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The Construction of Self-dual Solutions to SU(2) Gauge Theory

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Abstract. Ignoring the problem of sources and singularities, explicit expressions are constructed for the ansätze of Atiyah and Ward. These take an especially simple form in the R gauge of Yang. Some non-linear transformation properties of the self-duality equations in this gauge provide an inductive proof of the ansätze. There is a six-parameter family of these Bäcklund transformations. They take real SU(2) gauge fields into real SU(1, 1) gauge fields and vice versa.

1. Introduction

During the past year a great deal of progress has been made towards an understanding of classical gauge field theory, particularly for an SU(2) gauge group in a four dimensional Euclidean space. If it is assumed that the field strengths are self-dual (or anti-self-dual) the analysis can be taken much further, though this assumption has, as yet, no particular physical motivation. Indeed Ward [1] has shown how all the information contained in a self-dual gauge field can be "coded" into the structure of certain analytic complex vector bundles, the isomorphism class of the bundle being determined by the gauge fields. More importantly, the isomorphism class of the bundle determines the gauge fields up to a gauge transformation and Ward showed how, in principle, the fields may be extracted from the bundles.

Taking this approach further, Atiyah and Ward [2] used basic theorems in geometry to argue that these bundles are necessarily algebraic and to restrict further the bundle structures to be considered to obtain all self-dual SU(2) gauge fields. Thus the problem of finding the $(8k-3)$-parameter family [3] of solutions was reduced to one in algebraic geometry. (Here k denotes the instanton number.) Their construction leads to a hierarchy of ansätze, $A_l(l=1,2,\ldots)$, the l-th one of which can be expressed in a form which has as input the components of a "spin $(l-1)$" massless anti-self-dual linear field. The first ansatz, A_1, had been known for

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some time [4]. Apart from this, Atiyah and Ward gave an explicit form only for A_2. In a recent letter [5] we reported simple explicit forms for A_l for larger values of l. In the present paper we will discuss more fully the derivation of those results and report further developments.

Our simple explicit form depended on performing the construction of Atiyah and Ward in a suitable gauge. Independently of the work of Atiyah and Ward, though exploiting basically related ideas, Yang [6] had shown the existence of a particularly convenient gauge, the R gauge. In this gauge, the potentials A^a_κ take the form,

$$A_\alpha = -\frac{1}{2f}\begin{pmatrix} \eta^3_{\alpha\beta}\partial_\beta f & \eta^{1-i2}_{\alpha\beta}\partial_\beta e \\ \eta^{1+i2}_{\alpha\beta}\partial_\beta g & -\eta^3_{\alpha\beta}\partial_\beta f \end{pmatrix} \tag{1.1}$$

where $A_\alpha = \frac{1}{2}A^a_\alpha \sigma^a$ (the σ's denote the Pauli matrices), $\eta^a_{\alpha\beta}$ is the tensor introduced by 't Hooft [7] and $\eta^{1\pm i2} = \eta^1 \pm i\eta^2$. Explicitly $\eta^a_{\alpha\beta}$ $(a=1,2,3)$ is a suitably oriented and normalized basis for anti-self-dual tensors which may be taken to be

$$\eta^a_{\alpha\beta} = \varepsilon_{0a\alpha\beta} + \delta_{a\alpha}\delta_{0\beta} - \delta_{a\beta}\delta_{0\alpha}\,. \tag{1.2}$$

The conditions that the field strengths computed from the potentials (1.1),

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta] \tag{1.3}$$

be self dual, i.e.

$$^*F_{\alpha\beta} \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta} = F_{\alpha\beta}\,, \tag{1.4}$$

are

$$f(f_{y\bar{y}} + f_{z\bar{z}}) - f_y f_{\bar{y}} - f_z f_{\bar{z}} - e_y g_{\bar{y}} - e_z g_{\bar{z}} = 0\,, \tag{1.5a}$$

$$f(e_{y\bar{y}} + e_{z\bar{z}}) - 2e_y f_{\bar{y}} - 2e_z f_{\bar{z}} = 0\,, \tag{1.5b}$$

$$f(g_{y\bar{y}} + g_{z\bar{z}}) - 2g_{\bar{y}} f_y - 2g_{\bar{z}} f_z = 0\,, \tag{1.5c}$$

where

$$\sqrt{2}\begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix} = x^0 - i\boldsymbol{x}\cdot\boldsymbol{\sigma} = x\,, \tag{1.6}$$

say, so that

$$\det x = x_\alpha x_\alpha\,. \tag{1.7}$$

(As usual $f_y = \partial f/\partial y$, etc.). Here $\bar{y}$, $\bar{z}$ denote variables independent of the complex conjugates, y^*, z^*, of y and z, so that real Euclidean space is specified by $\bar{y} = y^*$, $\bar{z} = z^*$. We are following Ward [1] and Yang [6] in considering the field equations over complexified space. Equations (1.5) are just like those found by Yang [6], becoming precisely his if we set $e = \varrho$, $f = \varphi$ and $g = -\bar{\varrho}$; we shall discuss further the relationship of the formalism used in this paper to his in Section 4.

The simplest form R_l, we have found for A_l may be specified by the following procedure for calculating the functions e, f and g from the $(2l-1)$ components $\Delta_r(x)$, $|r| \leqq l-1$, of a spin $(l-1)$ massless anti-self-dual linear field. The linear equations satisfied by the Δ_r are

$$\frac{\partial \Delta_r}{\partial y} = -\frac{\partial \Delta_{r+1}}{\partial \bar{z}}; \quad \frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \bar{y}}, \qquad -l+1 \leqq r \leqq l-2. \tag{1.8}$$

For $l \geqq 2$ these equations imply $\partial^2 \Delta_r = 0$ for each r, and for $l=1$ we replace Equation (1.8) by the single requirement that $\partial^2 \Delta_0 = 0$.

The ansatz A_1 is then given by [4]

$$e = f = g = 1/\Delta_0. \tag{1.9}$$

For $l \geqq 2$ to specify the form, R_l, of the ansatz A_l, it is convenient to define the $l \times l$ matrix $D^{(l)}$ by

$$D^{(l)}_{rs} = \Delta_{r+s-l-1} \qquad 1 \leqq r, s \leqq l. \tag{1.10}$$

If $\delta^{(l)} = [D^{(l)}]^{-1}$, the inverse matrix, e, f and g are provided by

$$e = \delta^{(l)}_{11}, \quad f = \delta^{(l)}_{1l} = \delta^{(l)}_{l1}, \quad g = \delta^{(l)}_{ll}. \tag{1.11}$$

In other words, Equations (1.8) and the construction of e, f, and g through Equations (1.10) and (1.11) guarantee that Equations (1.5) are satisfied, and hence that the vector potential of Equation (1.1) leads to self-dual field strengths.

