\Lambda|}.$$

By definition of the AC model (model (2), Section I.A.) there exists some finite $\beta_0$ such that for all $\beta \geqq \beta_0$

$$g(\beta) = \int e^{-4\beta\lambda\phi_1(x)}d^N x < \infty.$$

Obviously $g(\beta)$ is monotone decreasing in $\beta$, as $\phi_1$ is positive. Thus there exists a finite constant $c$ such that

$$g(\beta)/v_N(\delta) \leqq e^c, \text{ for all } \beta \geqq \beta_0.$$

Hence

$$R^{AC}(\beta) = R_+^{AC}(\beta, \alpha/2) \leqq \exp[-\beta\alpha/4 + c]|\Lambda|, \tag{4.16}$$

which tends to 0 as $\beta \to \infty$. Recalling condition (4.1) (resp. Theorem 1.1 of Section I.C and Section I.D, inequality (1.38)) we observe that *inequality* (4.16) *completes the proof of* LRO *for the* AC *model for large enough* $\beta$.

(c′) *Models* (*3*) *and* (*4*) (*Quantum Heisenberg Models*). In order to estimate $R_+(\beta, \Delta)$ we need a lower bound on the partition function $Z_\Lambda(\beta)$. This is done by comparing it with the partition function of the corresponding spin $S$ Ising model (anisotropy $\alpha = 0$) by means of the *Peierls-Bogolyubov inequality*.

**Lemma 4.1.** *For models* (*3*) *and* (*4*) *the partition function satisfies*

$$Z_\Lambda \geqq Z_\Lambda^I. \tag{4.17}$$

*Where* $Z_\Lambda^I$ *is the partition function of the spin* $S$ *Ising model* (*i.e.* $\alpha = 0$ *in* (*1.3*) *and* (*1.5*)).

*Proof.* By the Peierls-Bogolyubov inequality,

$$Z_\Lambda \geqq \sum_j \exp[-\beta(\psi_j, H\psi_j)] \tag{4.18}$$

for any set $\{\psi_j\}$ of orthonormal vectors. Choose the $\psi_j$ to be eigenvectors of all the $S_i^z, i\in\Lambda$. Then the right side of (4.18) is precisely $Z_\Lambda^I$ because $(\psi_j, H^{xy}\psi_j)=0$ for all $j$.

**Lemma 4.2.** *For models (3) and (4)*

$$Z_\Lambda \geqq [(\delta/8)(2S+1)]^{|\Lambda|}\exp\{2\beta|\Lambda|(1-\delta)\}$$

*for any* $0\leqq\delta\leqq 1$.

*Proof.* Using Lemma 4.1,

$$Z_\Lambda \geqq Z_\Lambda^I \geqq {\textstyle\sum}'\exp(-\beta S^{-2}H^z)$$

where $\sum'$ means a restricted summation in which each $S_i^z \geqq S(1-\delta)^{1/2}$. (Note: the partition function for the Ising ferro and antiferromagnet are identical.) Then $H^z \leqq -2|\Lambda|S^2(1-\delta)$. To complete the proof we have to bound $\sum' 1 \equiv \mu^{|\Lambda|}$.

$$\mu = [S - S(1-\delta)^{1/2}+1]_+ \geqq [1+S\delta/2]_+ \geqq S\delta/2 \geqq (2S+1)\delta/8$$

for $S\geqq 1/2$, and where $[\ ]_+$ means integral part. To complete the bound on $R_+$ we use the fact that $\mathrm{Tr}\,1=(2S+1)^{|\Lambda|}$. For the ferromagnet, $e_0=-2|\Lambda|$. Thus, provided

$$\Delta > 4\delta \text{ (ferromagnet)}, \tag{4.19}$$

(1.50) and (1.51) are established for $\beta$ sufficiently large. For the antiferromagnet, $e_0 > -2|\Lambda|(1+1/4S)$. Thus, provided

$$\Delta > 4\delta + S^{-1} \text{ (antiferromagnet)}, \tag{4.20}$$

(1.50) and (1.51) are established for $\beta$ sufficiently large.

