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Stationary Solutions of the Bogoliubov Hierarchy Equations in Classical Statistical Mechanics. 3

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Abstract. We continue the analysis of the "conjugate" equation for the generating function of a Gibbs random point field corresponding to a stationary solution of the classical BBGKY hierarchy. This equation was established and partially investigated in the preceding papers under the same title. In the present paper we reduce a general theorem about the form of solutions of the "conjugate" equation to a statement which relates to a special case where the interacting particles constitute a "quasi"—one dimensional configuration.

0. Introduction

This paper continues the preceding papers of the authors [1, 2]. We continue here the proof of Main Theorem, more precisely, of its part which was formulated as Theorem 2, **1**[1]. Theorem 2′ proved in [2] contains the assertion of Theorem 2, **1** for the case $n_0=2$ and is the initial step of the inductive proof for arbitrary $n_0 \geqq 2$ (for the notations used without definitions, see [1, 2]). The purpose of this part of the work is to reduce Theorem 2, **1** to a special case where the configuration of interacting particles is represented by a one-dimensional graph ("chain"). The corresponding assertion (Basic Lemma) is formulated in Section 2 and will be proved in a separate paper.

In this Section we follow the assumptions of [1]. On account of Theorem 2′, **2** as the initial inductive step w.r.t. n_0, it is not hard to see that Theorem 2, **1** follow from:

Theorem 0.1. *Let $U(r)$ obey $(I_1, \mathbf{1}-I_4, \mathbf{1})$ and $f(\bar{x})$ obey $(G_1, \mathbf{1}-G_6, \mathbf{1})$ with $n_0 \geqq 3$. Suppose U and f satisfy Equation* (2.8, **1**)*:*

$$\{f(\bar{x}), H(\bar{x})\} + \sum_{y\in\bar{x}} \{f(\bar{x}\backslash y), U(\bar{x}\backslash y|y)\} = 0, \qquad \bar{x}\in D^0. \tag{0.1}$$

Then $f(\bar{x})=0$ for any $\bar{x}\in M_{n_0}\cap D^0$.

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[1] As in [2], we mark the references to [1] by the index **1**. The references to [2] are marked by the index **2**

1. Particle Configurations and Their Types

Definition. We say that $x=(q,v)\in\bar{x}$ is an *external* point in $\bar{x}$, $\bar{x}\in D^0$, if q is an extremal point of the convex hull spanned by $\{\tilde{q}\in R^\nu:\tilde{q}\in\bar{x}\}$[2].

For the sake of brevity we say sometimes "$q\in\bar{x}$ is an external point" instead of "$x=(q,v)\in\bar{x}$ is an external point". The same is related to the definitions which follow.

For every external point $q\in\bar{x}$ there exists an open cone $B_q^e\subset R^\nu$ with the vertex q such that

1) for all $q'\in B_q^e$ and any $\tilde{q}\in\bar{x}$, $\tilde{q}\neq q$, the inequality $|q'-\tilde{q}|>|q-\tilde{q}|$ holds, and

2) every $q'\in B_q^e$ such that $|q'-q|>d_0$ is an external point in $\bar{x}\cup x'$ where $x'=(q',v')$, $v'\in R^\nu$.

Definition. We say that $x=(q,v)\in\bar{x}$ is an *accessible* point in $\bar{x}$, $\bar{x}\in D^0$, if there exists a non-empty open set $B_q^a\subset R^\nu$ such that $|q'-q|>d_0$, $U'(|q'-q|)\neq 0$ and $\min\limits_{\tilde{q}\in\bar{x}:\tilde{q}\neq q}|q'-\tilde{q}|>d_1$, whenever $q'\in B_q^a$[3].

It is clear that every external $q\in\bar{x}$ is accessible, and the corresponding B_q^a may be chosen as a subset of B_q^e. We suppose below that for external $q\in\bar{x}$, B_q^a is always chosen belonging to B_q^e.

Definition. We say that $x=(q,v)\in\bar{x}$ is an *isolated* point in $\bar{x}$, $\bar{x}\in D^0$, if $|q-\tilde{q}|>d_1$ for any $\tilde{q}\in\bar{x}$, $\tilde{q}\neq q$. We say that $x=(q,v)\in\bar{x}$ is an *end* point in $\bar{x}$ if there exists a unique $\tilde{q}\in\bar{x}$, $\tilde{q}\neq q$, such that $|q-\tilde{q}|\leqq d_1$.

Clearly, for any isolated $q\in\bar{x}$, the mutual energy $U(\bar{x}\setminus x|x)$ and its gradient $\partial_q U(\bar{x}\setminus x|x)$ vanish.

Definition. Let $\bar{x}\in D^0$, $\bar{y}\subseteq\bar{x}$, $n(\bar{y})=s\geqq 2$. We say that $\bar{y}$ is a *chain* in $\bar{x}$ if:

(i) the points $q\in\bar{y}$ can be labelled by $i=1,\dots,s$ so that $|q_i-q_j|\leqq d_1$ iff $|i-j|\leqq 1$, $1\leqq i,j\leqq s$;

(ii) for any $\tilde{q}\in\bar{x}\setminus\bar{y}$, $q\in\bar{y}$, $\min|q-\tilde{q}|>d_1$. In that case we write $\bar{y}=[q_1,\dots,q_s]$. The points q_1 and q_s are called the ends of the chain $\bar{y}$.

Definition. Let $x=(q,v)\in\bar{x}$, $\bar{x}\in D^0$. We say that x is a *c-point* in $\bar{x}$, if there exists a chain $[q_1,\dots,q_s]$ in $\bar{x}$ with $q_1=q$. We say that $x'=(q',v')\in\bar{x}\setminus x$ is a *c(q)-point* in $\bar{x}$ if there exists a chain $[q_1,\dots,q_s]$ in $\bar{x}$ with $q_1=q$, $q_s=q'$.

Definition. We say that the *order of* $\bar{x}\in D^0$ *w.r.t.* $q\in\bar{x}$ is zero in the following three cases:

1^0 there is no end point in $\bar{x}$,

2^0 q is the unique end point in $\bar{x}$,

3^0 q is a c-point in $\bar{x}$, and the set of the end points in $\bar{x}$ is exausted by q and the $c(q)$-point $q'\in\bar{x}$.

We say that the *order of* $\bar{x}$ *w.r.t.* $q\in\bar{x}$ equals k, $k=1,2,\dots$, if for every end non-$c(q)$-point $x'=(q',v')\in\bar{x}$, $q'\neq q$, the order of $\bar{x}\setminus x'$ w.r.t. q is $\leqq k-1$ and for some such a point x' the equality holds.