To establish this it is possible to proceed in one of two ways. We first obtained the form R_l of the ansatz by explicitly implementing the construction described by Atiyah and Ward [2], seeking to write it in Yang's R gauge. To follow this path we begin in Section 2 by reviewing their constructions, as outlined in [1] and [2], making them sufficiently explicit for our purposes. This enables us to construct R_l in Section 3. Some of the algebraic details of this construction are relegated to Appendix A. It will be seen that there is a second and less economical form, R'_l, for the ansatz A_l in an R gauge.

The other way to establish the validity of R_l we gave in bare outline before [5]. It relies on some astonishing features of Yang's equations, which we originally stumbled on by chance, but are in essence elementary. What we have found are first order non-linear partial differential equations which relate solutions of Yang's equations and, in particular, relate R_l to $R_{l\pm 1}$. This makes possible an inductive proof of R_l depending only on the manipulation of partial derivatives and simple algebraic properties of matrices. These partial differential equations are remminiscent of the Bäcklund transformations associated with non-linear two dimensional field theories such as the Sine-Gordon theory [8]. In Section 4 we shall give severeal sets of these transformations, which we shall call Bäcklund transformations by analogy, and describe their roles in relation to the ansätze R_l and R'_l. Some calculational details are given in Appendix B.

In Section 5 we shall discuss the reality conditions to be satisfied by the functions appearing in the ansätze R_l in order for the vector potentials to be real, so that we are indeed discussing an SU(2) rather than an SL(2, $\mathbb{C}$) gauge theory (although the physical significance of requiring that the potentials, A^a_α, rather than the field strengths, $F^a_{\alpha\beta}$, be real is not entirely clear to us). The reality conditions have the curious property with respect to the Bäcklund transformations that they are not preserved but reversed in the sense that the reality condition for SU(2) converts into that for SU(1, 1) and vice versa.

An important insolved problem is that of the singularity structure [2] permitted in the functions, Δ_r, occurring in the ansätze; this corresponds to inclusion of sources in the Equations (1.8). However we feel that the Bäcklund transformations, which can certainly introduce singularities, may indeed provide a natural method for generating singularities in the vector potentials which are not reflected in the field strengths.

2. The Procedure of Atiyah and Ward

The basic observation which was exploited by both Ward [1] and Yang [6] was that if $F_{\alpha\beta}$ is self-dual its components in any anti-self-dual plane vanish. Consequently restricted to such plane, the tangential components of the potentials are a gauge transformation from the vacuum. We shall consider this statement in greater detail. We will be working in $\mathbb{C}^4$, the complexification of four-dimensional Euclidean space.

An anti-self-dual (two-dimensional) plane [1] is one for which the tensor

$$\Omega_{\alpha\beta} = V_\alpha W_\beta - W_\alpha V_\beta$$

is anti-self-dual ($\Omega = -{}^*\Omega$) for any displacements, V, W, in the plane. It is easy to see that *every* displacement in such a plane is null. Conversely every null displacement is contained in two totally null two-dimensional planes, through any given point, one self-dual and the other anti-self-dual. Let us use α and β to refer to the families of self-dual and anti-self-dual planes, respectively. A convenient way of labelling β planes is by equations of the form

$$\omega = x\pi \tag{2.1}$$

where ω and π are 2-spinors and x is the matrix of Equation (1.6). For given ω and π, Equation (2.1) defines a β plane. It is easy to see that displacements in the plane it defines are always null. For, if in addition to Equation (2.1), $\omega = y\pi$,

$$(x-y)\pi = 0 \tag{2.2}$$

and

$$\det(x-y) = 0$$

which, by Equation (1.7) implies that $(x_\alpha - y_\alpha)$ is null. Anti-self-duality is not difficult to check by explicit calculation. Let θ stand for the 4-spinor formed from π and ω,

$$\theta = \begin{pmatrix} \pi \\ \omega \end{pmatrix}. \tag{2.3}$$

It is clear that $\lambda\theta$ and θ both define the same β plane for every $\lambda \in \mathbb{C}$. In fact the β planes in $\mathbb{C}^4$ are in one-one correspondence with the points of complex projective three-dimensional space $\mathbb{C}P_3$. [To make this statement accurate we have to consider (complexified) compactified Euclidean space. If we consider only $\mathbb{C}^4$ the complex projective line given by $\pi = 0$ is missing from $\mathbb{C}P_3$.] This is the dual projective twistor space [9, 1]. Let us denote the point of $\mathbb{C}P_3$ corresponding to θ, $\in\mathbb{C}^4$, by $[\theta]$, so that

$$[\theta]=\{\lambda\theta:\lambda\in\mathbb{C}\} \tag{2.4}$$

and denote the corresponding β plane by

$$\beta_{[\theta]}=\{x:\omega=x\pi\}. \tag{2.5}$$

For any given β plane, $\beta_{[\theta]}$, we may integrate the equation

$$A_\alpha dx_\alpha=-ig^{-1}\partial_\alpha g dx_\alpha, \qquad (dx)\pi=0, \tag{2.6}$$

between any two points, x and y, of the plane to obtain a group element $g_{[\theta]}(x,y)\in\mathrm{SL}(2,\mathbb{C})$, the complexification of SU(2), provided that the field strength, $F_{\alpha\beta}$, obtained from A_α is self-dual. This is because the integrability conditions for Equation (2.5) over the plane $\beta_{[\theta]}$ amount to the vanishing of the components of $F_{\alpha\beta}$ in that plane. Formally,

$$g_{[\theta]}(x,y)=\mathscr{T}\exp\left\{i\int_x^y A_\alpha dx_\alpha\right\} \tag{2.7}$$

where the integral is path ordered and the path of integration lies entirely within $\beta_{[\theta]}$. Given $g_{[\theta]}(x,y)$, we may reconstruct the components of A_α in the plane $\beta_{[\theta]}$ using Equation (2.6).

What Equation (2.7) principally enables us to do is to define the parallel transport of 2-spinors over the plane $\beta_{[\theta]}$. We may introduce the vector space, $V_{[\theta]}$, of 2-spinor fields, $\psi_{[\theta]}(x)$ defined over $\beta_{[\theta]}$, whose values at different points are related by the SL(2, $\mathbb{C}$) element defined by Equation (2.7):

$$\psi_{[\theta]}(x)=g_{[\theta]}(x,y)\psi_{[\theta]}(y), \qquad x,y\in\beta. \tag{2.8}$$

$V_{[\theta]}$ is a two-dimensional vector space since the value of $\psi_{[\theta]}$ at any point $x\in\beta_{[\theta]}$ determines its value throughout $\beta_{[\theta]}$. The family of spaces $V_{[\theta]}$ forms a two-dimensional analytic vector bundle over $\mathbb{C}P_3$ [1].

