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The Onsager-Machlup Function as Lagrangian for the Most Probable Path of a Diffusion Process

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Abstract. By application of the Girsanov formula for measures induced by diffusion processes with constant diffusion coefficients it is possible to define the Onsager-Machlup function as the Lagrangian for the most probable tube around a differentiable function. The absolute continuity of a measure induced by a process with process depending diffusion w.r.t. a quasi translation invariant measure is investigated. The orthogonality of these measures w.r.t. quasi translation invariant measures is shown. It is concluded that the Onsager-Machlup function cannot be defined as a Lagrangian for processes with process depending diffusion coefficients.

1. Introduction

In the preceeding years a lot of work has been done concerning the Onsager-Machlup (OM) function [1–6]. Onsager and Machlup were the first to consider the probability of paths of a diffusion process as the starting point of a theory of fluctuations [7]. Their work was restricted to processes with linear drift and constant diffusion coefficients, the generalization to nonlinear equations was undertaken by Tisza and Manning [8]. The central point was to express the transition probability of a diffusion process by means of a functional integral over paths of the process. A certain part of the integrand was then called the OM function.

Recent works are concerned with finding the correct integrand. It has often been overlooked however, that dealing with paths of a diffusion process requires an almost sure calculus. Otherwise ambiguous results may occur [3, 4, 9, 10], as was pointed out in [2, 6].

Here the OM function can only be understood as a shortened form in the functional integral mentioned above. Therefore all forms of the OM function, containing the derivative of a path, are formal expressions. This is because of the fact that almost all paths of a diffusion process are nowhere differentiable.

Still another, more physical, meaning can be given to the term OM function. Some workers have taken up the idea of Tisza and Manning to interprete the OM

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function as a Lagrangian for determining the most probable path of the diffusion process by a variational principle. This cannot hold for a path of a diffusion process, because a solution of the variational principle should be twice differentiable. Besides, the probability of a single path is zero anyhow. Instead one can ask for the probability that a path lies within a certain region, which may be a tube along a differentiable function. So the OM function may be defined as the Lagrangian giving the most probable tube. Comparing the probabilities of different tubes of the same "thickness" requires a measure in function space, from which the probabilities can be derived. This is achieved by the induced measure. While in the problem of the OM function the theory of induced measures was referred to only twice [6, 11], there are other fields of physics where this theory is the basis for the connection between quantum mechanics and stochastic processes [12–14].

So hereafter we shall give an introduction to the theory of induced measures and try to motivate the mathematical concept. In Section 3 a definition of the OM function as a Lagrangian in the sense stated above will be given, and subsequently in Section 4 the OM function for processes with constant diffusion coefficients is calculated. We shall see that it is of the same form as was supposed in [2, 5, 6].

In Sections 5 and 6 we will consider processes with process depending diffusion. It is shown that the induced measure of any such process cannot be absolutely continuous w.r.t. a quasi translation invariant measure. The subsequent Section 7 consists in proving the orthogonality of measures induced by processes with process depending diffusion w.r.t. quasi translation invariant measures. In the conclusion the equation for the most probable tube of processes with constant diffusion is stated. As our results in the case of process depending diffusion differ considerably from others we finally give a critical comparison with these works.

2. Mathematical Introduction [15–18]

Let X_t denote a nonexploding diffusion process on $\tau=[s,u]$ defined by the stochastic differential equation w.r.t. the probability space (Ω, σ, P)

$$\begin{aligned} &dX_t=f(X_t)dt+g(X_t)dW_t \\ &g(x)>0, \quad X_s=x_0\in\mathbb{R}. \end{aligned} \tag{2.1}$$

The space of paths of such a diffusion process is the space $C_\tau^{x_0}$ of continuous functions

$$C_\tau^{x_0}=\{x(t)|x:\tau\to\mathbb{R},\ x(t)\text{ continuous},\ x(s)=x_0\}. \tag{2.2}$$

With the uniform norm $\|\cdot\|$

$$\|x\|=\max_{t\in\tau}|x(t)|, \qquad x(t)\in C_\tau^{x_0}, \tag{2.3}$$

$C_\tau^{x_0}$ is a Banach space. Using this norm to induce the uniform topology on $C_\tau^{x_0}$ we get the Borel field $\mathbb{B}_\tau^{x_0}$ of $C_\tau^{x_0}$.

A subset I_n of $C_\tau^{x_0}$ of the following form

$$I_n=\{x\in C_\tau^{x_0}|(x(t_1),\ldots,x(t_n))\in E\}, \tag{2.4}$$

where $s<t_1<\dots t_n\leqq u$ and E is a Borel set of $\mathbb{R}^n$ will be called n-dimensional cylinder set. The collection of all n-dimensional cylinder sets is a σ-field and the class of all finite-dimensional cylinder sets is a field, which we denote by I. In [17] it is shown that the σ-field $\sigma(I)$ generated by I is the Borel field $\mathbb{B}_\tau^{x_0}$ i.e.

$$\sigma(I)=\mathbb{B}_\tau^{x_0}. \tag{2.5}$$

Now let us define the measure μ_X on $\mathbb{B}_\tau^{x_0}$ induced by the diffusion process (2.1) by

$$\mu_X(B)=P(\{\omega\in\Omega|X_t(\omega)\in B\}), \qquad B\in\mathbb{B}_\tau^{x_0}. \tag{2.6}$$

If $B=I_n$ we have

$$\mu_X(I_n)=P_n^X(\{(x(t_1),\dots,x(t_n))\in E\}), \tag{2.7}$$

where E is the Borel set of $\mathbb{R}^n$ considered and P_n^X is the n-dimensional probability of the diffusion process X_t.

To introduce the following theorems in a heuristic way we call the attention to the problem of finding the most probable path of X_t. As just mentioned it only makes sense to ask for the probability that a path lies within the open tube $K(z,\varepsilon)$ which is defined as

$$K(z,\varepsilon)=\{x\in C_\tau^{x_0}|z\in C_\tau^{x_0},\ \|x-z\|<\varepsilon,\ \varepsilon>0\}. \tag{2.8}$$

Once an $\varepsilon>0$ is given one can compare the probabilities of tubes for all $z\in C_\tau^{x_0}$ using

$$\mu_x(K(z,\varepsilon))=P(\{\omega\in\Omega|X_t(\omega)\in K(z,\varepsilon)\}), \tag{2.9}$$

as $K(z,\varepsilon)\in\mathbb{B}_\tau^{x_0}$.