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[2] As in [1, 2], $q\in\bar{x}$ (resp., $v\in\bar{x}$) means that $(q,v)\in\bar{x}$ for some $v\in R^\nu$ (resp., $q\in R^\nu$)

[3] We suppose below that d_1 is chosen so that $d_1=\min[d:U(r)\equiv 0$ for $r\geqq d]$.
It is not hard to check that for every $\varepsilon>0$ there exists $r\in(d_1-\varepsilon,d_1)$ such that $U'(r)\neq 0$

Proposition 1.1. *There exists a unique non-negative integer-valued function $k(\bar{x}, q)$, $\bar{x} \in D^0$, $q \in \bar{x}$, with the properties indicated in the definition of the order of $\bar{x}$ w.r.t. q.*

Proof. We use the induction w.r.t. $n(\bar{x})$, the number of points $q \in \bar{x}$. Clearly, $k(\bar{x}, q) = 0$ for $n(\bar{x}) = 1$. Assume $k(\bar{x}', q')$ is defined for all $\bar{x}' \in D^0$ and $q' \in \bar{x}'$ with $n(\bar{x}') < n$, and consider $\bar{x} \in D^0 \cap M_n$ and $q \in \bar{x}$. If neither of conditions $1^0 - 3^0$ holds then one can find an end non-$c(q)$-point $q' \in \bar{x}$. Let $E_q(\bar{x})$ denote the set of all such points q', and

$$k(\bar{x}, q) = \max_{q' \in E_q(\bar{x})} k(\bar{x} \backslash q', q) + 1 .$$

It is easy to check that the function $k(\bar{x}, q)$ defined by this relation has the properties claimed in the definition of the order. The uniqueness is evident.

Definition. The minimum of $k(\bar{x}, q)$, $\bar{x} \in D^0$, over the set of external points $q \in \bar{x}$ is called the *order* of $\bar{x}$ and denoted by $k(\bar{x})$. The *type* of $\bar{x} \in D^0$ is the triple $(n(\bar{x}), m(\bar{x}), k(\bar{x}))$ where $m(\bar{x})$ is the number of (unordered) pairs $q, q' \in \bar{x}$ such that $|q - q'| \leqq d_1$. We say that a triple of non-negative integers (n, m, k) is *admissible* if there exists $\bar{x} \in D^0$ such that $n(\bar{x}) = n$, $m(\bar{x}) = m$, $k(\bar{x}) = k$.

Clearly, the type of $\bar{x}$ does not depend on v, $v \in \bar{x}$, and, in the case where $|q - q'| \neq d_1$ for any pair q, $q' \in \bar{x}$, it does not change for small shifts of $q \in \bar{x}$. Our inductive assertion giving the passage from $n_0 = n - 1$ to $n_0 = n$ may be reformulated as follows.

Theorem 1.2. *Let the conditions of Theorem 0.1 hold and (n, m, k) be an admissible triple with $n = n_0$. Then*

$$f(\bar{x}) = 0 \quad \text{if} \quad \bar{x} \in D^0 \quad \text{and the type of} \quad \bar{x} \quad \text{is} \quad (n, m, k) . \tag{0.2}$$

In this paper we prove Theorem 1.2 assuming that some auxiliary statement (Basic Lemma) is true (see the next section). This statement will be proved in a separate paper. Henceforth it is convenient to suppose that the interaction potential $U(r)$ and the generating function $f(\bar{x})$, $\bar{x} \in D^0$, satisfy conditions $(I'_1, \mathbf{2} - I'_3, \mathbf{2})$ and $(G'_1, \mathbf{2} - G'_4, \mathbf{2})$ respectively except that C^2 in $(I'_1, \mathbf{2})$ is replaced by C^3. In what follows we suppose these conditions to be valid as well as Equation (0.1) and do not specify this in the assertions formulated below.

2. Basic Lemma

We start with formulating the auxiliary statement from which Theorem 1.2 will be deduced below.

Basic Lemma. *Let $\bar{x} \in D^0$ and $n(\bar{x}) \geqq 3$. Suppose one can choose in $\bar{x}$ a chain $[q_1, \ldots, q_s]$, $s \geqq 2$, where q_1 is an external and q_s an accessible point, and the following holds. For any $\bar{x}' = (\bar{x} \backslash x_1) \cup x'_1$ with $x'_1 = (q'_1, v'_1) \in B^e_q \times R^\nu$ there are a non-empty open $B^c_{q'_1} \subset B^a_{q'_1}$ and a non-empty open set $B^c_{q_s} \subset B^a_{q_s}$ such that for all $x_0 = (q_0, v_0) \in B^c_{q'_1} \times R^\nu$ and*

$x_{s+1}=(q_{s+1}, v_{s+1})\in B^c_{q_s}\times R^\nu$, *Equations* (2.1a–e) *below are valid*

$$\begin{aligned}&\{f(\bar{x}'), U(|q_0-q_1|)\}\\&+\{f((\bar{x}'\cup x_0)\backslash x_s), U(|q_{s-1}-q_s|)\}=0,\end{aligned}\tag{2.1a}$$

$$\begin{aligned}&\{f(\bar{x}'), U(|q_s-q_{s+1}|)\}\\&+\{f((\bar{x}\cup x_{s+1})\backslash x_1), U(|q_1'-q_2|)\}=0,\end{aligned}\tag{2.1b}$$

$$\begin{aligned}&\{f(\bar{x}'), H(\bar{x}')\}+\{f(\bar{x}\backslash x_1), U(|q_1'-q_2)\}\\&\qquad+\{f(\bar{x}'\backslash x_s), U(|q_{s-1}-q_s|)\}=0,\end{aligned}\tag{2.1c}$$

$$\begin{aligned}&\{f(\bar{x}\backslash x_1), H(\bar{x}\backslash x_1)\}\\&+\{f(\bar{x}\backslash(x_1\cup x_2)), U(|q_2-q_3|)\}\\&+\{f(\bar{x}\backslash(x_1\cup x_s)), U(|q_{s-1}-q_s|)\}=0,\end{aligned}\tag{2.1d}$$

$$\begin{aligned}&\{f(\bar{x}'\backslash x_s), H(\bar{x}'\backslash x_s)\}\\&+\{f(\bar{x}'\backslash(x_{s-1}\cup x_s)), U(|q_{s-2}-q_{s-1}|)\}\\&+\{f(\bar{x}\backslash(x_1\cup x_s)), U(|q_1'-q_2|)\}=0\end{aligned}\tag{2.1e}$$

[*for* $s=2$, *Equations* (2.1d, e) *have the form* $\{f(\bar{x}\backslash x_1), H(\bar{x}\backslash x_1)\}=0$ *and* $\{f(\bar{x}'\backslash x_s), H(\bar{x}'\backslash x_s)\}=0$ *respectively*]. *Moreover, suppose Equations* (2.1a–e) *remain valid if* x_i, $i=1,\dots,s$, *are slightly changed. Then* $f(\bar{x}')=0$, $x_1'\in B^e_q\times R^\nu$.