To set up coordinates on this analytic vector bundle we need to pick an $x_{[\theta]}\in\beta_{[\theta]}$ for each $[\theta]$ and specify the value of $\psi_{[\theta]}(x_{[\theta]})$. This then determines $\psi_{[\theta]}(x)$ everywhere, given $g_{[\theta]}(x,y)$. But it is not possible to choose $x_{[\theta]}\in\beta_{[\theta]}$ smoothly throughout $\mathbb{C}P_3$ or even the space $\mathbb{C}P_3\sim\mathbb{C}P_1$ we obtain by omitting those $[\theta]$ for which $\pi=0$. In the latter case we may conveniently cover the space by two coordinate patches

$$x^1_{[\theta]}=\begin{pmatrix}\omega_1/\pi_1 & 0\\ \omega_2/\pi_1 & 0\end{pmatrix}, \qquad \pi_1\neq0, \tag{2.9a}$$

$$x^2_{[\theta]}=\begin{pmatrix}0 & \omega_1/\pi_2\\ 0 & \omega_2/\pi_2\end{pmatrix}, \qquad \pi_2\neq0. \tag{2.9b}$$

The coordinates in these two patches are related by

$$\psi_{[\theta]}(x^1_{[\theta]})=g_{[\theta]}(x^1_{[\theta]},x^2_{[\theta]})\psi_{[\theta]}(x^2_{[\theta]}).$$

The isomorphism class of the bundle is specified by the transition function

$$g(\omega, \pi)=g_{[\theta]}(x^1_{[\theta]}, x^2_{[\theta]}) \tag{2.10}$$

which is manifestly homogeneous: $g(\omega, \pi)=g(\lambda\omega, \lambda\pi)$, $\lambda\in\mathbb{C}$. This class will be unchanged if g is multiplied by elements of SL(2, $\mathbb{C}$) on the left and right, analytic in $\pi_1 \neq 0$ and $\pi_2 \neq 0$ respectively.

Ward [1] pointed out that the potentials A_α may be regained from $g(\omega, \pi)$ by exploiting the fact that

$$g(\omega, \pi)=g_{[\theta]}(x^1_{[\theta]}, x)g_{[\theta]}(x, x^2_{[\theta]}) \quad \text{for each} \quad x\in\beta_{[\theta]}.$$

Thus, writing

$$h(x, \zeta)=g_{[\theta]}(x^1_{[\theta]}, x) \quad \text{and} \quad k(x, \zeta)=g_{[\theta]}(x^2_{[\theta]}, x),$$

where $\zeta=\pi_1/\pi_2$,

$$g(x\pi, \pi)=h(x, \zeta)k(x, \zeta)^{-1}, \tag{2.11}$$

where, for fixed ζ, $h(x, \zeta)$ is analytic away from $\zeta=0$ and $k(x, \zeta)$ is analytic away from $\zeta=\infty$. With these analyticity requirements, it follows from Liouville's theorem that Equation (2.11) uniquely determines h and k up to a gauge transformation:

$$\begin{aligned} &h(x, \zeta)\to h(x, \zeta)\gamma(x), \\ &k(x, \zeta)\to k(x, \zeta)\gamma(x). \end{aligned} \tag{2.12}$$

Now, from Equation (2.6),

$$A_{i1}(x)-\zeta A_{i2}(x)=-ih(x, \zeta)^{-1}\left(\frac{\partial}{\partial x_{i1}}-\zeta\frac{\partial}{\partial x_{i2}}\right)h(x, \zeta) \tag{2.13a}$$

$$=-ik(x, \zeta)^{-1}\left(\frac{\partial}{\partial x_{i1}}-\zeta\frac{\partial}{\partial x_{i2}}\right)k(x, \zeta) \tag{2.13b}$$

since $(dx)\pi=0$ is satisfied if $dx_{i2}=-\zeta dx_{i1}$. [The components of A in Equations (2.13) are defined by $A_{ij}dx_{ij}=A_\alpha dx_{\alpha}$.] Thus the isomorphism class of the bundle determines the gauge potentials, A_α, up to gauge equivalence.

Conversely, we may also argue, again following Ward [1], that any homogeneous function, $g(\omega, \pi)$, with a suitable domain of analyticity, will yield an anti-self-dual field strength, provided that g may be split as in Equation (2.11). For then the equality of the right hand sides of Equations (2.13) follows from

$$D_i g=0, \quad \text{where} \quad D_i=\frac{\partial}{\partial x_{i1}}-\zeta\frac{\partial}{\partial x_{i2}} \tag{2.14}$$

and g is considered as a function of x and ζ. That these expression have the form of $A_{i1}(x)-\zeta A_{i2}(x)$ and that this potential does indeed yield a self-dual field strength follows from Liouville's theorem.

Atiyah and Ward [2] argue that it is sufficient to take

$$g=\begin{pmatrix}\zeta^{l} & \varrho(x,\zeta)\\ 0 & \zeta^{-l}\end{pmatrix} \tag{2.15}$$

where ϱ depends on x and ζ only through the variables $x_{11}\zeta+x_{12}$, $x_{21}\zeta+x_{22}$ and ζ. Consequently,

$$D_i\varrho=0. \tag{2.16}$$

We shall explicitly extract the gauge potentials from Equation (2.15) in the next section.

3. Explicit Construction of the Atiyah-Ward Ansätze

Let us write the matrices occuring in Equation (2.11) as

$$h=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad k=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \tag{3.1}$$

where $\alpha\delta-\beta\gamma=ad-bc=1$ and $\alpha,\beta,\gamma,\delta$ are regular as functions of ζ, except at $\zeta=0$ whilst a,b,c,d are regular except at $\zeta=\infty$. Then from Equations (2.11) and (2.15),

$$c=\gamma\zeta^{l}, \quad d=\delta\zeta^{l}, \tag{3.2a}$$

$$a\zeta^{l}+\varrho c=\alpha, \quad b\zeta^{l}+\varrho d=\beta. \tag{3.2b}$$

It follows from Equations (3.2a) that c and d are polynomials of degree at most l:

$$c(x,\zeta)=\sum_{r=0}^{l} c_r(x)\zeta^{r}, \quad d(x,\zeta)=\sum_{r=0}^{l} d_r(x)\zeta^{r}. \tag{3.3}$$

Further the coefficient of ζ^r in the Laurent series of $\alpha-a\zeta^l$ vanishes for $0<r<l$. Thus

$$\oint\frac{d\zeta}{\zeta}\varrho c\zeta^{-r}=\oint\frac{d\zeta}{\zeta}\varrho d\zeta^{-r}=0, \quad 0<r<l, \tag{3.4}$$

where the integration contours encircle the origin. These equations may be written,

$$\sum_{s=0}^{l} c_s\Delta_{s-r}=\sum_{s=0}^{l} d_s\Delta_{s-r}=0, \quad 0<r<l, \tag{3.5}$$

where

$$\Delta_r(x)=\frac{1}{2\pi i}\oint\frac{d\zeta}{\zeta}\varrho(x,\zeta)\zeta^{r}. \tag{3.6}$$

(This is the Penrose transform.) As a consequence of Equation (2.16) the functions Δ_r satisfy,

$$\frac{\partial \Delta_r}{\partial y} = -\frac{\partial \Delta_{r+1}}{\partial \bar{z}}, \quad \frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \bar{y}}. \tag{3.7}$$