Principly the same is done in $\mathbb{R}^n$, if one asks for the most probable value x_m of the stochastic variable X. Instead of the induced measure μ_x we have the distribution $P(x)$, possessing a density $p(x)$ w.r.t. the Lebesgue-measure μ_L, such that $P(A)=\int_A p(x)\mu_L(dx)$ and $A=[x,y]$. In the limit $A=[x,x+dx]$ we get $P(x\in A)=p(x)\mu_L(A)=p(x)dx$, where dx is independent of x. This property is the wellknown translation invariance of the Lebesgue-measure. Hence calculating the most probable value x_m means to maximize $p(x)$.

The concept of a translation invariant measure cannot be transferred to function space. Instead the role of the small interval dx may be taken over by the tube $K(z,\varepsilon)$ and that of the Lebesgue-measure by a quasi translation invariant measure. $\mu_X(K(z,\varepsilon))$ can be represented as a functional integral with the range of integration $K(z,\varepsilon)$. If μ_X is absolutely continuous w.r.t. a quasi translation invariant measure μ, it is possible to shift the range of integration to $K(x_0,\varepsilon)$, which no longer depends on z [19].

Our aim is to approximate the integrand of the new functional integral for ε small, such that we get $\mu_X(K(z,\varepsilon))\propto M(z)\mu(K(x_0,\varepsilon))$. This defines $M(z)$ as a certain functional. For that some theorems are needed.

We will call measures μ_X, μ_Y equivalent ($\mu_X\sim\mu_Y$) if μ_X is absolutely continuous w.r.t. μ_Y ($\mu_X\ll\mu_Y$) and if $\mu_Y\ll\mu_X$ [17]. Our main theorem, as in [6], is the Girsanov formula [15, 20] which states a sufficient condition for the absolute continuity (or equivalence) of an induced measure w.r.t. another. Here we need

Theorem 2.1 [15, 20]. *Let X_t and Y_t be two diffusion processes defined by the stochastic differential equations*

$$dX_t = f(X_t)dt + G(X_t)dW_t\,, \tag{2.10}$$

$$dY_t = k(Y_t)dt + G(Y_t)dW_t\,, \tag{2.11}$$

$$c>0\,, \quad X_s = Y_s = x_0 \in \mathbb{R}\,, \quad t\in\tau\,; \quad k, f, G \in C^2\,.$$

Then we have $\mu_X \sim \mu_Y$ (as the diffusions are equal) and the Radon-Nikodym derivative (RND) *of μ_X* w.r.t. *μ_Y is given by*

$$\frac{d\mu_X}{d\mu_Y}[Y_t(\omega)] = \exp\left\{\int_s^u a(Y_t(\omega))dW_t(\omega) - \frac{1}{2}\int_s^u (a(Y_t(\omega))^2 dt\right\}, \tag{2.12}$$

where

$$a(x) = \frac{f(x)-k(x)}{G(x)}\,. \tag{2.13}$$

We transform the stochastic integral in (2.12) using the Ito formula [15]. Firstly we consider $G(x)=c$.

Setting

$$V(x) = \frac{1}{c}\int^x dy a(y) \tag{2.14}$$

we get

$$dV(Y_t) = \left\{\frac{1}{c}a(Y_t)k(Y_t) + \frac{c}{2}\frac{da(x)}{dx}(Y_t)\right\}dt + a(Y_t)dW_t \tag{2.15}$$

or

$$\int_s^u a(Y_t)dW_t = V(Y_u) - V(x_0) - \frac{1}{2}\int_s^u \left\{\frac{2}{c}a(Y_t)k(Y_t) + c\frac{da(x)}{dx}(Y_t)\right\}dt\,. \tag{2.16}$$

Replacing the Ito integral in (2.12) by the above relation we get an expression $F[y(t)]$ for (2.12), which clarifies the functional property of the RND on $C_\tau^{x_0}$ ($y(t)\in C_\tau^{x_0}$)

$$F[y(t)] = \frac{d\mu_X}{d\mu_Y}[y(t)] = \exp\left\{V(y(u)) - V(x_0) - \frac{1}{2}\int_s^u dt b(y(t))\right\}, \tag{2.17}$$

where

$$b(y(t)) = \{a(y(t))\}^2 + c\frac{da(x)}{dx}(y(t)) + \frac{2}{c}a(y(t))k(y(t))\,. \tag{2.18}$$

It is well known that the uniqueness of the Lebesgue measure μ_L on $\mathbb{R}^n$ is given by its translation invariance. That means, if T is a translation on $\mathbb{R}^n$, then for each $E \in \mathbb{B}^n$

$$TE \in B^n \quad \text{and} \quad \mu_L(TE) = \mu_L(E) \tag{2.19}$$

holds. In $C_\tau^{x_0}$ such a one to one mapping is a translation by any function $z_0(t) \in C_\tau^0 = \{x(t) | x : \tau \to \mathbb{R},\ x(t) \text{ continuous},\ x(s) = 0\}$ and it is easy to see that a translation invariant measure μ_X on $C_\tau^{x_0}$ does not exist [16].