By induction it easely follows from Basic Lemma that if $\bar{x}$ contains a chain $\bar{y}=[q_1,\dots,q_s]$ where q_1 is an external and q_s an accessible point, and $|q-\tilde{q}|>d_1$ for any pair $q,\tilde{q}\in\bar{x}\backslash\bar{y}$, then $f(\bar{x})=0$.

Proposition 2.1. *Let* $\bar{x}\in D^0$ *and* $n(\bar{x})\geqq 3$. *Suppose one can choose* $x_i=(q_i,v_i)\in\bar{x}$, $i=1,2$, *where* q_1 *is an external point such that the following holds. For any* $x_1'=(q_1',v_1')\in B^e_{q_1}\times R^\nu$ *one can find a non-empty open set* $B^c_{q_1'}\subset B^a_{q_1'}$ *such that for all* $x_0=(q_0,v_0)\in B^c_{q_1'}\times R^\nu$ *Equations* (2.2a, b) *below are valid:*

$$\{f(\bar{x}'), U(|q_0-q_1'|)\}=0,\tag{2.2a}$$

$$\{f(\bar{x}'), H(\bar{x}')\}+\{f(\bar{x}\backslash x_1), U(|q_1'-q_2|)\}=0,\tag{2.2b}$$

where, as above, $\bar{x}'=(\bar{x}\backslash x_1)\cup x_1'$. *Then* $f(\bar{x}')=0$, $x_1'\in B^e_q\times R^\nu$.

Notice that Proposition 2.1 yields the assertion of Basic Lemma under the additional assumption $U'(|q_{s-1}-q_s|)=0$.

Proof of Proposition 2.1. We use the elementary identity $\partial_q U(|q|)=U'(|q|)q/|q|$ and the properties of the potential $U(r)$. It follows from (2.2a) that for any $x_1'\in B^e_{q_1}\times R^\nu$,

$$\langle\partial_{v_1'}f(\bar{x}'), q_1'-q_0\rangle=0,\qquad q_0\in B^c_{q_1'}.$$

Since $B^c_{q_1'}$ is open, this implies that $\partial_{v_1'} f(\bar{x}')=0$. Then Equation (2.2b) takes the form

$$\langle \partial_{q_1'} f(\bar{x}'), v_1' \rangle + \sum_{\substack{x=(q,v)\in \bar{x}' \\ q \neq q_1'}} (\langle \partial_q f(\bar{x}'), v \rangle - \langle \partial_v f(\bar{x}'), \partial_q U(\bar{x}' \backslash x | q) \rangle) - \langle \partial_{v_2} f(\bar{x} \backslash x_1), \partial_{q_2} U(|q_1' - q_2|) \rangle = 0.$$

Apply to this equation the operator $\partial_{v_1'}$. We get

$$\partial_{q_1'} f(\bar{x}')=0, \quad q_1' \in B^e_{q_1}. \tag{2.3}$$

Hence $f(\bar{x}')$ does not depend on $x_1' \in B^e_{q_1} \times R^\nu$. The set $B^e_{q_1}$ is unbounded, and thus condition $(G_4', 2)$ implies that $f(\bar{x}')=0$ on $B^e_{q_1} \times R^\nu$.

We prove here two more auxiliary statements used below.

Proposition 2.2. *Let the conditions of Basic Lemma hold. Then*

$$\partial^2_{v_2, v_s} f(\bar{x} \backslash x_1)=0. \tag{2.4}$$

Proof. According to Basic Lemma, $f(\bar{x}')=0$, $\bar{x}'=(\bar{x} \backslash x_1) \cup x_1'$, $x_1' \in B^e_{q_1} \times R^\nu$. Applying to (2.1c) the operator ∂_{v_s} we have the equation

$$\partial^2_{v_2, v_s} f(\bar{x} \backslash x_1) \partial_{q_2} U(|q_1' - q_2|)=0, \quad q_1' \in B^e_{q_1}.$$

According to the definition of $B^e_{q_1}$ and the fact that $|q_1 - q_2| \leqq d_1([q_1, \ldots, q_s]$ is a chain) we can choose a nonempty open $\tilde{B} \subset B^e_q$ such that $U'(|q' - q_2|) \neq 0$ for all $q' \in \tilde{B}$ (see footnote 3, p. 226). Hence $\partial^2_{v_2, v_s} f(\bar{x} \backslash x_1)$ $(q_2 - q_1')=0$, $q_1' \in \tilde{B}$, and the matrix $\partial^2_{v_2, v_s} f(\bar{x} \backslash x_1)$ is zero.

Proposition 2.3. *Let $\bar{x} \in D^0$ and $n(\bar{x}) \geqq 3$. Suppose one can choose in $\bar{x}$ a chain $[q_1, \ldots, q_s]$, $s \geqq 2$, where q is an external point and the following holds. For any $x_1' = (q_1', v_1') \in B^e_{q_1} \times R^\nu$ one can find a non-empty open set $B^c_{q_1'} \subset B^a_{q_1'}$ such that for all $x_0 = (q_0, v_0) \in B^c_{q_1'} \times R^\nu$ Equations* (2.1c), (2.2a), *and* (2.5) *below are valid:*

$$\partial^2_{v_1, v_{s-1}} f(\bar{x}' \backslash x_s)=0. \tag{2.5}$$

Then $f(\bar{x}')=0$, $\bar{x}'=(\bar{x} \backslash x_1) \cup x_1'$, $x_1' \in B^e_{q_1} \times R^\nu$.

Proof of Proposition 2.3. As above, Equation (2.2a) implies that $\partial_{v_1'} f(\bar{x}')=0$. On account of this and of (2.5) we apply to (2.1c) the operator ∂_{v_1}. Then we get (2.3) whence it follows that $f(\bar{x}')=0$.