We may express the functions a and b in terms of c and d,

$$a(x,\zeta) = -\frac{1}{2\pi i}\oint \frac{d\xi}{\xi-\zeta}\xi^{-l}\varrho(x,\xi)c(x,\xi) \tag{3.8a}$$

$$b(x,\zeta) = -\frac{1}{2\pi i}\oint \frac{d\xi}{\xi-\zeta}\xi^{-l}\varrho(x,\xi)d(x,\xi) \tag{3.8b}$$

where the integration contours positively encircle the origin in $|\xi|>|\zeta|$. The Equation (3.5) leaves four of the $2l+2$ functions c_r, d_r, $0\leqq r\leqq l$ arbitrary and the constraint $ad-bc=1$ further reduces this by one. The remaining freedom is exactly that reflecting gauge invariance. From functions a, b, c, d satisfying these constraints we may construct the potentials using Equation (2.13a):

$$i[A_{i1}(x)-\zeta A_{i2}(x)] = \begin{pmatrix} dD_i a - bD_i c & dD_i b - bD_i d \\ aD_i c - cD_i a & -dD_i a + bD_i c \end{pmatrix}. \tag{3.9}$$

Making use of calculations performed in Appendix A we obtain

$$dD_i a - bD_i c = d_0\frac{\partial a_0}{\partial x_{i1}} - b_0\frac{\partial c_0}{\partial x_{i1}} - \zeta\left\{d_l\frac{\partial \alpha_0}{\partial x_{i2}} - \beta_0\frac{\partial c_l}{\partial x_{i2}}\right\} \tag{3.10}$$

$$\begin{aligned} aD_i c - cD_i a &= \alpha_0\frac{\partial c_l}{\partial x_{i1}} - c_l\frac{\partial \alpha_0}{\partial x_{i1}} - \alpha_0\frac{\partial c_{l-1}}{\partial x_{i2}} + c_{l-1}\frac{\partial \alpha_0}{\partial x_{i2}} \\ &\quad - \alpha_{-1}\frac{\partial c_l}{\partial x_{i2}} + c_l\frac{\partial \alpha_{-1}}{\partial x_{i2}} - \zeta\left\{\alpha_0\frac{\partial c_l}{\partial x_{i2}} - c_l\frac{\partial \alpha_0}{\partial x_{i2}}\right\} \end{aligned} \tag{3.11}$$

$$\begin{aligned} dD_i b - bD_i d &= d_0\frac{\partial b_0}{\partial x_{i1}} - b_0\frac{\partial d_0}{\partial x_{i1}} - \zeta\left\{d_0\frac{\partial b_0}{\partial x_{i2}} - b_0\frac{\partial d_0}{\partial x_{i2}}\right. \\ &\quad \left. - d_0\frac{\partial b_1}{\partial x_{i1}} + b_1\frac{\partial d_0}{\partial x_{i1}} - d_1\frac{\partial b_0}{\partial x_{i1}} + b_0\frac{\partial d_1}{\partial x_{i1}}\right\} \end{aligned} \tag{3.12}$$

where α_s, β_s, a_s, and b_s stand for the coefficient of ζ^s in the Laurent series of α, β, a, and b, respectively, and may be expressed in terms of Δ_s, c_s, and d_s using Equations (3.2b). The constraint that $ad-bc=1$ yields

$$\alpha_0 d_l - \beta_0 c_l = a_0 d_0 - b_0 c_0 = 1. \tag{3.13}$$

It is now clear that the way of removing the remaining gauge freedom, which yields the simplest results, is to take

$$d_0 = c_l = 0, \tag{3.14}$$

and

$$c_0^2 = d_l^2 = f, \tag{3.15}$$

say. Then

$$dD_ia - bD_ic = \frac{1}{2f}\left(\frac{\partial f}{\partial x_{i1}} + \zeta \frac{\partial f}{\partial x_{i2}}\right), \tag{3.16a}$$

$$aD_ic - cD_ia = -\frac{1}{f}\frac{\partial g}{\partial x_{i2}} \tag{3.16b}$$

$$dD_ib - bD_id = \frac{\zeta}{f}\frac{\partial e}{\partial x_{i1}} \tag{3.16c}$$

where

$$e = c_0 d_1 \quad \text{and} \quad g = c_{l-1} d_l . \tag{3.17}$$

Equations (3.9) and (3.16) are exactly of the form of Equation (1.1) and so we have succeeded in writing the ansatz A_l in Yang's R gauge. The Equation (3.14) reduces Equation (3.5) to two sets of $(l-1)$ homogeneous equations in l variables making the calculation of the ratios c_r/c_0 and d_r/d_l in terms of determinants of the Δ's a simple exercise in linear algebra. Then, using Equations (3.13) and (3.15), we obtain the expressions for e, f, and g given in Equations (1.11) which constitute the form R_l of the ansatz A_l. These expressions involve only $\Delta_r(x)$ for $|r| \leqq l-1$, on which the only restrictions are Equations (3.7).

Another way of simplifying Equations (3.10)–(3.12) is to take

$$\alpha_0 = b_0 = 0 \tag{3.18}$$

instead of Equation (3.14). Then we obtain again expressions of the form of Equation (3.16) but with

$$c_l^2 = d_0^2 = 1/f \tag{3.19}$$

and

$$g = \alpha_{-1}/c_l , \qquad e = b_1/d_0 . \tag{3.20}$$

Again we may explicitly evaluate e, f, and g in terms of the Δ's. This time Equation (3.18) supplements Equation (3.5), so that there r runs from 0 to l inclusive, providing two sets of $l+1$ homogeneous equations in $l+2$ unknowns. Proceeding as before we obtain a second form, R_l' of the ansatz A_l:

$$e = (\mathrm{adj}\, D^{(l+2)})_{l+2,l+2}/\det D^{(l)} \tag{3.21a}$$

$$f = (\mathrm{adj}\, D^{(l+2)})_{l+2,1}/\det D^{(l)} \tag{3.21b}$$

$$g = (\mathrm{adj}\, D^{(l+2)})_{1,1}/\det D^{(l)} \tag{3.21c}$$

where $\mathrm{adj}\, D^{(l)}$ denotes the adjugate matrix of $D^{(l)}$.

Note that the elegant forms R_l and R_l' of the ansatz are artefacts of the choice of gauge; the general expressions given in Equations (3.10)–(3.12) are much less concise. As we remarked in the introduction, this gauge is special because of the rather magical properties of the self-duality equations in it, Yang's equations. These properties, which take the form of transformations on e, f, and g, are discussed in the next section.