The adequate measure here is the quasi translation invariant (q.t.i.) measure defined as follows:

Definition 2.1. Let T be a transformation $T: C_\tau^{x_0} \to C_\tau^{x_0}$ such that

$$Tx \to x + z_0, \tag{2.20}$$

where $z_0 \in C_\tau^0$, z_0 twice differentiable and bounded. Consider the diffusion processes X_t and

$$TX_t = X_t + z_0(t). \tag{2.21}$$

If the induced measures μ_X and μ_{TX} are equivalent, μ_X and μ_{TX} will be called q.t.i.. Next we show

Theorem 2.2. *Each diffusion process X_t with constant diffusion $c > 0$ induces a* q.t.i. *measure μ_X.*

Proof. Let us take (2.10) and the translation (2.20). Then we get in combination with (2.21)

$$\begin{aligned} dTX_t &= \{f(X_t) + \dot{z}_0(t)\} dt + c dW_t \\ &= \{f(TX_t - z_0(t)) + \dot{z}_0(t)\} dt + c dW_t . \end{aligned} \tag{2.22}$$

As the diffusion has not changed under T we have by Theorem 2.1 $\mu_{TX} \sim \mu_X$ and the RND of μ_{TX} w.r.t. μ_X is given by (2.12), where $a(x)$ is replaced by

$$a_X(x, z_0) = \frac{f(x - z^0) + \dot{z}_0 - f(x)}{c}. \quad \square \tag{2.23}$$

The index X [cf. (2.23)] indicates that the term belongs to the RND of a measure μ_{TX} w.r.t. μ_X; if we replace z_0 by $-z_0$ in these functionals they refer to the RND of $\mu_{T^{-1}X}$ w.r.t. μ_X.

If we want to eliminate the stochastic integral now, we have to take into account that V_X, given by (2.14), is a function, explicitely depending on time. Now the Ito formula yields

$$\begin{aligned} dV_X(X_t, z_0(t)) = \Bigg\{ & \frac{\partial V_X(x, z_0(t))}{\partial t}\bigg|_{x = X_t} + \frac{1}{c} a_X(X_t, z_0(t)) f(X_t) \\ & + \frac{c}{2} \frac{\partial a_X(x, z_0(t))}{\partial x}\bigg|_{x = X_t} \Bigg\} dt + a_X(X_t, z_0(t)) dW_t . \end{aligned} \tag{2.24}$$

We will denote the RND of μ_{TX} w.r.t. μ_X by $J_X[X_t, z_0(t)]$. Setting

$$d_X(x(t), z_0(t)) = \{a_X(x(t), z_0(t))\}^2 + 2\left.\frac{\partial V_X(x, z_0(t))}{\partial t}\right|_{x=x(t)} + \frac{2}{c} a_X(x(t), z_0(t)) f(x(t)) + c \left.\frac{\partial a_X(x, z_0(t))}{\partial x}\right|_{x=x(t)} \tag{2.25}$$

we get the following expression for $J_X[x(t), z_0(t)]$

$$J_X[x(t), z_0(t)] = \exp\left\{ V_X(x(u), z_0(u)) - V_X(x_0, z_0(s)) - \frac{1}{2}\int_s^u dt d_X(x(t), z_0(t)) \right\}. \tag{2.26}$$

The q.t.i. measures are important because of the following theorem.

Theorem 2.3[1] [19]. *If we take the translation* (2.20) *then for* $B \in \mathbb{B}_\tau^{x_0}$ *we have* $T^{-1}B \in \mathbb{B}_\tau^{x_0}$. *If* $\Phi[x]$ *is a measurable functional on* $C_\tau^{x_0}$ *and* μ_X *is a* q.t.i. *measure the following equation holds*

$$\int_B \Phi[y] d\mu_X(y) = \int_{T^{-1}B} \Phi[x + z_0] J_X[x, -z_0] d\mu_X(x). \tag{2.27}$$

This is easily seen: Based on the definition of $J_X[x, z_0]$ as RND of $\mu_{T^{-1}X}$ w.r.t. μ_X we have

$$J_X[x, -z_0] d\mu_X(x) = d\mu_{T^{-1}X}(x). \tag{2.28}$$

Now for any $B \in \mathbb{B}_\tau^{x_0}$

$$\mu_{T^{-1}X}(T^{-1}B) = P(\{\omega | T^{-1}X_t(\omega) \in T^{-1}B\}) = P(\{\omega | X_t(\omega) \in B\}) = \mu_X(B) \tag{2.29}$$

holds, which yields (2.27). □

3. The Definition of the Onsager-Machlup Function

In the preceeding section we have considered the tube $K(z, \varepsilon)$ as was given by (2.8). We are now interested in the most probable tube. As this tube depends on a function $z(t)$ we have to look for that function $z(t)$ which maximizes (2.9). If we restrict ourselves on bounded functions $z(t)$, twice differentiable with bounded derivatives, the following definition makes sense.

Definition 3.1. Let $\varepsilon > 0$ be given. Let $z_m(t)$ be a function that maximizes $\mu_X(K(z, \varepsilon))$. If for $\varepsilon \to 0$ $z_m(t)$ can be found by variation of a functional $\int_s^u \mathrm{OM}(\dot{z}, z) dt$, the integrand $\mathrm{OM}(\dot{z}, z)$ will be called Onsager-Machlup function.

The question of uniqueness of the OM function depends on the formulation of the variational principle and will be discussed later (cf. Section 8).

[1] In [16] this theorem is proven for the Wiener measure

4. The Onsager-Machlup Function for a Diffusion Process with Constant Diffusion

Let X_t be given by the stochastic differential equation

$$dX_t = f(X_t)dt + c\,dW_t, \quad c>0, \quad X_s = x_0 \in \mathbb{R}, \quad f \in C^2. \tag{4.1}$$

For any bounded function $z(t) \in C_\tau^{x_0}$, twice differentiable and both derivatives bounded, we can find a function $z_0(t) \in C_\tau^0$ as in (2.20) such that

$$z(t) = x_0 + z_0(t), \qquad \dot{z}(t) = \dot{z}_0(t). \tag{4.2}$$

Following Definition 3.1 we have to consider

$$\mu_X(K(z,\varepsilon)) = \int_{K(z,\varepsilon)} d\mu_X(x). \tag{4.3}$$

Now we have

$$T^{-1}K(z,\varepsilon) = K(x_0,\varepsilon), \tag{4.4}$$

where T is given by (2.20).