3. The Case $k=0$

Assuming Basic Lemma to be proved we establish some technical results which, together with Lemma 3.1, enable us to prove Theorem 1.2. We start with the case $k=0$.

Lemma 3.1. *Let $(n, m, 0)$, $n \geqq 3$, be an admissible triple. Suppose* (0.2) *holds for the following triples (if they are admissible): $(n+1, m+1, 0)$, (n, m_1, k_1), $(n-1, m_2, k_2)$, $(n-2, m_3, k_3)$, where $0 \leqq m_i \leqq m-i$, $k_i \geqq 0$, $i=1,2,3$. Then* (0.2) *holds for the triple $(n, m, 0)$.*

Proof. Let $\bar{x}\in D^0$, $n(\bar{x})=n$, $m(\bar{x})=m$, $k(\bar{x})=0$, and $x=(q,v)\in\bar{x}$ be an external point with $k(\bar{x},q)=0$. Then, according to the definition of the order, q may be of one of the following four kinds:

a$_0$) q is isolated in $\bar{x}$,

b$_0$) q is a non-isolated and non end point in $\bar{x}$, i.e., there are q', $q''\in\bar{x}\backslash x$, $q'\neq q''$, such that $|q-q'|\leqq d_1$, $|q-q''|\leqq d_1$,

c$_0$) q is an end non-c-point *in* $\bar{x}$,

d$_0$) q is a c-point in $\bar{x}$.

We shall consider successively all these cases. Due to condition $(G_1', \mathbf{2})$, we may assume that $|q'-q''|\neq d_1$ for any pair q', $q''\in\bar{x}$, and so, we can slightly change each $(q,v)\in\bar{x}$ conserving the type of $\bar{x}$ and the kind of q.

a$_0$) In that case $\bar{x}$ has no end points. Let $\bar{x}'=(q',v')\in B_q^e\times R^\nu$, $\bar{x}'=(\bar{x}\backslash x)\cup x'$; clearly, $k(\bar{x}',q')=0$, i.e., the type of $\bar{x}'$ is $(n,m,0)$.

If $q_0\in B_{q'}^a\subset B_{q'}^e$, then q_0 is an external point in $\bar{x}'\cup x_0$, $x_0=(q_0,v_0)$. The type of $\bar{x}'\cup x_0$ is $(n+1,m+1,0)$, and hence, $f(\bar{x}'\cup x_0)=0$. Due to the above assumption, the gradients $\partial f(\bar{x}'\cup x_0)$ also vanish.

Furthermore, for every $\tilde{x}=(\tilde{q},\tilde{v})\in\bar{x}\backslash x$ one has two possibilities: either $\tilde{q}$ is isolated in $\bar{x}'\cup x_0$ or there exist $q'',q'''\in\bar{x}\backslash(x\cup\tilde{x})$, $q''\neq q'''$, such that $|q''-\tilde{q}|<d_1$, $|q'''-\tilde{q}|<d_1$. In the first case $\partial_{\tilde{q}}U((\bar{x}'\cup x_0)\backslash\tilde{x}|\tilde{q})=0$, and in the second one the type of $(\bar{x}'\cup x_0)\backslash\tilde{x}$ is (n,m_1,k_1) with $m_1\leqq m-1$. Hence, in the second case $f((\bar{x}'\cup x_0)\backslash\tilde{x})$ and the gradients $\partial f((\bar{x}'\cup x_0)\backslash\tilde{x})$ vanish.

These arguments show that Equation (0.1) for $\bar{x}'\cup x_0$ takes the form

$$\begin{aligned}&\langle\partial_{v'}f(\bar{x}'),\partial_{q'}U(|q_0-q'|)\rangle\\&+\langle\partial_{v_0}f((\bar{x}\backslash x)\cup x_0),\partial_{q_0}U(|q_0-q'|)\rangle=0.\end{aligned}\tag{3.1}$$

Equation (3.1) coincides with (3.8b, **2**). By repeating arguments used in the proof of Lemma 3.1, **2** we obtain from (3.1) the following representation:

$$\begin{aligned}f(\bar{x}')&=a_1(\bar{x}\backslash x,q')\\&\quad+\langle\boldsymbol{a}_1(\bar{x}\backslash x,q'),v'\rangle,\qquad(q',v')\in B_q^e.\end{aligned}\tag{3.2}$$

Notice that for $q_0\in(\bar{x}\backslash x)\cup x_0$ all the conditions indicating the Case a$_0$) hold. Hence the same arguments give

$$f((\bar{x}\backslash x)\cup x_0)=\boldsymbol{a}_1(\bar{x}\backslash x,q_0)+\langle\boldsymbol{a}_1(\bar{x}\backslash x,q_0),v_0\rangle.$$

Now Equation (3.1) takes the form

$$\begin{aligned}&\langle\boldsymbol{a}_1(\bar{x}\backslash x,q'),\partial_{q'}U(|q_0-q'|)\rangle\\&+\langle\boldsymbol{a}_1(\bar{x}\backslash x,q_0),\partial_{q_0}U(|q_0-q'|)\rangle=0,\end{aligned}$$

or, on account of Proposition 2.1(i), **2** and our choice of q_0,

$$\langle\boldsymbol{a}_1(\bar{x}\backslash x,q'),q'-q_0\rangle+\langle\boldsymbol{a}_1(\bar{x}\backslash x,q_0),q_0-q'\rangle=0.\tag{3.3}$$

Applying to (3.3) successively the operators ∂_{q_0} and $\partial_{q'}$ we obtain

$$\partial_{q'}\boldsymbol{a}_1(\bar{x}\backslash x,q')+\partial_{q_0}\boldsymbol{a}_1(\bar{x}\backslash x,q_0)^*=0.$$

This means that, for fixed $\bar{x}\backslash x$, the matrix $\partial_{q'}\boldsymbol{a}_1(\bar{x}\backslash x, q')$ is locally constant w.r.t. $q' \in B_q^e$. The set B_q^e is linearly connected; hence, $\partial_{q'}\boldsymbol{a}_1(\bar{x}\backslash x, q')$ does not depend on q'. Therefore,

$$\boldsymbol{a}_1(\bar{x}\backslash x, q') = A_1(\bar{x}\backslash x)q' + \boldsymbol{a}_2(\bar{x}\backslash x), \qquad q' \in B_q^e.$$

Using condition $(G_4', \mathbf{2})$ and the fact that B_q^e is unbounded, it is not hard to show that both $A_1(\bar{x}\backslash x)$ and $\boldsymbol{a}_2(\bar{x}\backslash x)$ vanish. Thus the analysis of Equation (3.1) gives that $f(\bar{x}')$ does not depend on v'.