4. Bäcklund Transformations

It is well known [8] that for certain non-linear partial differential equations there exist transformations, called Bäcklund transformations, (which take the form of non-linear first order partial differential equations) yielding solutions of the equation from solutions of the same equation. For example, for the Sine-Gordon equation

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=\sin\varphi \tag{4.1}$$

the Bäcklund transformation takes the form

$$\begin{aligned}\frac{\partial\varphi}{\partial\xi}+\frac{\partial\psi}{\partial\xi}&=k\sin\left(\frac{\varphi-\psi}{2}\right)\\ \frac{\partial\varphi}{\partial\eta}-\frac{\partial\psi}{\partial\eta}&=\frac{1}{k}\sin\left(\frac{\varphi+\psi}{2}\right)\end{aligned} \tag{4.2}$$

where $\xi=t+x$, $\eta=t-x$. The consistency of Equations (4.2) requires both that φ and that ψ satisfy the Sine-Gordon Equation (4.1). In practice it enables a solution φ to be calculated from a given solution ψ of Equation (4.1). The parameter k is a reflection of the Lorentz invariance of Equation (4.1): $\xi\to\xi/k$; $\eta\to k\eta$. This parameter may be varied each time the transformation is applied, so that the solutions generated by N applications of the transformation depend on at least N parameters, $k_1,k_2,\dots,k_N$, as well as constants of integration.

The Bäcklund transformations we have found for Yang's Equations (1.5) have a dependence on a large number of parameters corresponding to the large group of algebraic transformations that can be performed on (e,f,g) whilst preserving Equations (1.5). Here we can only point out how these parameters arise and we do not understand if they are related in any systematic way to the $(8k-3)$ parameters of a k instanton solution to the self-dual Yang-Mills equations. The problem of understanding these $(8k-3)$ parameters is clearly intimately related to the problem of singularities in Equations (1.8) which remains open.

These transformations share with Equations (4.2) the property of taking one solution of a set of non-linear differential equations into another. They do not possess the stronger property that the mere consistency of the transformation implies that both the input functions and the outpout functions satisfy the equations in question. It is not clear that this is a serious disadvantage in practice. We shall present our emperical findings in the form of three lemmas. Suppose that (e,f,g) provide a solution of Yang's Equations (1.5).

Lemma α. *A solution of Yang's equation is given by* $(\varepsilon,\varphi,\gamma)$ *provided that* $\varepsilon\gamma\neq\varphi^2$ *and*

$$f=-(\varepsilon\gamma-\varphi^2)/\varphi\,, \tag{4.3a}$$

$$\frac{\partial}{\partial\bar{y}}\left(\frac{\varepsilon}{\varepsilon\gamma-\varphi^2}\right)=-\frac{1}{f^2}\frac{\partial e}{\partial z},\qquad \frac{\partial}{\partial\bar{z}}\left(\frac{\varepsilon}{\varepsilon\gamma-\varphi^2}\right)=\frac{1}{f^2}\frac{\partial e}{\partial y}, \tag{4.3b}$$

$$\frac{\partial}{\partial z}\left(\frac{\gamma}{\varepsilon\gamma-\varphi^2}\right)=-\frac{1}{f^2}\frac{\partial g}{\partial\bar{y}},\qquad \frac{\partial}{\partial y}\left(\frac{\gamma}{\varepsilon\gamma-\varphi^2}\right)=\frac{1}{f^2}\frac{\partial g}{\partial\bar{z}}. \tag{4.3c}$$

Lemma β. *A solution of Yang's equations is given by* $(\varepsilon, \varphi, \gamma)$ *provided that*

$$\varphi = 1/f, \tag{4.4a}$$

$$\frac{\partial \varepsilon}{\partial z} = -\frac{1}{f^2}\frac{\partial g}{\partial \bar{y}}, \quad \frac{\partial \varepsilon}{\partial y} = \frac{1}{f^2}\frac{\partial g}{\partial \bar{z}}, \tag{4.4b}$$

$$\frac{\partial \gamma}{\partial \bar{z}} = \frac{1}{f^2}\frac{\partial e}{\partial y}, \quad \frac{\partial \gamma}{\partial \bar{y}} = -\frac{1}{f^2}\frac{\partial e}{\partial z}. \tag{4.4c}$$

Lemma γ. *A solution of Yang's equations is given by* $(\varepsilon, \varphi, \gamma)$ *where*

$$\begin{pmatrix} \varepsilon & \varphi \\ \varphi & \gamma \end{pmatrix} = (aP+b)(cP+d)^{-1} \tag{4.5a}$$

if $P = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ *and a, b, c and d are diagonal* 2×2 *matrices such that*

$$ad - bc = 1. \tag{4.5b}$$

The proofs of the lemmas are completely straightforward. Lemma α may be obtained from β by using the particular case, γ_0 say, of γ in which the right hand side of Equation (4.5a) is just P^{-1}. The symmetry of the left hand side of Equation (4.5a) is guaranteed by Equation (4.5b). We have not been able to show that the combination of β with γ produces all possible Bäcklund transformations. The transformation α is given a separate status because it plays an important role: it provides an inductive proof of R_l.

To see how this inductive proof works consider the first step from R_1 to R_2. For R_1 we have $e = f = g = 1/\Delta_0$ with $\partial^2 \Delta_0 = 0$. Lemma α yields a new solution $(\varepsilon, \varphi, \gamma)$ defined by Equations (4.3), which become

$$\varphi/(\varepsilon\gamma - \varphi^2) = -\Delta_0 \tag{4.6a}$$

$$\frac{\partial}{\partial \bar{y}}\left(\frac{\varepsilon}{\varepsilon\gamma - \varphi^2}\right) = \frac{\partial \Delta_0}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}\left(\frac{\varepsilon}{\varepsilon\gamma - \varphi^2}\right) = -\frac{\partial \Delta_0}{\partial y}, \tag{4.6b}$$

$$\frac{\partial}{\partial z}\left(\frac{\gamma}{\varepsilon\gamma - \varphi^2}\right) = \frac{\partial \Delta_0}{\partial \bar{y}}, \quad \frac{\partial}{\partial y}\left(\frac{\gamma}{\varepsilon\gamma - \varphi^2}\right) = -\frac{\partial \Delta_0}{\partial \bar{z}}. \tag{4.6c}$$

Thus we are led to introduce two new functions Δ_1, Δ_{-1} such that

$$\frac{\partial \Delta_0}{\partial z} = \frac{\partial \Delta_1}{\partial \bar{y}}, \quad \frac{\partial \Delta_0}{\partial \bar{y}} = -\frac{\partial \Delta_1}{\partial \bar{z}} \tag{4.7a}$$

and

$$\frac{\partial \Delta_0}{\partial \bar{z}} = -\frac{\partial \Delta_{-1}}{\partial y}, \quad \frac{\partial \Delta_0}{\partial \bar{y}} = \frac{\partial \Delta_{-1}}{\partial z}. \tag{4.7b}$$

We may then integrate Equations (4.7) to obtain what is indeed the ansatz R_2:

$$\begin{pmatrix} \varepsilon & \varphi \\ \varphi & \gamma \end{pmatrix} = \begin{pmatrix} \Delta_{-1} & \Delta_0 \\ \Delta_0 & \Delta_1 \end{pmatrix}^{-1}. \tag{4.8}$$