We know that each diffusion process Y_t with diffusion c and initial value $Y_s = x_0$ induces a q.t.i. measure μ_Y with $\mu_X \sim \mu_Y$. Hence combining (2.15) and (2.27) we get

$$\mu_X(K(z,\varepsilon)) = \int_{K(x_0,\varepsilon)} F[x+z_0]J_Y[x,-z_0]d\mu_Y(x). \tag{4.5}$$

As μ_X is already q.t.i. we may take μ_X instead of μ_Y in (4.5). Then $F=1$ and instead of (4.5) we get

$$\mu_X(K(z,\varepsilon)) = \int_{K(x_0,\varepsilon)} J_X[x,-z_0]d\mu_X(x). \tag{4.6}$$

As we want to get $K(0,\varepsilon)$ as integration area we define

$$Y_t^0 = Y_t - x_0 \tag{4.7}$$

and denote by μ_{Y^0} the measure induced on C_τ^0.

Applying Equation (2.27) again with translation parameter x_0 yields

$$\mu_X(K(z,\varepsilon)) = \int_{K(0,\varepsilon)} F[y+z]J_Y[y+x_0,-z_0]d\mu_{Y^0}(y). \tag{4.8}$$

Combining (2.17) and (2.26) we can write for the integrand of (4.8)

$$\begin{aligned} F[y+z]J_Y[y+x_0,-z_0] = \exp\Big\{ & V(y(u)+z(u)) \\ & - V(x_0) - \frac{1}{2}\int_s^u dt\,b(y(t)+z(t))\Big\} \\ \cdot\exp\Big\{ & V_Y(y(u)+x_0,-z_0(u)) - V_Y(x_0,-z_0(s)) \\ & - \frac{1}{2}\int_s^u dt\,d_Y(y(t)+x_0,-z_0(t))\Big\}. \end{aligned} \tag{4.9}$$

We remark that the integrals in (4.9) are Riemann integrals. They can easily be estimated by well known methods. We expand the exponent of (4.9) into a Taylor series around $y(t)=0$ and split off the terms of zero order. The remaining terms can be made arbitrarily small if we choose ε small enough, as for $y(t)\in K(0,\varepsilon)$

$$\|y(t)\| \leqq \varepsilon \tag{4.10}$$

holds.

Denoting the remaining terms by $\Delta[y,z]$ we have

$$\begin{aligned} F[y+z]J_Y[y+x_0,-z_0] &= \exp(\Delta[y,z]) \\ &\exp\left\{V(z(u))-V(x_0)-\frac{1}{2}\int_s^u dtb(z(t))\right\} \\ &\exp\left\{V_Y(x_0,-z_0(u))-V_Y(x_0,-z_0(s)) \right. \\ &\left. -\frac{1}{2}\int_s^u dtd_Y(x_0,-z_0(t))\right\}. \end{aligned} \tag{4.11}$$

Inserting this into (4.8) we get

$$\mu_X(K(z,\varepsilon))=F[z]J_Y[x_0,-z_0]\int_{K(0,\varepsilon)}\exp(\Delta[y,z])d\mu_{Y^0}(y). \tag{4.12}$$

We recall that for a functional $\Psi[y]$ on C_τ^0 with $\|\Psi[y]\|\leqq\delta$ the following relation holds

$$\int_B \Psi[y]d\mu(y)\leqq\delta\mu(B), \qquad B\in\mathbb{B}_\tau^0. \tag{4.13}$$

Now we choose an $\varepsilon>0$ such that $\Delta[y,z]<\delta$ for $\delta\to 0$. Expanding the exponential in (4.12) we can approximate (4.12) neglecting terms smaller than $\delta(\delta\ll 1)$

$$\mu_X(K(z,\varepsilon))=F[z]J_Y[x_0,-z_0]\mu_{Y^0}(K(0,\varepsilon)). \tag{4.14}$$

Using $\mu_{Y^0}(K(0,\varepsilon))=\mu_Y(K(x_0,\varepsilon))$ we finally get

$$\mu_X(K(z,\varepsilon))=F[z]J_Y[x_0,-z_0]\mu_Y(K(x_0,\varepsilon)). \tag{4.15}$$

To find a $z(t)$ which maximizes (4.15) we have to maximize the functional

$$M[z]=F[z]J_Y[x_0,-z_0]. \tag{4.16}$$

X_t is given by (4.1) and let Y_t be given by (2.11). To get (4.16) in terms of $f(x)$ and $k(x)$ we need the following equations for which we used (2.13), (2.14), (2.18), (2.23), (2.25), (4.2) and the differentiability of $z(t)$

$$V(z(u))-V(x_0)=\frac{1}{c^2}\int_s^u dt[\dot{z}(t)\{f(z(t))-k(z(t))\}], \tag{4.17}$$

$$b(z(t))=\left\{\frac{f(z(t))-k(z(t))}{c}\right\}^2+\left.\frac{df(x)}{dx}\right|_{x=z(t)}$$
$$-\left.\frac{dk(x)}{dx}\right|_{x=z(t)}+\frac{2}{c^2}\{f(z(t))-k(z(t))\}k(z(t)), \tag{4.18}$$

$$V_Y(x_0,-z_0(u))-V_Y(x_0,-z_0(s))=\frac{1}{c^2}\int_s^u dt\{\dot{z}(t)k(z(t))-\ddot{z}(t)x_0\}, \tag{4.19}$$

$$d_Y(x_0,-z_0(t))=-\frac{k^2(x_0)}{c^2}-\left.\frac{dk(x)}{dx}\right|_{x=x_0}+\frac{\{\dot{z}(t)\}^2}{c^2}$$
$$+\frac{k^2(z(t))}{c^2}-\frac{2\ddot{z}(t)x_0}{c^2}+\left.\frac{dk(x)}{dx}\right|_{x=z(t)}. \tag{4.20}$$

With (2.17) and (2.26) we get for $M[z]$ combining the last four formulas

$$\ln(M[z])=-\frac{1}{2}\int_s^u dt\left\{\left(\frac{f(z)-\dot{z}}{c}\right)^2+f'(z)-\left(\frac{k^2(x_0)}{c^2}+k'(x_0)\right)\right\}. \tag{4.21}$$

In accordance with Definition 3.1 we define the following OM function

$$\mathrm{OM}(\dot{z},z)=\left(\frac{f(z)-\dot{z}}{c}\right)^2+f'(z)-\left(\frac{k^2(x_0)}{c^2}+k'(x_0)\right). \tag{4.22}$$