Now consider Equation (0.1) for $\bar{x}'$. As above, for every $\tilde{x} = (\tilde{q}, \tilde{v}) \in \bar{x}'$ either $\tilde{q}$ is isolated in $\bar{x}'$ or there exist $q'', q''' \in \bar{x}'\backslash\tilde{x}$, $q'' \neq q'''$, such that $|q'' - \tilde{q}| < d_1$, $|q''' - \tilde{q}| < d_1$. In the first case $\partial_{\tilde{q}} U(\bar{x}'\backslash\tilde{x}|\tilde{q}) = 0$, and in the second one the type of $\bar{x}'\backslash\tilde{x}$ is $(n-1, m_2, k_2)$ with $m_2 \leqq m-2$. Hence in the second case $f(\bar{x}'\backslash\tilde{x})$ and the gradients $\partial f(\bar{x}'\backslash\tilde{x})$ vanish. Equation (0.1) for $\bar{x}'$ takes the form

$$\{f(\bar{x}'), H(\bar{x}')\} = 0, \tag{3.4}$$

or, due to the fact that $f(\bar{x}')$ does not depend on v',

$$\begin{aligned} \langle \partial_{q'} f(\bar{x}'), v' \rangle + \sum_{\tilde{x} = (\tilde{q}, \tilde{v}) \in \bar{x}\backslash x} (\langle \partial_{\tilde{q}} f(\bar{x}'), \tilde{v} \rangle \\ - \langle \partial_{\tilde{v}} f(\bar{x}'), \partial_{\tilde{q}} U(\bar{x}'\backslash\tilde{x}|\tilde{q}) \rangle) = 0. \end{aligned} \tag{3.5}$$

Apply to (3.5) the operator $\partial_{v'}$. We get

$$\partial_{q'} f(\bar{x}') = 0, \qquad q' \in B_q^e.$$

Hence $f(\bar{x}')$ is locally constant w.r.t. $q' \in B_q^e$. From this and the fact that B_q^e is connected, it follows that $f(\bar{x}')$ does not depend on q'. Now condition $(G_4', \mathbf{2})$ and the fact that B_q^e is unbounded imply that $f(\bar{x}') = 0$.

b_0) As in the Case a_0), $\bar{x}$ has no end points. Let B_q denote the subset of B_q^e consisting of the points $\hat{q}$ such that for any $\tilde{q} \in \bar{x}\backslash x$, $|\hat{q} - \tilde{q}| < d_1$ iff $|q - \tilde{q}| < d_1$. It is clear that B_q is a bounded open set, and q is a limit point for B_q. We denote the closure of B_q in B_q^e by $\bar{B}_q$. Let $x' = (q', v') \in R^\nu \times R^\nu$, and $\bar{x}' = (\bar{x}\backslash x) \cup x'$. If $q' \in B_q$, then, clearly, $\bar{x}'$ is of the type $(n, m, 0)$. At the same time the complement $B_q^e\backslash\bar{B}_q$ consists of the points q' for which the type of $\bar{x}'$ is (n, m_1, k) with $0 \leqq m_1 \leqq m-1$. Hence $f(\bar{x}') = 0$ for $q' \in B_q^e\backslash\bar{B}_q$. To prove this equality for $q' \in B_q$, we have to check that $f(\bar{x}')$ is locally constant w.r.t. $q' \in B_q$.

Let $x' = (q', v') \in B_q \times R^\nu$, $x_0 = (q_0, v_0) \in B_{q'}^a \times R^\nu$. Then $k(\bar{x}' \cup x_0, q_0) = 0$, and hence, the type of $\bar{x}' \cup x_0$ is $(n+1, m+1, 0)$. Thus $f(\bar{x}' \cup x_0)$ and the gradients $\partial f(\bar{x}' \cup x_0)$ vanish.

Furthermore, for any $\tilde{x} = (\tilde{q}, \tilde{v}) \in \bar{x}'$ one has two above mentioned possibilities: either $\tilde{q}$ is isolated in $\bar{x}' \cup x_0$ or there exist $q'', q''' \in \bar{x}'\backslash\tilde{x}$, $q'' \neq q'''$, such that $|q'' - \tilde{q}| < d_1$, $|q''' - \tilde{q}| < d_1$. The analysis of the both possibilities is similar to that in the Case a_0). Finally, Equation (0.1) for $\bar{x}' \cup x_0$ takes the form

$$\langle \partial_{v'} f(\bar{x}'), \partial_{q'} U(|q' - q_0|) \rangle = 0, \tag{3.6}$$

i.e., coincides with (2.2a). From Equation (3.6) it follows that $\partial_{v'} f(\bar{x}') = 0$, $x' = (q', v') \in B_q \times R^\nu$ (see the proof of Proposition 2.1).

As above, a simple analysis shows that Equation (0.1) for $\bar{x}'$ takes the form (3.5), whence we get

$$\partial_{q'} f(\bar{x}')=0, \quad q'\in B_q. \tag{3.7}$$

This completes the proof for the Case b_0).

c_0) In that case q_0 is the unique end point in $\bar{x}$. We define the set $B_q \subset B_q^e$ in the same way as in b_0). Consider Equation (0.1) for $\bar{x}'\cup x_0$, $x_0=(q_0, v_0)\in B_{q'}^a \times R^\nu$, $x'=(q', v')\in B_q \times R^\nu$. Repeating the above arguments one can show that Equation (0.1) for $\bar{x}'\cup x_0$ is of the form (3.6). Next we establish that Equation (0.1) for $\bar{x}'$ takes the form (3.4). But we have seen above that (3.4) and (3.6) imply (3.7), and it follows from (3.7) that $f(\bar{x}')=0$.

d_0) Let $\bar{y}=[q_1, \ldots, q_s]$, $s\geqq 2$, be a chain in $\bar{x}$ with $q_1=q$.

In the Case d_0) $\bar{x}$ has just two end points: q_1 and q_s. As in the Cases b_0) and c_0), consider the set $B_q\subset B_q^e$ which is defined by the same conditions. It is enough to prove that $f(\bar{x}')=0$ for $\bar{x}'=(\bar{x}\backslash x)\cup x_1'$, $x_1'=(q_1', v_1')\in B_{q_1}\times R^\nu$.