To prove the general inductive step that Lemma α takes us from R_{l-1} to R_l we need to use a result due to Jacobi [10] on the sub-determinants of an adjugate matrix and a differential equation (4.11) satisfied by elements of $\tilde{D}^{(l)} = \operatorname{adj} D^{(l)}$ proved in Appendix B. If $(\varepsilon, \varphi, \gamma)$ are given by R_l,

$$\begin{aligned} \varepsilon\varphi - \gamma^2 &= \{\tilde{D}^{(l)}_{11}\tilde{D}^{(l)}_{ll} - \tilde{D}^{(l)}_{1l}\tilde{D}^{(l)}_{l1}\}/\{\det D^{(l)}\}^2 \\ &= \det D^{(l-2)}/\det D^{(l)} \end{aligned} \tag{4.9}$$

using Jacobi's result (stated in Appendix B). Now, since

$$\begin{aligned} \varphi &= (-1)^{l+1} \det D^{(l-1)}/\det D^{(l)}, \\ (\varepsilon\gamma - \varphi^2) &= -(-1)^l \det D^{(l-2)}/\det D^{(l-1)} = -f \end{aligned} \tag{4.10}$$

if (e, f, g) are given by R_{l-1}. Now, further, it is proved in Appendix B that

$$\tilde{D}^{(l)}_{1l} \frac{\partial \tilde{D}^{(l-1)}_{11}}{\partial y} - \tilde{D}^{(l-1)}_{11} \frac{\partial D^{(l)}_{1l}}{\partial y} = \tilde{D}^{(l)}_{11} \frac{\partial D^{(l-1)}_{1,l-1}}{\partial \bar{z}} - \tilde{D}^{(l-1)}_{1,l-1} \frac{\partial \tilde{D}^{(l)}_{11}}{\partial \bar{z}} \tag{4.11}$$

which implies

$$\left[\frac{\tilde{D}^{(l)}_{1l}}{\tilde{D}^{(l-1)}_{1,l-1}}\right]^2 \frac{\partial}{\partial y}\left[\frac{\tilde{D}^{(l-1)}_{11}}{\tilde{D}^{(l)}_{1l}}\right] = -\frac{\partial}{\partial \bar{z}}\left[\frac{\tilde{D}^{(l)}_{11}}{\tilde{D}^{(l-1)}_{1,l-1}}\right]. \tag{4.12}$$

From R_l we see that

$$\varepsilon = \tilde{D}^{(l)}_{11}/\det D^{(l)}$$

and so

$$\begin{aligned} \varepsilon/(\varepsilon\gamma - \varphi^2) &= \tilde{D}^{(l)}_{11}/\det D^{(l-2)} \\ &= (-1)^l \tilde{D}^{(l)}_{11}/\tilde{D}^{(l-1)}_{1,l-1}, \end{aligned} \tag{4.13}$$

whilst from R_{l-1},

$$e = (-1)^{l+1}\tilde{D}^{(l-1)}_{11}/\tilde{D}^{(l)}_{1l} \tag{4.14}$$

and

$$f = (-1)^{l+1}\tilde{D}^{(l-1)}_{1,l-1}/\tilde{D}^{(l)}_{1l}. \tag{4.15}$$

From Equations (4.12)–(4.15) we see that the second of Equations (4.3b) is satisfied with $(\varepsilon, \varphi, \gamma)$ given by R_l and (e, f, g) given by R_{l-1}. That the same is true of the rest of Equations (4.3) follows by similar arguments. This completes the inductive proof of R_l using Bäcklund transformations.

Applying the transformation of Lemma β to R_{l+1} yields R'_l, the less economical form of the ansatz A_l in an R gauge, provided by Equations (3.21). The transformation α applied to R'_l yields R'_{l-1} rather than R'_{l+1}. The proofs of these statements use exactly the same techniques as we have just used to show α takes us from R_{l-1} to R_l. The transformation α followed by β may be integrated to give the gauge transformation relating R_l and R'_l. On the other hand, β followed by α may be integrated to give the special case γ_0 of γ.

We may summarise these relationships in the diagram:

$$\begin{tikzcd}
R_{l+2} \arrow[dr, leftrightarrow, "\beta", "\gamma_0"' near end] \arrow[ddr, leftrightarrow, "\gamma_0" near end] & R'_{l+2} \arrow[d, "\alpha"] \\
R_{l+1} \arrow[u, "\alpha"] \arrow[dr, leftrightarrow, "\beta"] \arrow[ddr, leftrightarrow, "\gamma_0" near end] & R_{l+1} \arrow[d, "\alpha"] \\
R_l \arrow[u, "\alpha"] \arrow[dr, leftrightarrow, "\beta"] & R'_l \arrow[d, "\alpha"] \\
R_{l-1} \arrow[u, "\alpha"] & R'_{l-1} .
\end{tikzcd}$$

The transformation β also plays another role. It relates the functions used by Yang [6] to ours, if we set $\varepsilon=\varrho$ and $\gamma=-\bar{\varrho}$. Since $\beta\gamma_0\beta$ may be integrated to give a gauge transformation, we see that the application of the purely algebraic transformation γ_0 on Yang's variables may be a gauge transformation. It has been explicitly verified that all the transformations γ effect gauge transformations on Yang's variables.

5. Further Developments

In this section we shall make some brief remarks about the reality conditions and possible singularities of the Δ's. In order to obtain real SU(2) gauge vector potentials, A^a_α, we impose the condition $A_\alpha = A^\dagger_\alpha$ for real x on Equation (1.1). In particular, this gives

$$\eta^3_{\alpha\beta}\partial_\beta f/f = \eta^3_{\alpha\beta}\partial_\beta f^*/f^* \tag{5.1}$$

which implies that f must be real apart from a constant phase factor. Since A_α is unchanged if we scale e, f, and g by the same constant we may assume that f is real for real SU(2) potentials, without loss of generality. The remaining condition is that

$$\eta^{1+i2}_{\alpha\beta}\partial_\beta e^*/f^* = \eta^{1+i2}_{\alpha\beta}\partial_\beta g/f \tag{5.2}$$

which yields $e^*=g$ assuming f to be real.

However, the Bäcklund transformations α and β are not consistent with the maintenance of the reality conditions for SU(2):

$$f=f^*, e^*=g \quad \text{when} \quad \bar{y}=y^* \quad \text{and} \quad \bar{z}=z^*. \tag{5.3}$$

Rather, they are consistent with the resulting solution satisfying the different reality conditions:

$$\varphi=\varphi^*, \varepsilon^*=-\gamma \quad \text{when} \quad \bar{y}=y^* \quad \text{and} \quad \bar{z}=z^*. \tag{5.4}$$

Since, just as σ_1, σ_2, σ_3 satisfy an SU(2) algebra, $i\sigma_1$, $i\sigma_2$, and σ_3 satisfy an SU(1, 1) algebra, (5.4) gives the reality conditions appropriate to real SU(1, 1) potentials. As regards reality conditions these Bäcklund transformations alternate between SU(2) and SU(1, 1).