The function $z_m(t)$ that maximizes (4.15) must be independent of the choice of the q.t.i. measure μ_Y i.e. independent of $k(z)$. As we see this is fulfilled. The term

$$-\left(\frac{k^2(x_0)}{c^2}+k'(x_0)\right) \tag{4.23}$$

is a constant and depends on the measure chosen. As it is a constant it cannot influence $z_m(t)$. So we can take as OM function the expression

$$\mathrm{OM}(\dot{z},z)=\left(\frac{f(z)-\dot{z}}{c}\right)^2+f'(z). \tag{4.24}$$

With (4.24) we give some versions of (4.15):

$$\mu_X(K(z,\varepsilon))=\exp\left\{\frac{1}{2}\left(\frac{k^2(x_0)}{c^2}+k'(x_0)\right)(u-s)\right\}$$

$$\cdot\mu_Y(K(x_0,\varepsilon))\exp\left\{-\frac{1}{2}\int_s^u dt\,\mathrm{OM}(\dot z,z)\right\}, \tag{4.25}$$

$$\mu_X(K(z,\varepsilon))=\exp\left\{\frac{1}{2}\left(\frac{f^2(x_0)}{c^2}+f'(x_0)\right)(u-s)\right\}$$

$$\cdot\mu_X(K(x_0,\varepsilon))\exp\left\{-\frac{1}{2}\int_s^u dt\,\mathrm{OM}(\dot z,z)\right\}, \tag{4.26}$$

$$\mu_X(K(z,\varepsilon))=\mu_{W^c}(K(x_0,\varepsilon))\exp\left\{-\frac{1}{2}\int_s^u dt\,\mathrm{OM}(\dot z,z)\right\}. \tag{4.27}$$

In the last expression we used the modified Wiener process

$$W_t^c=c(W_t-W_s)+x_0\,. \tag{4.28}$$

Until now our considerations were restricted to processes with constant diffusion. In the next section we treat the case of process depending diffusion. Our aim is to show that the way described above is not possible in this case, i.e. that the induced measure of a process with process depending diffusion (p.d.d.) is not absolutely continuous w.r.t. a quasi translation invariant measure. This purpose is achieved in two steps:

Firstly we prove, that, if two measures are absolutely continuous w.r.t. another, the diffusion coefficients of the underlying processes are the same. This is the conversion of the Girsanov theorem (Theorem 2.1).

Secondly we show, using the first step, that only processes with constant diffusion induce quasi translation invariant measures. The first step is rather mathematical in nature, but there seems to be no reference concerning this point.

5. The Converse of the Girsanov Theorem

Let X_t and Y_t be given by (2.10) and (2.11). Theorem 2.1 then states

$$\mu_X\sim\mu_Y\,. \tag{5.1}$$

Let us define the measure $\hat{P}$ by the absolutely continuous transformation of the measure P:

$$\hat{P}(A)=\int_A \varrho(\omega)dP(\omega)\,, \tag{5.2}$$

where $A\in\sigma$ and

$$\varrho(\omega)=\frac{d\mu_Y}{d\mu_X}(X_t(\omega))\,. \tag{5.3}$$

Then Theorem 2.1 is equivalent to the following statement:

Theorem 5.1 [15]. *The process*

$$\hat{W}_t = W_t + \int_s^t a(X_{t'})dt' \tag{5.4}$$

is a Wiener process w.r.t. $\hat{P}$, *and a is given by* (2.13).

Proof of the Equivalence. Combining (2.10) and (5.4), we get

$$dX_t = k(X_t)dt + G(X_t)d\hat{W}_t \,. \tag{5.5}$$

Now if Theorem 5.1 holds, then (5.5) is the SDE of X_t w.r.t. $\hat{P}$ and we have the equality of μ_Y and $\hat{\mu}_X$, induced by $(X_t, \hat{P})$, on $\mathbb{B}_\tau^{x_0}$.

For any $B \in \mathbb{B}_\tau^{x_0}$ we have:

$$P(\{\omega | Y_t(\omega) \in B\}) = \mu_Y(B) = \hat{\mu}_X(B) = \hat{P}(\{\omega | X_t(\omega) \in B\})$$
$$= \int_{\{\omega | X_t(\omega) \in B\}} \frac{d\hat{P}}{dP}(\omega) dP(\omega) = \int_B \frac{d\hat{P}}{dP}(\omega) d\mu_X(X_t(\omega)) \,, \tag{5.6}$$

because $\hat{P}$ is by definition absolutely continuous w.r.t. P. So

$$\frac{d\mu_Y}{d\mu_X}(X_t(\omega)) = \frac{d\hat{P}}{dP}(\omega) \,.$$

If Theorem 2.1 holds we have for any $B \in \mathbb{B}_\tau^{x_0}$:

$$\mu_Y(B) = \int_B \frac{d\mu_Y}{d\mu_X} d\mu_X = \int_{\{\omega | X_t(\omega) \in B\}} \frac{d\hat{P}}{dP}(\omega) dP(\omega) = \int_{\{\omega | X_t(\omega) \in B\}} d\hat{P}(\omega)$$
$$= \hat{P}(\{\omega | X_t(\omega) \in B\}) = \hat{\mu}_X(B) \,. \tag{5.7}$$

Thus X_t induces the same measure w.r.t. $\hat{P}$ as Y_t w.r.t. P, so that $(X_t, \hat{P})$ is equivalent to (Y_t, P). Therefore the SDE of X_t w.r.t. $\hat{P}$ is (5.5). Then $(\hat{W}_t, \hat{P})$ is the Wiener process. □

Note 5.1. Let $H_u^w = \sigma(x_0, W_t, t \leqq u)$ denote the σ-algebra of all events $W_t(\omega)$, $t \leqq u$. As each diffusion process is a non anticipating functional of the Wiener process we can choose [18]: $H_u^w = \mathbb{B}_\tau^{x_0}$. In [20] it is shown, that Theorem 5.1 also holds if we replace $a(X_u)$ by any H_u^w-measurable function $\alpha_u(\omega)$.