Let $x_1'=(q_1', v_1')\in B_{q_1}\times R^\nu$, and $x_0=(q_0, v_0)\in B_{q_1'}^a\times R^\nu$. The type of $\bar{x}'\cup x_0$ is $(n+1, m+1, 0)$, and hence all gradients $\partial f(\bar{x}'\cup x_0)$ vanish. For any $\tilde{x}=(\tilde{q}, \tilde{v})\in \bar{x}'$, $\tilde{q}\neq q_s$, there are two possibilities which we have discussed before, and so, Equation (0.1) for $\bar{x}'\cup x_0$ takes the form

$$\{f(\bar{x}'), U(|q_0-q_1'|)\}+\{f((\bar{x}'\cup x_0)\backslash x_s), U(|q_s-q_{s-1}|)\}=0\,. \tag{3.8}$$

Equation (3.8) coincides with (2.1a).

A similar analysis shows that Equation (0.1) for $\bar{x}'$, $\bar{x}\backslash x_1$ and $\bar{x}'\backslash x_s$ takes the form (2.1c), (2.1d), and (2.1e) respectively.

The remaining part of the proof for the Case d_0) proceeds in two stages. First, consider the particular case where q_s is an accessible point in $\bar{x}'$. This assumption allows us to consider $\bar{x}'\cup x_{s+1}$, where $x_{s+1}=(q_{s+1}, v_{s+1})\in B_{q_s}^a\times R^\nu$. As above, we conclude that Equation (0.1) for $\bar{x}'\cup x_{s+1}$ takes the form (2.1b). Thus, all the assumptions of Basic Lemma are valid. Using Basic Lemma, we obtain that $f(\bar{x}')$, and hence the gradients $\partial f(\bar{x}')$ vanish. Moreover, due to Proposition 2.2, Equation (2.4) holds.

Now consider the general Case d_0). If $U'(|q_{s-1}-q_s|)=0$, Equation (3.8) [i.e., (2.1a)] coincides with (2.2a), and Equation (0.1) for $\bar{x}'$ [i.e., (2.1c)] takes the form (2.2b). Due to Proposition 2.1 we get $f(\bar{x}')=0$. Thus we may assume that $U'(|q_{s-1}-q_s|)\neq 0$. This means that q_{s-1} is an accessible point in $(\bar{x}'\cup x_0)\backslash x_s$. Since q_0 is an external point in $(\bar{x}'\cup x_0)$, we obtain that $(\bar{x}'\cup x_0)\backslash x_s$ with the chain $[q_0, \ldots, q_{s-1}]$ satisfies the conditions of the particular case just considered. Hence the gradients $\partial f((\bar{x}'\cup x_0)\backslash x_s)$ vanish, and Equation (3.8) takes the form (2.2a). Furthermore, due to Proposition 2.2, Equation (2.5) holds. Since it was established above that Equation (0.1) for $\bar{x}'$ is of the form (2.1c), the assumptions of Proposition 2.3 are valid. Hence $f(\bar{x}')=0$. This completes the proof for the Case d_0) and the proof of Lemma 3.1 in the whole.

Remarks. a) The C^3-property of U is used only in the Case d_0) (via Basic Lemma), otherwise it is enough to have the C^1-property.

b) Relation (0.2) for $(n-2, m_3, k_3)$ with $0\leqq m_3\leqq m-3$ is used only in the Case d_0).

c) There are two triples, namely $(2,1,0)$ and $(1,0,0)$, for which (0.2) is known to be false. We shall refer to them as exceptional. For $n=3,4$ there are exceptional triples among those listed in the conditions of Lemma 3.1. However using the previous remark one can easily check that for any $(n,m,0)$ except $(3,3,0)$ these exceptional triples may be omitted from the conditions of Lemma 3.1 without detriment for the proof.

The case $(3,3,0)$ requires a special analysis.

Lemma 3.2. *Suppose* (0.2) *holds for the triples* $(4,4,0)$, $(3,2,0)$, $(3,1,0)$, *and* $(2,0,0)$. *Then* (0.2) *holds for the triple* $(3,3,0)$.

Proof. First we show that the assumptions of Lemma 3.2 imply that the function $f(\bar{x})$ for $n(\bar{x})=2$ does not depend on $v, v\in\bar{x}$. We may suppose that $\bar{x}=(x_1,x_2)$, $x_i=(q_i,v_i)$, $i=1,2$, where $|q_1-q_2|<d_1$. Clearly, both q_1 and q_2 are external points in $\bar{x}$. Taking $x_0=(q_0,v_0)\in B^a_{q_1}\times R^\nu$ we may write Equation (0.1) for $\bar{x}\cup x_0$ in the form

$$\langle \partial_{v_1} f(\bar{x}), \partial_{q_1} U(|q_0-q_1|)\rangle + \partial_{v_1} f(x_1\cup x_0), \partial_{q_1} U(|q_1-q_2|)\rangle = 0. \tag{3.9}$$

This follows from the fact that $\bar{x}\cup x_0$ is of the type $(3,2,0)$. Equation (3.9) coincides with (3.12, **2**). Further, Equation (0.1) for $\bar{x}$ is of the form (3, 2b, **2**). The analysis of this pair of equations does not differ from that in the proof of Lemma 3.2(i), **2**. As a result we get the assertion claimed.

Now let $\bar{x}\in D^0$ is of the type $(3,3,0)$, and $|q'-q''|\neq d_1$ for any pair $q',q''\in\bar{x}$. Clearly, any $x=(q,v)\in\bar{x}$ is external in $\bar{x}$. Consider the set $B_q\subset B^e_q$, $q\in\bar{x}$, defined as above. Let $q'\in B_q$, $x'=(q',v')$ and $\bar{x}'=(\bar{x}\backslash x)\cup x'$. Then the type of $\bar{x}'$ is $(3,3,0)$. Again we have to prove that $f(\bar{x}')$ is locally constant w.r.t. $q'\in B_q$.

Let $x'\in B_q\times R^\nu$, $x_0=(q_0,v_0)\in B^a_{q'}\times R^\nu$. Due to the conditions of Lemma 3.2, Equation (0.1) for $\bar{x}'\cup x_0$ takes the form (3.6), whence we obtain that $\partial_{v'} f(\bar{x}')=0$. Due to the assertion above, Equation (0.1) for $\bar{x}'$ takes the form (3.4) which gives (3.7).