Just as in the case of the Bäcklund transformation (4.2) for the Sine-Gordon equation, there are parameters that can be adjusted each time the transformation is performed. In the case of Yang's equations, these parameters result from the combination of transformation β with the six dimensional group of algebraic invariances of Yang's equations given in Lemma γ. The parameter k in the Sine-Gordon Bäcklund transformation may be regarded as a reflection of the Lorentz invariance of Equation (4.1). We shall see that only one of the six parameters present in the most general Bäcklund transformation we have found for Yang's equation may be obtained in an analogous way.

Because we have chosen a gauge, Yang's equations lack the full SO(4) invariance, let alone conformal invariance, of the original Euclidean Yang-Mills field theory, or the self duality Equation (1.4). However they are invariant under the subgroup of orthogonal transformations which preserve each two-dimensional plane in the two families y, z constant and $\bar{y}$, $\bar{z}$ constant, as well as being invariant under dilations. To see this let us define

$$\partial=\begin{pmatrix} \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \quad \text{and} \quad \bar{\partial}=\begin{pmatrix} \dfrac{\partial}{\partial \bar{y}} \\ \dfrac{\partial}{\partial \bar{z}} \end{pmatrix}, \tag{5.5}$$

so that the Laplacian $\partial^2=\partial\cdot\bar{\partial}$ and Yang's equations may be written

$$\begin{aligned} &f\partial\cdot\bar{\partial}f-\partial f\cdot\bar{\partial}f-\partial e\cdot\bar{\partial}g=0, \\ &f\partial\cdot\bar{\partial}e-2\partial e\cdot\bar{\partial}f=0, \\ &f\partial\cdot\bar{\partial}g-2\bar{\partial}g\cdot\partial f=0. \end{aligned} \tag{5.6}$$

The condition that these equations be invariant under the transformation

$$\begin{pmatrix} y' \\ z' \end{pmatrix}=H\begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} \bar{y}' \\ \bar{z}' \end{pmatrix}=\bar{H}\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix}, \tag{5.7}$$

where H, $\bar{H}$ are 2×2 matrices, is that $\bar{H}^T H$ be a multiple, λ, of the identity. The Bäcklund transformation, Equations (4.4), may be written

$$\partial\varepsilon=\frac{1}{f^2}\chi\bar{\partial}g, \qquad \bar{\partial}\gamma=-\frac{1}{f^2}\chi\partial e \tag{5.8}$$

where

$$\chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Performing the change of variables (5.7) on Equations (5.8) we obtain

$$\partial' \varepsilon = \frac{1}{f^2} H\chi \bar{H}^{-1} \bar{\partial}' g, \qquad \bar{\partial}' \gamma = -\frac{1}{f^2} \bar{H} \chi H^{-1} \partial' e. \tag{5.9}$$

But since $H\chi H^T = \chi \det H$ we may rewrite Equations (5.9) as

$$\partial' \varepsilon = \frac{k}{f^2} \chi \bar{\partial}' g, \qquad \bar{\partial}' \gamma = -\frac{1}{kf^2} \chi \partial' e, \tag{5.10}$$

where $k = \det H/\lambda = \lambda/\det \bar{H}$. So, perhaps surprisingly, the spatial symmetries of Yang's equations leads to only one parameter in the Bäcklund transformation. It occurs in just the same way as for the Sine-Gordon equations. Further, it is amongst those provided by Lemma γ. In addition to those six parameters there will be constants of integration.

We do not yet know what happens to the action integral under a Bäcklund transformation. For finite action solutions, e, f, and g are singular on specific curves in the four-dimensional complexied Euclidean space we are considering. It is not known how to classify these curves [2], but consideration of explicit examples shows that the Bäcklund transformations change the nature of these singularities. One may hope that they will provide a route for a better understanding of this problem. The singularities generated by the Bäcklund transformation are consistent with the linear Equations (1.8) satisfied by the Δ's provided sources are added. Much further work needs to be done in this direction.

The Bäcklund transformations further the analogy between the self-dual Yang-Mills equations and the exactly soluble two dimensional theories. An analogue of the inverse scattering method has been given by Belavin and Zakharov [11]. Further development of this analogy may lead to the uncovering of an infinity of conservation laws for the self-duality equations.

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Appendix A. Calculation of $h^{-1}D_i h$

We shall first discuss Equations (3.13). Using Equations (3.8),

$$\begin{aligned} ad - bc &= -\frac{1}{2\pi i} \oint \frac{d\xi}{\xi - \zeta} \xi^{-l} \varrho(x,\xi) \{c(\xi)d(\zeta) - d(\xi)c(\zeta)\} \\ &= \frac{1}{2\pi i} \oint d\xi \xi^{-1} \varrho(x,\xi) \left\{ c(\xi) \left[\frac{d(\xi) - d(\zeta)}{\xi - \zeta} \right] - d(\xi) \left[\frac{c(\xi) - c(\zeta)}{\xi - \zeta} \right] \right\} \end{aligned} \tag{A1}$$

where the integration, as throughout this Appendix, is as in Equations (3.8). Each of the terms in square brackets in Equation (A1) is a polynomial in ξ of degree

$(l-1)$. By Equation (3.4) none of these terms will contribute except those involving ξ^{l-1}. Consequently using Equation (3.2b) to express these contribution in terms of α_0 and β_0,

$$ad-bc=\alpha_0 d_l-\beta_0 c_l\,. \tag{A2}$$

To obtain the second expression in Equation (3.13) we follow a slightly different procedure

$$\begin{aligned}ad-bc=&\frac{1}{2\pi i}\oint d\xi\xi^{l-1}\varrho(x,\xi)\left\{c(\xi)\left[\frac{\zeta d(\xi)-\xi d(\zeta)}{\xi-\zeta}\right]\right.\\&\left.-d(\xi)\left[\frac{\zeta c(\xi)-\xi c(\zeta)}{\xi-\zeta}\right]\right\}.\end{aligned} \tag{A3}$$

Again the terms in square brackets are polynomials of degree $(l-1)$ in ξ. This time Equation (3.4) implies that only the constant terms in these polynomials can contribute. Again using Equation (3.26),

$$ad-bc=a_0 d_0-b_0 c_0\,. \tag{A4}$$

We use similar manipulations to establish Equations (3.10)–(3.12). To avoid ambiguities write

$$D_i^\zeta=\frac{\partial}{\partial x_{i1}}-\zeta\frac{\partial}{\partial x_{i2}}\,. \tag{A5}$$

$$\begin{aligned}\mathrm{d}D_i^\zeta \mathrm{a}-bD_i^\zeta c=&-\frac{1}{2\pi i}\oint\frac{d\xi}{\xi-\zeta}\xi^{-l}\{d(\zeta)D_i^\zeta[\varrho(\xi)c(\xi)]-\varrho(\xi)c(\xi)D_i^\zeta c(\zeta)\}\\=&\frac{1}{2\pi i}\oint d\xi\xi^{-l}\left\{D_i^\zeta[\varrho(\xi)c(\xi)]\left[\frac{d(\xi)-d(\zeta)}{\xi-\zeta}\right]\right.\\&\left.-\varrho(\xi)d(\xi)\left[\frac{D_i^\zeta c(\xi)-D_i^\zeta c(\zeta)}{\xi-\zeta}\right]-\frac{c(\xi)d(\xi)}{\xi-\zeta}D_i^\zeta\varrho(\xi)\right\}.\end{aligned} \tag{A6}$$