We now prove the converse of the Girsanov theorem.

Theorem 5.2. *Let* μ_X *and* μ_Y *denote the measures induced by the diffusion processes* X_t *and* Y_t *respectively. If* $\mu_X \sim \mu_Y$ *then the diffusions of* X_t *and* Y_t *are the same.*

For the proof we need the lemma:

Lemma 5.1. *If* $X/_t$ *and* $Y/_t$ *denote the restrictions of* X_t *and* Y_t *on* $\tau' = [s, t]$, *then clearly* $\mu_{X/_t} \sim \mu_{Y/_t}$ *if* $\mu_X \sim \mu_Y$.

We consider the RND's

$$\varrho_u(\omega) = \frac{d\mu_Y}{d\mu_X}(X_{t'}(\omega)) \tag{5.8}$$

defined on $C_\tau^{x_0}$ *and*

$$\varrho_t(\omega)=\frac{d\mu_{Y/_t}}{d\mu_{X/_t}}(X_{t'}(\omega)) \tag{5.9}$$

defined on $C_\tau^{x_0}$.

Then $(\varrho_t, \mathbb{B}_{\tau'}^{x_0})_{t\leq u}$ *is a martingale.*

Proof. For any $B\in\mathbb{B}_{\tau'}^{x_0}$

$$\int_B \varrho_u(\omega)d\mu_X(X_{t'}(\omega))=\int_B \frac{d\mu_Y}{d\mu_X}d\mu_X$$

$$=\int_B d\mu_Y=\mu_{Y/_t}(B)=\int_B \varrho_t d\mu_{X/_t}=\int_B \varrho_t(\omega)d\mu_X(X_{t'}(\omega))\,. \tag{5.10}$$

This yields $E(\varrho_u(\omega)|\mathbb{B}_{\tau'}^{x_0})=\varrho_t(\omega)$ which shows $(\varrho_t, \mathbb{B}_{\tau'}^{x_0})$ is martingale. □

For the proof of Theorem 5.2 we consider the two diffusion processes

$$dX_t=f(X_t)dt+G(X_t)dW_t \qquad X_s=x_0\in\mathbb{R} \tag{5.11}$$

and

$$dY_t=k(Y_t)dt+\bar{G}(Y_t)dW_t \qquad Y_s=x_0\,. \tag{5.12}$$

Let $\varrho_u(\omega)$ be defined as in (5.8). With Note 5.1 and Lemma 5.1 we have (ϱ_t, H_t^w) is a martingale. Now by a result of Kunita and Watanabe [21], there exists a H_t^w-measurable function $\alpha_t(\omega)$ such that a positive martingale can be written as

$$E(\varrho_u|H_t^w)-E(\varrho_u|H_s^w)=\int_s^t \alpha_{t'}dW_{t'}\,. \tag{5.13}$$

Because of $E(\varrho_u|H_s^w)=E(\varrho_u|x_0)=E(\varrho_u)=1$ we get

$$\varrho_t(\omega)-1=\int_s^t \alpha_{t'}(\omega)dW_{t'}(\omega) \quad \text{or}$$

$$d\varrho_t=\alpha_t dW_t\,. \tag{5.14}$$

By the Ito formula $d\ln\varrho_t=-(1/2)\left(\frac{\alpha_t}{\varrho_t}\right)^2 dt+\left(\frac{\alpha_t}{\varrho_t}\right)dW_t$ and (5.8) we get

$$\frac{d\mu_Y}{d\mu_X}(X_t(\omega))=\exp\left\{\int_s^u\left(\frac{\alpha_t}{\varrho_t}(\omega)\right)dW_t(\omega)-(1/2)\int_s^u\left(\frac{\alpha_t}{\varrho_t}(\omega)\right)^2dt\right\}. \tag{5.15}$$

So

$$\Phi_t(\omega)=\frac{\alpha_t}{\varrho_t}(\omega) \tag{5.16}$$

defines a H_t^w-measurable function. Now if we take (5.15) to define a new measure $\hat{P}$ as was done in (5.4), we have by virtue of Note 5.1

$$\hat{W}_t = W_t - \int_s^t \Phi_{t'} dt' \tag{5.17}$$

is the Wiener process w.r.t. $\hat{P}$. Introducing this into (5.11) yields:

$$dX_t = (f(X_t) + G(X_t)\Phi_t)dt + G(X_t)d\hat{W}_t \tag{5.18}$$

which is the SDE of X_t w.r.t. $\hat{P}$. The induced measure will be called $\hat{\mu}_X$. As $\mu_X \sim \mu_Y$, we have the equality $\hat{\mu}_X = \mu_Y$ as was shown in (5.7). So $(X_t, \hat{P})$ is equivalent to (Y_t, P). This gives the equality of the drifts of (5.18) and (5.12)

$$k(Y_t(\omega)) = f(Y_t(\omega)) + G(Y_t(\omega))\Phi_t(\omega) \tag{5.19}$$

and the equality of the diffusions of (5.12) and (5.18), so that

$$G(x) = \bar{G}(x) . \quad \square \tag{5.20}$$

Combining Theorems 2.1 and 5.2 gives the final result:

$$\mu_X \sim \mu_Y \Leftrightarrow G(x) = \bar{G}(x) . \tag{5.21}$$

6. The Quasi Translation Invariance of Induced Measures

Now we are able to show that a measure induced by a diffusion process with p.d.d. cannot be absolutely continuous w.r.t. a q.t.i. measure. With respect to (5.21) it is sufficient to show:

Theorem 6.1. *Only processes with constant diffusion induce* q.t.i. *measures.*

For the proof we consider the process (2.10) and the transformation T, defined by (2.20). Let us consider the new process $Y_t = TX_t$, which is governed by the stochastic differential equation

$$dY_t = (f(Y_t - z_0) + \dot{z}_0)dt + G(Y_t - z_0)dW_t . \tag{6.1}$$

Now quasi translation invariance requires $\mu_Y \sim \mu_X$. With (5.21)

$$G(x) = G(x - z_0) \tag{6.2}$$

holds. This can only be fulfilled for any z_0 if $G(x) = \text{constant}$. $\square$

As a consequence we have that the proceeding in Section 4 is not possible in the case of process depending diffusion.

