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Amenability of Crossed Products of C^*-Algebras

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Abstract. It is shown that the class of amenable (resp. strongly amenable) C^*-algebras is closed under the process of taking crossed products with discrete amenable groups. Under certain circumstances, amenability is also preserved under taking a "crossed product" with an amenable semigroup of linear endomorphisms. These facts are used to show that certain simple C^*-algebras $\mathcal{O}_n$ studied by J. Cuntz are amenable but not strongly amenable (thus answering a question of B. E. Johnson), yet are stably isomorphic to strongly amenable algebras.

The purpose of this note is to prove stability of the classes of amenable and strongly amenable C^*-algebras under the process of taking crossed products with discrete amenable groups (and under certain circumstances, with discrete amenable semigroups). These facts are used to show that certain simple C^*-algebras $\mathcal{O}_n$, studied by Cuntz in [4], are amenable but not strongly amenable, yet are stably isomorphic to strongly amenable algebras. We thus answer a question of Johnson [7], who wondered if every amenable C^*-algebra is strongly amenable, and at the same time show that strong amenability is not preserved under "cutting down" by projections. The questions of whether amenability is a stable isomorphism invariant, and whether every nuclear C^*-algebra is amenable, remain open, and the example of the algebras $\mathcal{O}_n$ seems to indicate that positive answers to these are at least plausible. The present results also show that a strongly amenable C^*-algebra need not be an inductive limit of Type I algebras.

We use the following notation. If X is a Banach space, X^* denotes its dual. For $K \subseteq X^*$, $\operatorname{co} K$ denotes its weak-* closed convex hull. If A is a C^*-algebra with unit, $U(A)$ denotes the unitary group of A. If A is any C^*-algebra, we use the notation $A^\sim$ to mean A if A has a unit, A with identity adjoined otherwise. Recall that a C^*-algebra A is *amenable* if and only if for every 2-sided unital Banach $A^\sim$-module X and for every continuous derivation $d: A \to X^*$, d is the coboundary of some element of X^* [7, p. 61]. (One may drop the continuity assumption on d since, by [8], every

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derivation into X^* is continuous.) A is *strongly amenable* if and only if for every 2-sided unital Banach $A^\sim$-module X and for every $f\in X^*$, there exists $g\in \operatorname{co}\{ufu^* : u\in U(A)\}$ with $ag=ga$ for all $a\in A$ [2, Proposition 1b].

Proposition 1. *If A is a strongly amenable C^*-algebra with* 1, *then A admits at least one finite trace.*

Proof. Regard A as a 2-sided A-module and let f be a state of A. Then there exists $g\in \operatorname{co}\{ufu^* : u\in U(A)\}$ with $ag=ga$ for all $a\in A$. Since $g(1)=1$, g is a tracial state on A.

Corollary 1. *An infinite (in the sense of* [5]*) simple C^*-algebra with* 1 *cannot be strongly amenable.*

Corollary 2. *An infinite simple C^*-algebra with* 1 *cannot be an inductive limit of Type I C^*-algebras. (This was proved in* [4] *by a different method, using a slightly stronger definition of infinite C^*-algebras.)*

Proof. By [7, Theorem 7.9 and Proposition 7.6], an inductive limit of Type I C^*-algebras is strongly amenable.

Our first main result is motivated by [3], which essentially contains a von Neumann algebra version of the same theorem. Since we limit ourselves to actions of *discrete* groups G [only when G is discrete is A a subalgebra of the crossed product $C^*(G,A)$], Connes' proof works here with only minor modifications. One could probably generalize the theorem to the case where G is non-discrete, but since we have no applications in mind for such a more general result, it doesn't seem worth the additional technical complications.

Theorem 1. *Let G be a discrete amenable group acting by automorphisms on an amenable C^*-algebra A. Then the crossed product $C^*(G,A)$ is amenable.*

Proof. First we extend the action of G to $A^\sim$ and show that $B=C^*(G,A^\sim)$ is amenable. Note that we may view $A^\sim$ as a subalgebra of B and G as a subgroup of $U(B)$. Let X be a 2-sided unital Banach B-module and let $d:B\to X^*$ be a (continuous) derivation. We may regard X as an $A^\sim$-module, and by amenability of A, $d|A^\sim$ is inner. So subtracting a coboundary if necessary, we may assume $d|A^\sim=0$. For $s\in G$, let $f(s)=s\cdot d(s^{-1})$, and let m be a left invariant mean on G. Since f is uniformly bounded and takes values in a dual Banach space, we may form $f_0=\int f(s)dm(s)$. [We define this weakly by $\langle x,f_0\rangle=\int\langle x,f(s)\rangle\, dm(s)$ for each $x\in X$.] We claim $d(b)=bf_0-f_0b$ for all $b\in B$. It is enough to prove this for $b\in A^\sim$ or $b\in G$ since the algebra generated (algebraically) by $A^\sim$ and G is dense in B.