$$\begin{aligned}aD_i^\zeta c-cD_i^\zeta a=&-\frac{1}{2\pi i}\oint d\xi\xi^{-l}\left\{D_i^\zeta[\varrho(\xi)c(\xi)]\left[\frac{c(\xi)-c(\zeta)}{\xi-\zeta}\right]\right.\\&\left.-\varrho(\xi)c(\xi)\left[\frac{D_i^\zeta c(\xi)-D_i^\zeta c(\zeta)}{\xi-\zeta}\right]-\frac{c(\xi)^2}{\xi-\zeta}D_i^\zeta\varrho(\xi)\right\}.\end{aligned} \tag{A7}$$

$$\begin{aligned}dD_i^\zeta b-bD_i^\zeta d=&\frac{1}{2\pi i}\oint d\xi\xi^{-l-1}\left\{D_i^\zeta[\varrho(\xi)d(\xi)]\left[\frac{\zeta d(\xi)-\xi d(\zeta)}{\xi-\zeta}\right]\right.\\&\left.-\varrho(\xi)d(\xi)\left[\frac{\zeta D_i^\zeta d(\xi)-\xi D_i^\zeta d(\zeta)}{\xi-\zeta}\right]-\frac{\zeta d(\xi)^2}{\xi-\zeta}D_i^\zeta\varrho(\xi)\right\}.\end{aligned} \tag{A8}$$

The integrals in Equations (A6)–(A8) are evaluated using the techniques applied to (A1) and (A3) together with

$$D_i^\zeta\varrho(\xi)=(\xi-\zeta)\frac{\partial\varrho(\xi)}{\partial x_{i2}}=\frac{(\xi-\zeta)}{\xi}\frac{\partial\varrho(\xi)}{\partial x_{i1}}\,. \tag{A9}$$

Appendix B. Proof of a Differential Equation

In this appendix we will prove the differential Equation (4.11) which, together with result of Jacobi stated below, forms the key step in establishing that the transformation α yields R_{l+1} from R_l. Let us define the $l \times l$ matrix $B^{(l)}$ by

$$B^{(l)}_{rs} = \Delta_{r+s-l}. \tag{B1}$$

Then

$$b^{(l)} = \det B^{(l)} = \tilde{D}^{(l+1)}_{11}. \tag{B2}$$

We also have that

$$d^{(l)} = \det D^{(l)} = (-1)^l \tilde{D}^{(l+1)}_{1,l+1}. \tag{B3}$$

To establish Equation (4.11) we need to prove that

$$d^{(l)} \frac{\partial b^{(l-1)}}{\partial y} - b^{(l-1)} \frac{\partial d^{(l)}}{\partial y} = d^{(l-1)} \frac{\partial b^{(l)}}{\partial \bar{z}} - b^{(l)} \frac{\partial d^{(l-1)}}{\partial \bar{z}}. \tag{B4}$$

Now

$$\begin{aligned} d^{(l-1)} \frac{\partial b^{(l)}}{\partial \bar{z}} + b^{(l-1)} \frac{\partial d^{(l)}}{\partial y} &= d^{(l-1)} \sum_{r,s=1}^{l} \frac{\partial B^{(l)}_{rs}}{\partial \bar{z}} \tilde{B}^{(l)}_{rs} + b^{(l-1)} \sum_{r,s=1}^{l} \frac{\partial D^{(l)}_{rs}}{\partial y} \tilde{D}^{(l)}_{rs} \\ &= \sum_{r,s=1}^{l} \frac{\partial B^{(l)}_{rs}}{\partial \bar{z}} (d^{(l-1)} \tilde{B}^{(l)}_{rs} - b^{(l-1)} \tilde{D}^{(l)}_{rs}). \end{aligned} \tag{B5}$$

Using Equation (1.8) for Δ_r. But we may now use Equation (B2) together with

$$d^{(l-1)} = \tilde{B}^{(l)}_{ll}$$

and

$$\tilde{D}^{(l)}_{r1} = (-1)^{l-1} \tilde{B}^{(l)}_{rl}, \tilde{D}^{(l)}_{1s} = (-1)^{l-1} \tilde{B}^{(l)}_{ls}$$

to write the right hand side of (B5) in the form,

$$\sum_{r,s=1}^{l} \frac{\partial B^{(l)}_{rs}}{\partial \bar{z}} \{(\tilde{B}^{(l)}_{ll} \tilde{B}^{(l)}_{rs} - \tilde{B}^{(l)}_{rl} \tilde{B}^{(l)}_{ls}) + (\tilde{D}^{(l)}_{r1} \tilde{D}^{(l)}_{1s} - \tilde{D}^{(l)}_{11} \tilde{D}^{(l)}_{rs})\}. \tag{B6}$$

The expression (B6) involves subdeterminants of the adjugate matrices $\tilde{B}^{(l)}$ and $\tilde{D}^{(l)}$. A result of Jacobi [10] states that if $\tilde{M}^{(r)}$ is an $r \times r$ submatrix of the adjugate matrix $\tilde{M}$ of the $N \times N$ matrix M, and $M^{(N-r)}$ denotes the $(N-r) \times (N-r)$ matrix obtained from M by striking out the rows and columns similarly placed to the rows and columns of $\tilde{M}$ which contain the elements of $\tilde{M}^{(r)}$, then

$$\det \tilde{M}^{(r)} = (\det M)^{r-1} \det M^{(N-r)}. \tag{B7}$$

Applying this result to expression (B6) we obtain

$$\begin{aligned} d^{(l-1)}\frac{\partial b^{(l)}}{\partial z}+b^{(l-1)}\frac{\partial d^{(l)}}{\partial y} &= \sum_{r,s=1}^{l-1}\left\{\frac{\partial B_{rs}^{(l)}}{\partial \bar{z}}b^{(l)}\tilde{D}_{rs}^{(l-1)}-\frac{\partial B_{r+1,s+1}^{(l)}}{\partial \bar{z}}d^{(l)}\tilde{B}_{rs}^{(l-1)}\right\} \\ &= b^{(l)}\sum_{r,s=1}^{l-1}\frac{\partial D_{rs}^{(l-1)}}{\partial \bar{z}}\tilde{D}_{rs}^{(l-1)}+d^{(l)}\sum_{r,s=1}^{l-1}\frac{\partial B_{rs}^{(l-1)}}{\partial y}\tilde{B}_{rs}^{(l-1)} \\ &= b^{(l)}\frac{\partial d^{(l-1)}}{\partial \bar{z}}+d^{(l)}\frac{\partial b^{(l-1)}}{\partial y}, \end{aligned} \tag{B8}$$

establishing Equation (B4).

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