7. Orthogonality of Measures

Before giving up the search for the OM function we investigate the last possibility which could lead us to the solution of the problem, i.e. we investigate the component of a measure which is absolutely continuous to a q.t.i. measure. This will be specified as follows:

Definition 7.1 [22]. Two measures μ_X and μ_Y are called orthogonal $\mu_X \perp \mu_Y$ if there exists an $A \in C_\tau^{x_0}$ such that $\mu_X(A) = 0$ and $\mu_Y(C_\tau^{x_0} - A) = 0$. The RND of μ_X w.r.t. μ_Y is zero μ_Y-almost everywhere. For each μ_X and μ_Y exists a unique representation

$$\mu_X = a\nu_X + b\nu_Y, \tag{7.1}$$

where $\nu_X \perp \mu_Y$ and $\nu_Y \sim \mu_Y$.

$a + b = 1$ and ν_X, ν_Y are probability measures. $a\nu_X$ is called the orthogonal component of μ_X w.r.t. μ_Y and $b\nu_Y$ is called the absolutely continuous component of μ_X w.r.t. μ_Y. Hence

$$\mu_X \perp \mu_Y$$

if $b\nu_Y = 0$. We are interested to know which values $a\nu_X$ and $b\nu_Y$ can take. For example the orthogonal component of μ_X might be much smaller than the absolutely continuous one. To solve this problem, we cite the following theorem:

Theorem 7.1 [22]. *Let μ_X and μ_Y be two measures on $\mathbb{B}_\tau^{x_0}$. Denote by $\mathscr{B}_n^{x_0}$ the σ-algebra generated by the collection of the n-dimensional cylindersets I_n and denote by $\mu_{X/n}$ and $\mu_{Y/n}$ the restrictions of μ_X and μ_Y on $\mathscr{B}_n^{x_0}$ respectively. It is clear that $\mathscr{B}_n^{x_0}$ is an increasing sequence of σ-algebras, such that $\sigma \bigcup_n \mathscr{B}_n^{x_0} = \mathbb{B}_\tau^{x_0}$. Assume that on $\mathscr{B}_n^{x_0}$ $\mu_{X/n} \sim \mu_{Y/n}$ holds. Then $(\varrho_n, \mathscr{B}_n^{x_0})$ is a martingale, where*

$$\varrho_n = \frac{d\mu_{X/n}}{d\mu_{Y/n}}. \tag{7.2}$$

From the theorem on the limit of martingales follows that [22]

$$\varrho(x) = \lim_{n\to\infty} \varrho_n(x) \tag{7.3}$$

exists μ_Y-almost everywhere and

$$\varrho(x) = \frac{d\nu_Y}{d\mu_Y}(x). \tag{7.4}$$

If $\mu_X \perp \mu_Y$ then $\varrho(x) = 0$.

Let us call a process with diffusion $G(x)$ a $G(x)$-process and its measure $G(x)$-measure. As we consider only nonexploding processes, all $G(x)$-measures are equivalent. We can use this equivalence if we want to investigate whether a $G(x)$-measure has an absolutely continuous component w.r.t. a $K(x)$-measure. We can choose a $G(x)$- or $K(x)$-measure, such that we can easily apply Theorem 7.1.

To apply Theorem 7.1 we need $\mu_{X/n} = P_x^n$ which is the n-dimensional probability distribution of the process X_t. If X_t is given by (2.10) we can take as equivalent $G(x)$-process the process

$$\bar{X}_t = u(W_t) \tag{7.5}$$

the n-dimensional probability $P_{\bar{x}}^n$ of which is known. $u(x)$ is determined by

$$u' = G(u). \tag{7.6}$$

This is seen at once if we write down the SDE of $\bar{X}_t$.

$$d\bar{X}_t = (1/2)u''(u^{-1}(\bar{X}_t))dt + u'(u^{-1}(\bar{X}_t)dW_t\,. \tag{7.7}$$

If the process (2.10) has an absolutely continuous component w.r.t. a q.t.i. measure, then the measure $\mu_{\bar{X}}$ induced by (7.5) has one too. As q.t.i. measure we take the modified Wiener measure μ_{W^c} induced by

$$W_t^c = c(W_t - W_s) + x_0 \tag{7.8}$$

the n-dimensional probability of which is also wellknown. $P_{\bar{x}}^n$ and $P_{w^c}^n$ are both absolutely continuous w.r.t. n-dimensional Lebesgue measure. Hence $\mu_{\bar{X}/n} \sim \mu_{W^c/n}$. The RND

$$\varrho_n(\omega) = \frac{d\mu_{\bar{X}/n}}{d\mu_{W^c/n}}(W_t^c(\omega)) \tag{7.9}$$

is given by

$$\varrho_n(\omega) = \exp\left\{\sum_{i=1}^{n} -\ln\frac{G(w_i)}{c} - (1/2\Delta t)(u^{-1}(w_i) - u^{-1}(w_{i-1}))^2 \right.$$
$$\left. + (1/2\Delta t)\left(\frac{w_i - w_{i-1}}{c^2}\right)^2\right\}, \tag{7.10}$$

where we set $\Delta t = \dfrac{u-s}{n}$ and $w_i = W_{s+i\Delta t}^c(\omega)$, $i = 1, 2, \ldots, n$.