If $a\in A^\sim$,

$$\begin{aligned} a\cdot f(s)-f(s)\cdot a &= as\cdot d(s^{-1})-s\cdot d(s^{-1})\cdot a\\ &= s\cdot(s^{-1}as\cdot d(s^{-1}))-s\cdot(d(s^{-1})\cdot a)=s\cdot[d((s^{-1}as)s^{-1})-d(s^{-1})\cdot a]\end{aligned}$$

[since $d(s^{-1}as)=0$] $=s\cdot[d(s^{-1}a)-d(s^{-1})\cdot a]=0$, since $d(a)=0$. This being true for all s, we also have $a\cdot f_0-f_0\cdot a=0=d(a)$.

If $s,t\in G$, we have

$$\begin{aligned} t\cdot f(s) &= ts\cdot d(s^{-1})=(ts)\cdot d((ts)^{-1}t)\\ &= (ts)\cdot[d((ts)^{-1})\cdot t+(ts)^{-1}\cdot d(t)]=f(ts)\cdot t+d(t)\,. \end{aligned}$$

So $d(t)=t\cdot f(s)-f(ts)\cdot t$, and by left invariance of m, $d(t)=t\cdot f_0-f_0\cdot t$, as desired. Thus d is inner and B is amenable.

Now note finally that $C^*(G,A)$ is a closed 2-sided ideal of B, so $C^*(G,A)$ is amenable by [7, Proposition 5.1].

Next we have a strong amenability version of Theorem 1.

Theorem 2. *Let G be a discrete amenable group acting by automorphisms on a strongly amenable C^*-algebra A. Then the crossed product $C^*(G,A)$ is strongly amenable.*

Proof. Form $B=C^*(G,A^\sim)$ as before. Again, we first show B is strongly amenable. Let X be a 2-sided unital Banach B-module, and let $f\in X^*$. Since X is also an $A^\sim$-module, strong amenability of A implies that there exists $g\in \mathrm{co}\{ufu^*:u\in U(A^\sim)\}$ with $ag=ga$ for all $a\in A$. Now by amenability of G (cf. [7, Proposition 7.8]), there exists $h\in\mathrm{co}\{tgt^*:t\in G\}$ with $sh=hs$ for all $s\in G$. Clearly $h\in\mathrm{co}\{ufu^*:u\in U(B)\}$, so it is enough to show that $ah=ha$ for all $a\in A$. But if $t\in G$ and $a\in A$, $a\cdot(tgt^{-1})=t(t^{-1}at)\cdot gt^{-1}=tg\cdot(t^{-1}at)t^{-1}$ (since $t^{-1}at\in A$) $=(tgt^{-1})\cdot a$. So every tgt^* is centralized by A and hence so is h. Thus B is strongly amenable.

Again, as $C^*(G,A)$ is an ideal in B, it, too, is strongly amenable [7, pp. 74–75].

For our applications, we need a modified version of Theorem 1 in which the amenable group G is replaced by an amenable *semigroup* of linear *endo*morphisms of A. As the proof indicates, the structure of the argument remains the same.

Theorem 3. *Let B be a C^*-algebra with 1 generated by an amenable C^*-subalgebra A and by an element S satisfying the conditions $1\in A$, $S^*S=1$, and $SAS^*\subseteqq A$. Then B is amenable.*

Proof. Let $e=SS^*$, which is a projection in A, let X be a 2-sided unital Banach B-module, and let $d:B\to X^*$ be a (continuous) derivation. We must show that d is inner. Since X is also an A-module and A is amenable, we may subtract a coboundary from d and suppose that $d|A=0$. This implies in particular that $d(e)=0$, so that $0=d(SS^*)=d(S)S^*+Sd(S^*)$, and $d(S^*)=-S^*d(S)S^*$.

We claim it suffices to show the existence of some $f_0\in X^*$ such that $af_0-f_0a=0$ for all $a\in A$ and such that $d(S)=Sf_0-f_0S$. Indeed, for such f_0 we would have $d(S^*)=-S^*d(S)S^*=-S^*(Sf_0-f_0S)S^*=-f_0S^*+S^*f_0e=-f_0S^*+S^*ef_0$ (since $e\in A$ commutes with f_0) $=S^*f_0-f_0S^*$, so that d would coincide with the inner derivation defined by f_0 on S, S^*, and A, hence on the $*$-algebra generated (algebraically) by S and A. Since this algebra is dense in B and d is continuous, it would follow that $d(b)=bf_0-f_0b$ for all $b\in B$, as desired.

To construct f_0, first let $f(n)=S^{*n}d(S^n)$, and let m be an invariant mean on the semigroup of positive integers. We let $f_0=\int f(n)dm(n)$. $\left(\text{It would suffice to take for } f_0\right.$ a weak-* limit point of the sequence $\left.\left\{\sum_{i=1}^{n} f(i)/n\right\}.\right)$ For $a\in A$, we have

$$\begin{aligned} a\cdot f(n)-f(n)\cdot a &= aS^{*n}d(S^n)-S^{*n}d(S^n)a \\ &= S^{*n}S^n(aS^{*n}d(S^n))-S^{*n}d(S^n)a \\ &= S^{*n}[(S^naS^{*n})d(S^n)-d(S^n)a]\,. \end{aligned}$$

But $d(S^n a) = d(S^n a S^{*n} S^n) = (S^n a S^{*n}) d(S^n) + d(S^n a S^{*n}) S^n$, and since $S^n a S^{*n} \in A$, we get $d(S^n a) = (S^n a S^{*n}) d(S^n)$. Substituting above,

$$a \cdot f(n) - f(n) \cdot a = S^{*n}[d(S^n a) - d(S^n) a] = 0,$$

since $d(a) = 0$. Applying m, we get $a \cdot f_0 - f_0 \cdot a = 0$ for all $a \in A$.