The final estimate for the ferromagnet is obtained from (4.6) and (1.51). Choose $n=(1+\alpha)/(1-\alpha)<2/(1-\alpha)$. Thus $\sigma<(1+\alpha)/2<1$. Choose $\Delta = K\beta^{-2/3}$ where $K$ is chosen such that $\sigma^{(1,8)S(10nK)^{-1/2}/16} < e^{-K/2}$. This can be done uniformly in $S>1/2$. For sufficiently large $\beta, n\Delta<1/40$. Furthermore, with this choice of $\Delta, R_+ > R_-$. Hence

$$\lim_{\Lambda\to\infty}\langle P_\Lambda\rangle^{1/|\Lambda|} \leqq \exp(-K\beta^{1/3}/2) \tag{4.21}$$

which tends to zero as $\beta$ tends to infinity. Note that there is a $\beta^{1/3}$, instead of a $\beta$ dependence in (4.21). This completes the proof of LRO for the quantum ferromagnet, except for the assumption that the chessboard estimate holds.

The calculation for the antiferromagnet is more complicated. We have to combine (4.5) with (1.51). As $\beta\to\infty, R_+(\beta,\Delta)^{1/|\Lambda|}\to 0$, provided $\xi<1$, by (1.51) and (4.5). The problem resides in $R_-(\beta,\Delta)$. This will *not* go to zero as $\beta\to\infty$, but for small enough $\alpha$ (depending on $S$), which we call $\alpha_c(S)$, we can make $R_-(\beta, \Delta)^{1/|\Lambda|}$ smaller than any given number, say $\mu$. Choose $\mu$ such that (4.1) is satisfied. We omit details, but note that $\alpha_c(S)$ tends to 1 as $S$ tends to infinity.

*IV.C. Estimate of the Spontaneous Magnetization*

Consider the order parameter which satisfies the previously derived inequality

$$\sigma(\beta) > \delta^2 - 6\varepsilon \tag{1.24}$$

provided $\langle P_0^+ P_j^- \rangle < \varepsilon/2$ and $\langle P_0^{<\delta} \rangle < \varepsilon$. In the classical models (1) and (6) these inequalities hold for all $\varepsilon > 0$ and $\delta < 1$ if $\beta$ is large enough. This follows from chessboard estimates applied to $\langle P_0^{<\delta} \rangle$, and the results of Section IV. Thus $\sigma(\beta) \to 1$ as $\beta \to \infty$.

For the quantum antiferromagnets (3), (6), an estimate on $\langle P_0^{<\delta} \rangle$ can be obtained using chessboard estimates and exponential localization, as before, with the following result: Given $\varepsilon > 0$, $\delta < 1$ and $\alpha < 1$ there exists an $S(\varepsilon, \delta, \alpha) < \infty$ such that (1.24) holds as $\beta \to \infty$ for $S > S(\varepsilon, \delta, \alpha)$. For the ferromagnet (4), chessboard estimates, if they could be shown to be true, would easily yield $\sigma(\beta) \to 1$ as $\beta \to \infty$ for all $S$ and all $\alpha < 1$. Without using chessboard estimates, we can show that $\langle P_0^{<\delta} \rangle \to 0$ as $\beta \to \infty$ for all $\delta < 1$ and all $\alpha < 1$. This is proved by means of the following thermodynamic argument: It is sufficient to show

$$\langle (S_0^z)^2 \rangle \to 1 \text{ as } \beta \to \infty.$$

By the Schwarz inequality and translation invariance

$$S^{-2} \langle (S_0^z)^2 \rangle \geqq \tfrac{1}{4} S^{-2} \sum_{|i-0|=1} \langle S_0^z S_i^z \rangle.$$

This latter quantity is half the $H^z$-energy per site. The ground state has the property that $S^{-2} \langle S_i^z S_j^z \rangle = 1$ for all $i, j$. If $S^{-2} \langle S_0^z S_i^z \rangle \not\to 1$ as $\beta \to \infty$, for $|i-0| = 1$, then the free energy would not approach the ground state energy as $\beta \to \infty$. This, it is easy to see by the previous arguments, would be a contradiction.

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**Note Added in Proof**

We are informed that an inequality similar to Theorem 2.2 was also proved independently by R. F. Streater and E. B. Davies (unpublished). Their proof is similar to ours.