4. The Case $k>0$

In this section we consider the case $k>0$. Assuming Basic Lemma to be proved we establish here the following

Lemma 4.1. *Let* (n,m,k), $n\geqq 3$, $k\geqq 0$, *be an admissible triple. Suppose* (0.2) *holds for the following triples (whenever they are admissible):* $(n+1,m+1,k)$, (n,m,k_1), (n,m_1,k_1'), $(n-1,m-1,k_2)$, $(n-1,m_2,k_2')$, $(n-2,m-2,k_3)$, $(n-2,m_3,k_3')$, *where* $0\leqq m_i\leqq m-i$, $0\leqq k_i<k$, $k_i'\geqq 0$. *Then* (0.2) *holds for the triple* (n,m,k).

Proof. Let $\bar{x}\in D^0$, $n(\bar{x})=n$, $m(\bar{x})=m$, $k(\bar{x})=k$, and $x=(q,v)\in\bar{x}$ be an external point with $k(\bar{x},q)=k$. As for $k=0$, the four cases are possible:

a$_k$) q is isolated in $\bar{x}$,

b$_k$) q is a non-isolated and non-end point in $\bar{x}$: there are $q',q''\in\bar{x}\backslash x$, $q'\neq q''$, such that $|q-q'|<d_1$, $|q-q''|<d_1$,

c$_k$) q is an end non-c-point in $\bar{x}$,

d$_k$) q is a c-point in $\bar{x}$.

As above, we consider these cases successively assuming that $|q'-q''|\neq d_1$ for any pair $q', q''\in\bar{x}$.

a_k) Let $x'=(q',v')\in B^e_q\times R^\nu$, $\bar{x}'=(\bar{x}\backslash x)\cup x'$; clearly, $k(\bar{x}',q')=k$ i.e., the type of $\bar{x}'$ is (n,m,k). If $q_0\in B^a_{q'}\subset B^e_{q'}$, then q_0 is an external point in $\bar{x}'\cup x_0$, $x_0=(q_0,v_0)$. We use the following

Proposition 4.2. *Let* $\bar{x}\in D^0$, $x=(q,v)\in\bar{x}$ *be an external point, and* $k(\bar{x})=k(\bar{x},q)$. *Then for all* $x_0=(q_0,v_0)\in B^a_q\times R^\nu$, $k(\bar{x}\cup x_0)=k(\bar{x}\cup x_0,q_0)=k(\bar{x})$.

Proof of Proposition 4.2. We first show by induction w.r.t. $k(\bar{x},q)$ that $k(\bar{x}\cup x_0,q_0)=k(\bar{x},q)$. For $k(\bar{x},q)=0$, this follows immediately from the definition of the order and the condition $q_0\in B^a_q$.

Assume the equality $k(\bar{x}\cup x_0,q_0)=k(\bar{x},q)$ is proved for all cases, where $k(\bar{x},q)\leqq\bar{k}-1$ and consider the case $k(\bar{x},q)=\bar{k}$. By definition, for any end non-$c(q)$-point $x'\in\bar{x}$, $x'\neq x$ the order $k(\bar{x}\backslash x',q)$ is no more than $\bar{k}-1$, and for some such x' (we denote it by x'_0) the equality holds. The inductive assumption gives that $k((\bar{x}\cup x_0)\backslash x',q_0)=k(\bar{x}\backslash x',q)\leqq\bar{k}-1$ for any end non-$c(q)$-point $x'\in\bar{x}$, $x'\neq x$, and for $x'=x'_0$ the equality holds.

It is obvious that every end non-$c(q_0)$-point $x'\in\bar{x}\cup x_0$ is non-$c(q)$-point. Thus from the definition of the order we get that $k((\bar{x}\cup x_0),q_0)=\bar{k}$.

The final remark is that $k(\bar{x}\cup x_0,q')\geqq k(\bar{x}\cup x_0,q_0)$ for any external $q'\in\bar{x}\cup x_0$. Indeed, the inequality $k(\bar{x}\cup x_0,q')\geqq k(\bar{x},q')+1$ holds for any q' such that x_0 is a non-$c(q')$-point. Hence for such q' $k(\bar{x}\cup x_0,q')\geqq k(\bar{x})+1=k(\bar{x},q)+1=k(\bar{x}\cup x_0,q_0)+1$. If, on the contrary, q' is $c(q_0)$-point then $k(\bar{x}\cup x_0,q')=k(\bar{x}\cup x_0,q_0)$ as a simple argument shows. This completes the proof of Proposition 4.2.

Using Proposition 4.2, we conclude that $k(\bar{x}'\cup x_0)=k$. Now the type of $\bar{x}'\cup x_0$ is $(n+1,m+1,k)$, and hence $f(\bar{x}'\cup x_0)$ and the gradients $\partial f(\bar{x}'\cup x_0)$ vanish.

For every $\tilde{x}=(\tilde{q},\tilde{v})\in\bar{x}\backslash x$ we have three possibilities: either $\tilde{q}$ is isolated in $\bar{x}'\cup x_0$, or there exist $q'',q'''\in\bar{x}\backslash(x\cup\tilde{x})$, $q''\neq q'''$, such that $|q''-\tilde{q}|<d_1$, $|q'''-\tilde{q}|<d_1$, or, finally, $\tilde{q}$ is an end point. The analysis of the two first cases is the same as that for $k=0$. In the third case we have

$$k((\bar{x}'\cup x_0)\backslash\tilde{x})\leqq k((\bar{x}'\cup x_0)\backslash\tilde{x},q_0)\leqq k(\bar{x}'\cup x_0,q_0)-1\,. \tag{4.1}$$

This follows from the definition of the order and the fact that $\tilde{x}\in\bar{x}\backslash x$ is a non-$c(q_0)$-point in $\bar{x}'\cup x_0$. Due to Proposition 4.2, the RHS of (4.1) is $(k-1)$. Hence, in the third case $f((\bar{x}'\cup x_0)\backslash\tilde{x})$ and the gradients $\partial f((\bar{x}'\cup x_0)\backslash\tilde{x})$ vanish. This means that Equation (0.1) for $\bar{x}'\cup x_0$ takes the form (3.1). The same arguments as for $k=0$, give that $f(\bar{x}')$ does not depend on v'.

Further, repeating the analysis of Equation (0.1) for $\bar{x}'$ given in the Case a_0) we arrive at Equation (3.4) and then (3.5). The final arguments do not differ from that in the Case a_0).

b_k) The arguments used in that case are essentially the same as in the Case b_0). First, the problem is reduced to checking that $f(\bar{x}')$ is locally constant w.r.t. $q'\in B_q$, where $\bar{x}'=(\bar{x}\backslash x)\cup x'$, $x'=(q',v')$, and $B_q\subset B^e_q$ is defined in the same way as in Section 3.