Next we observe that

$$\begin{aligned} ed(S) &= ed(S^{*n} S^{n+1}) = e[d(S^{*n}) S^{n+1} + S^{*n} d(S^{n+1})] \\ &= e[d(S^{*n}) S^n] S + S[S^{*n+1} d(S^{n+1})] \\ &= -ef(n)S + Sf(n+1) = e(Sf(n+1) - f(n)S), \end{aligned}$$

since $0 = d(1) = d(S^{*n} S^n) = d(S^{*n}) S^n + f(n)$. Then by invariance of m, we see that $ed(S) = e(Sf_0 - f_0 S) = Sf_0 - (ef_0)S = Sf_0 - (f_0 e)S = Sf_0 - f_0 S$, since $e \in A$ commutes with f_0 and since $eS = S$. But, on the other hand, $0 = d((1-e)S) = (1-e)d(S) + d(1-e)S = (1-e)d(S)$, since $1 - e \in A$. Thus $d(S) = ed(S) = Sf_0 - f_0 S$, as desired. This completes the proof.

Now let $\mathcal{O}_n$ denote the simple infinite C^*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ with $S_1 S_1^* + \ldots + S_n S_n^* = 1$, as analyzed in detail in [4]. Since $\mathcal{O}_n$ is infinite, it is not strongly amenable, by Corollary 1 to Proposition 1. On the other hand, it is shown in [4, § 1] that $B = \mathcal{O}_n$ satisfies the hypotheses of Theorem 3, with A a *UHF*-algebra (for n finite – when $n = \infty$, A is merely *AF*) and $S = S_1$. Thus $\mathcal{O}_n$ is amenable.

Let $\underline{K}$ denote the C^*-algebra of all compact operators on an infinite-dimensional separable Hilbert space. By [4, 2.1], $\mathcal{O}_n \otimes \underline{K}$ is isomorphic to a crossed product of an *AF*-algebra (matroid if n is finite) by an action of the integers. Hence $\mathcal{O}_n \otimes \underline{K}$ is strongly amenable by Theorem 2. (It was already observed in [4] that $\mathcal{O}_n \otimes \underline{K}$, and hence $\mathcal{O}_n$, is nuclear.) Thus with stable isomorphism defined as in [1], strong amenability is *not* a stable isomorphism invariant. On the other hand, nuclearity *is* a stable isomorphism invariant, and there are indications that the same may be true for amenability.

Finally, we note that by the proof of [4, 2.3], $\mathcal{O}_n \otimes \underline{K}$ is not an inductive limit of Type I C^*-algebras. (Alternatively, we know this for $\mathcal{O}_n$ by Corollary 2 to Proposition 1. But as Philip Green has indicated to us, it is easy to see that the class of inductive limits of Type I C^*-algebras is closed under stable isomorphism [1], for if A is an inductive limit of a directed family of Type I C^*-subalgebras A_α, and if $A \otimes \underline{K} \cong B \otimes \underline{K}$, then $B \cong p(A \otimes \underline{K})p$ for some projection p in $A \otimes \underline{K}$ [2]. However, by [6, Lemmas 1.6 and 1.8], there exists a projection e contained in some $A_\alpha \otimes \underline{K}$ and an inner automorphism of $(A \otimes \underline{K})^\sim$ carrying p to e. So $B \cong \varinjlim e(A_\alpha \otimes \underline{K})e$ and B is an inductive limit of Type I algebras. In particular, $\mathcal{O}_n \otimes \underline{K}$ cannot be an inductive limit of Type I algebras if $\mathcal{O}_n$ is not.) So a strongly amenable C^*-algebra need not be an inductive limit of Type I algebras. The known relations among various classes of "nice" C^*-algebras are summarized in the following Table 1.

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1 At least with suitable assumptions regarding existence of units

2 Assuming B has a unit. One could also extend the argument to the case where B has an approximate identity consisting of projections

Table 1

Property of C^*-algebras	Inductive limit of Type I C^*-algebras	[7] $\Longrightarrow$ $\not\Leftarrow$ $\mathcal{O}_n \otimes K$	Strongly amenable	[7] $\Longrightarrow$ $\not\Leftarrow$ $\mathcal{O}_n$	Amenable	[3] $\Longrightarrow$ $\Leftarrow ? =$	Nuclear
Stable isomorphism invariant?	yes[a]		no		?		yes

[a] At least with suitable assumptions regarding existence of units

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