The limit $n \to \infty$ of (7.10) exists and gives the absolutely continuous component ν_{W^c} of $\mu_{\bar{X}}$ w.r.t. μ_{W^c}:

$$\varrho(\omega) = \exp\left\{\lim_{n\to\infty} \sum_{i=1}^{n} -\ln\frac{G(w_i)}{c} - (1/2\Delta t)(u^{-1}(w_i) - u^{-1}(w_{i-1}))^2 \right.$$
$$\left. + (1/2\Delta t)\frac{(w_i - w_{i-1})^2}{c^2}\right\}. \tag{7.11}$$

Now we use the fact [21] that for a diffusion process Y_t with diffusion $B(y)$ and $\sum_{i=1}^{\infty} \Delta t_i < \infty$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (y_i - y_{i-1})^2 \overset{\text{a.s.}}{=} \int_s^u dt B^2(Y_t(\omega)) \quad \text{holds}, \tag{7.12}$$

or in a discretisized form

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{(y_i - y_{i-1})^2}{\Delta t} \overset{\text{a.s.}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} B^2(y_{i-1})\,. \tag{7.13}$$

To apply (7.13) we must determine the diffusion of the process

$$Z_t = u^{-1}(W_t^c)\,. \tag{7.14}$$

The Ito formula yields for the diffusion term

$$c\frac{du^{-1}}{dx}(x)|_{x=W_t^c}=c\frac{1}{G(W_t^c)}, \tag{7.15}$$

where we used (7.6). Inserting this into (7.11) we get for the exponential of (7.11)

$$\begin{aligned}&\lim_{n\to\infty}\sum_{i=1}^{n}\left\{-\ln\frac{G(w_i)}{c}-(1/2)\frac{c^2}{G(w_{i-1})}+(1/2)\right\}\\&\quad=-\ln\frac{c}{G(x_0)}+\ln\frac{c}{G(W_u^c(\omega))}\\&\qquad+(1/2)\lim_{n\to\infty}\sum_{i=0}^{n-1}\left\{\ln\frac{c^2}{G^2(w_i)}-\frac{c^2}{G^2(w_i)}+1\right\}.\end{aligned} \tag{7.16}$$

To discuss the r.h.s. of (7.16) we have to consider the function

$$r=\ln y-y+1. \tag{7.17}$$

It is easy to see that $r\leqq 0$, where $r=0$ holds for $y=1$ only. For example let us take

$$G(x)=d>0, \quad d\neq c \tag{7.18}$$

then the limit in (7.16) is infinite and negative, so that (7.11) becomes zero. Remembering (7.5) and (7.6) we see that the choice (7.18) gives another modified Wiener process besides (7.8):

$$W_t^d=d(W_t-W_s)+x_0. \tag{7.19}$$

Hence we have shown that

$$\varrho=\frac{d\nu_{W^c}}{d\mu_{W^c}}=0, \tag{7.20}$$

i.e. μ_{W^d} has no absolutely continuous component w.r.t. μ_{W^c} if $c\neq d$

$$\mu_{W^d}\perp\mu_{W^c}. \tag{7.21}$$

This is a special case of the wellknown dichotomy of Gaussian measures [17].

If we take instead of (7.18) a process depending diffusion $G(x)$, we can use the argument that a continuous function cannot vanish in a neighborhood of a point, where it is unequal to zero. Now $G(W_t(\omega))$ is a continuous function of t and if there exists a t_0, such that $G(W_{t_0}(\omega))\neq c$ then this holds for a neighborhood $U_\varepsilon(t_0)$ of t_0. For all $t_i\in U_\varepsilon(t_0)$ then (7.17) is negative. We can choose an ε, such that $r\leqq-\delta$, $\delta>0$, for all $t_i\in U_\varepsilon(t_0)$. So the limit in (7.16) becomes infinite and (7.20) becomes zero, which states the orthogonality of measures induced by processes with process depending diffusion w.r.t. quasi translation invariant measures.

8. The Most Probable Tube—Conclusion

Firstly we return to the case of constant diffusion, where an OM function has been stated. It seems to be more suitable for gaining some information of the diffusion process, if we consider paths with fixed initial and variable final point. In both cases the most probable tube $z_m(t)$ is given by a variational principle

$$\delta \int_s^u dt \mathrm{OM}(\dot{z}, z) = 0, \tag{8.1}$$

where either

$$z_m(s) = x_0, \quad z_m(u) = x_1, \quad x_1 \in \mathbb{R} \tag{8.2}$$

or

$$z_m(s) = x_0, \quad \left. \frac{\partial \mathrm{OM}(\dot{z}, z)}{\partial \dot{z}} \right|_{\dot{z}(u), z(u)} = 0. \tag{8.3}$$

(8.1) and (8.2) correspond to the variational problem of classical mechanics; according to the Lagrange function the OM function is not unique. It can be changed by a gauge transformation. We get the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial \mathrm{OM}(\dot{z}, z)}{\partial \dot{z}} = \frac{\partial \mathrm{OM}(\dot{z}, z)}{\partial z} \tag{8.4}$$

as differential equation for $z_m(t)$. We obtain with (4.24)

$$\begin{aligned} \ddot{z}_m &= \frac{c^2}{2} f''(z_m) + f'(z_m) f(z_m), \\ z_m(s) &= x_0, \quad z_m(u) = x_1. \end{aligned} \tag{8.5}$$

(8.1) and (8.3) give the equation of motion $z_m(t)$ if only the initial point is fixed. The condition (8.3) shows that the OM function cannot be changed by a total differential of a function of z without changing $z_m(t)$. We need the whole expression (4.24) to get the evolution equation of $z_m(t)$

$$\begin{aligned} \ddot{z}_m &= \frac{c^2}{2} f''(z_m) + f'(z_m) f(z_m), \\ z_m(s) &= x_0, \quad \dot{z}_m(u) = f(z_m(u)). \end{aligned} \tag{8.6}$$

Based on our Definition 3.1 we had to restrict ourselves to differentiable functions $z(t)$. As we have shown, we can determine a most probable tube $K(z_m, \varepsilon)$ by means of a variation principle (8.1) where ε must be smaller than a given δ. The Equations (8.5) and (8.6) then hold for each $\varepsilon < \delta$.

As was stated in the introduction, some attempts exist, where an OM function has been given and used as Lagrangian even in the case of process depending diffusion [5]. A certain part of the integrand of a functional integral from which the OM function has been derived is called measure. The term (4.12) in [5] is only a part of the usual "element of integration" in approximating a functional integral by a n-fold ordinary integral.

So it is worthwhile to state the fact that in the case of process depending diffusion the induced measure is not absolutely continuous w.r.t. a quasi translation invariant measure, which seems to result in the failure of defining an Onsager-Machlup function as a Lagrangian in this case.

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