For $x'=(q',v')\in B_q\times R^\nu$ and $x_0=(q_0,v_0)\in B^a_{q'}\times R^\nu$ we obtain in view of Proposition 4.2 that $k(\bar{x}'\cup x_0)=k$. Hence the type of $\bar{x}'\cup x_0$ is $(n+1,m+1,k)$, and $f(\bar{x}'\cup x_0)$ and the gradients $\partial f(\bar{x}'\cup x_0)$ vanish.

Now for every $\tilde{x}=(\tilde{q},\tilde{v})\in\bar{x}'$ there are three possibilities listed above, and the same analysis as above shows that Equation (0.1) for $\bar{x}'\cup x_0$ takes the form (3.6), Equation (0.1) for $\bar{x}'$ takes the form (3.5). This leads to (3.7). The proof for the Case b_0) is completed.

c_k) The proof for this case repeates the arguments we already used. So we omit it from the paper.

d_k) Let $\bar{y}=[q_1,\ldots,q_s]$, $s\geqq 2$, be a chain in $\bar{x}$ with $q_1=q$. As in Case d_0), the problem is reduced to checking that $f(\bar{x}')=0$ for $\bar{x}'=(\bar{x}\backslash x_1)\cup x_1'$, $x_1'=(q_1',v_1')\in B_{q_1}\times R^\nu$.

Let $x_1'=(q_1',v_1')\in B_{q_1}\times R^\nu$, $x_0=(q_0,v_0)\in B^a_{q_1'}\times R^\nu$. According to Proposition 4.2, the type of $\bar{x}'\cup x_0$ is $(n+1,m+1,k)$, and hence $f(\bar{x}'\cup x_0)$ and the gradients $\partial f(\bar{x}'\cup x_0)$ vanish. For every $\tilde{x}=(\tilde{q},\tilde{v})\in\bar{x}'$, $\tilde{q}\neq q_s$, we have three possibilities listed in the Case a_k). The same analysis as above shows that Equation (0.1) for $\bar{x}'\cup x_0$ takes the form (3.8) (with q' replaced by q_1'), while Equation (0.1) for $\bar{x}'$, and $\bar{x}'\backslash x_s$, $x_s=(q_s,v_s)\in\bar{x}$, takes the form (2.1c), and (2.1e), respectively.

Consider a particular case, where q_s is an accessible point in $\bar{x}'$, and q_2 is an external point in $\bar{x}\backslash x_1$. If $x_{s+1}=(q_{s+1},v_{s+1})\in B^a_{q_s}\times R^\nu$, then Equation (0.1) for $\bar{x}'\cup x_{s+1}$ takes the form (2.1b).

Now write down Equation (0.1) for $\bar{x}\backslash x_1$. Our assumption on q_2 and Proposition 4.2 together give

$$k(\bar{x}\backslash x_1)\leqq k(\bar{x}\backslash x_1,q_2)=k(\bar{x},q_1)=k\,.$$

Further arguments are the same as those for $\bar{x}'\backslash x_s$. As a result we get Equation (0.1) for $\bar{x}\backslash x_1$ in the form (2.1d). Thus for our particular case all conditions of Basic Lemma are fulfilled. Due to Basic Lemma and Propsosition 2.2, $f(\bar{x}')$ and the gradients $\partial f(\bar{x}')$ vanish, and Equation (2.4) holds.

In general case we apply the assertion stated just now to $(\bar{x}\cup x_0)\backslash x_s$ and then finish the proof as in Section 3, d_0).

Remarks. a) As for the case $k=0$, the C^3-property of U is used only in d_k).

b) Relation (0.2) for $(n-2,m-2,k_3)$ and $(n-2,m_3,k_3')$ with $0\leqq m_3\leqq m-3$, $0\leqq k_3<k$, and $k_3'\geqq 0$ is also used only in d_k).

c) As for the case $k=0$, the exceptional triples occur in conditions of Lemma 4.1 for $n=3$ and 4. The unique admissible triples (n,m,k) with $n=3,4$ and $k>0$ are $(4,3,1)$ and $(4,2,1)$. The previous remark shows that both of them can be treated without making use exceptional triples.

Summarizing all what we have said in Sections 3 and 4 we obtain

Lemma 4.3. *Let* (n,m,k), $n\geqq 3$, $k\geqq 0$, *be an admissible triple. Suppose* (0.2) *holds for all admissible non-exceptional triples indicated in Lemma 3.1 for* $k=0$ *and in Lemma 4.1 for* $k>0$. *Then* (0.2) *holds for the triple* (n,m,k).

5. Theorem 1.2 Follows from Basic Lemma

Assuming Basic Lemma to be proved we establish in this section Theorem 1.2. First suppose $\bar{x}\in D^0\cap M_n$ and $m(\bar{x})=0$ [and, consequently, $k(\bar{x})=0$]. Lemma 3.1′, **2** says that in that case $f(\bar{x})=0$; this is the first induction step.

Now we state an assertion which immediately follows from Lemma 4.3 and leads us to Theorem 1.2.

Proposition 5.1. *Fix* $n' \geqq 2$, $m' \geqq 0$, *and* $k' \geqq 0$. *Suppose* (0.2) *to be true for any admissible non-exceptional triple* (n, m, k) *with* $0 \leqq n \leqq n_0$, $m \geqq 0$, $k \geqq 0$, $n - m \geqq n_0 - m' + 1$ *and with* $n > n'$, $n - m = n_0 - m'$, $k = k'$. *Then* (0.2) *is true for the triple* $(n', n' - n_0 + m', k')$.

Having (0.2) established for all admissible triples (n, m, k) with $n - m \geqq c$, one uses Proposition 5.1 to prove (0.2) successively for $n - m \geqq c + 1$, $k = 0$, then for $n - m \geqq c + 1$, $k = 1$, and so on. Thus we establish (0.2) for admissible triples in the following order (we suppose that $n_0 > 4$):

$(n_0, 0, 0)$,

$(n_0, 1, 0)$, $(n_0 - 1, 0, 0)$, $(n_0, 1, 1)$,

$(n_0, 2, 0)$, $(n_0 - 1, 1, 0)$, $(n_0 - 2, 0, 0)$, $(n_0, 2, 1)$, $(n_0 - 1, 1, 1)$,

$(n_0, 3, 0)$, etc. ..., (n_0, m_0, k_0).

Here m_0, k_0 are maximal m and k for which (n_0, m, k) is admissible.

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