Dirichlet Forms and Markov Semigroups on C^*-Algebras[*]

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Abstract. We extend the classical theory of Dirichlet forms and associated Markov semigroups to the case of a C^*-algebra with a trace. Semigroups of completely positive maps are characterized by completely positive Dirichlet forms.

1. Introduction

A powerful method for the generation of Markov processes in the commutative case is given by the classical theory of Dirichlet forms and spaces. This theory has its roots in classical potential theory and has been developed particularly since the fundamental work of Beurling and Deny [6]. The theory is closely related with Dynkin's and Hunt's theory of stong Markov processes and has been greatly developed recently in its symmetric L_2-version particularly by Fukushima and Silverstein, see [20, 21, 30, 31] and [2–4].

Since the theory of Dirichlet forms in the commutative case deals with forms which are monotone with respect to a class of contractions applied to certain subalgebras of continuous functions, it is natural to expect a non commutative extension of the theory to the case of C^*-algebras. It is the purpose of this paper to show that, at least in the case of C^*-algebras with a trace, this idea can actually be carried through. The outcome are Markov semigroups, i.e. positivity preserving semigroups of maps, and completely Markov semigroups, i.e. semigroup of completely positive maps. Positive and completely positive maps of C^*-algebras have been the object of many investigations, standard references for foundational work are e.g. [5, 8, 32, 33]. More recently a considerable renewed interest in completely positive maps has arisen particularly in connection with certain foundational problems of non equilibrium statistical mechanics. We allude here to the large body of work on the so called quantum dynamical semigroups and quantum stochastic process, see e.g. [1, 7, 9, 10, 12–19, 22–28]. Notably a classification of norm continuous completely positive map on the C^*-algebra $\mathscr{B}(\mathscr{H})$ bounded operators on a Hilbert space has been achieved, [28], see also [22]. For

other work concerned with the construction and classification of completely positive maps see e.g. [7, 9, 10, 12–17, 19, 26]. Markov structures associated with free Fermi fields are studied in [29, 34]. In this paper we show in particular that the method of Dirichlet forms permits to obtain a large class of generators of positive and of completely positive maps, which goes beyond the classes obtained previously by other means.

We now summarize shortly the content of the paper. In Section 2 we introduce the Dirichlet forms on a C^*-algebra A with a lower semicontinuous trace τ, as positive quadratic forms E on the hermitian part of $L^2(A, \tau)$ which have a certain contraction property. We also introduce symmetric Markov semigroups as strong contraction semigroups $\Phi_t(\cdot)$ on $L^2(A, \tau)$, symmetric with respect to the scalar product given by τ and such that $0 \leq x \leq 1$ implies $0 \leq \Phi_t(x) \leq 1$. We show that the positive quadratic form given by the infinitesimal generator of a symmetric Markov semigroup Φ_t on $L^2(A, \tau)$ is a Dirichlet form E. Conversely we show that the symmetric contraction semigroup generated on $L^2(A, \tau)$ by a Dirichlet form is a Markov semigroup.

In Section 3 we introduce the concept of a completely Markov semigroup, as a Markov semigroup such that Φ_t is completely positive for all $t \geq 0$. We prove that Φ_t is a completely Markov semigroup on $L^2(A, \tau)$ if and only if there is a weight ϱ on the algebraic tensor product $A \otimes \bar{A}$, with the square integrable elements, such that $\tau(\Phi_t(x)y^*) = \varrho(x \otimes y)$, for all $x \in A$, $y \in \bar{A}$. We call a sesquilinear form E on $L^2(A, \tau)$ a completely Dirichlet form if $\sum_{ij} E(x_{ij}, x_{ij})$ is a Dirichlet form on the hermitian part of $L^2(A \otimes M_n, \tau \otimes \tau_n)$, where M_n are the $n \times n$ matrices and τ_n the corresponding trace, for all n. We show that a semigroup Φ_t is completely Markov if and only if the corresponding Dirichlet form E is completely Dirichlet. A criterium for this is that E be the monotone upwards limit of a sequence of positive bounded forms of the form $w(x^2) + \varrho((x \otimes 1 - 1 \otimes x)^2)$, where w and ϱ are weights.

In Section 4 we give another characterization of Dirichlet forms as forms that contract under so called normal contractions, which are the non commutative version of the classical concept of normal contraction. This characterization is then used to construct a class of completely Dirichlet forms and thus of completely Markov (in particular: completely positive) semigroups, for the case of a C^*-algebra contained in the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on some Hilbert space $\mathscr{H}$. The class is of the type

$$E(x, x) = \mathrm{tr}(x^2 M) + \sum_{i=1}^{\infty} \mathrm{tr}([x, m_i]^*[x, m_i]),$$

where M is an arbitrary positive self-adjoint operator (not necessarily bounded) on $\mathscr{H}$ and m_i are operators in $\mathscr{B}(\mathscr{H})$ such that $\mathrm{tr}(m_i^* m_i) < \infty$.

2. Dirichlet Forms and Markov Semigroups

Let A be a C^*-algebra with a lower semicontinuous faithful trace τ (for the definition see e.g. [11], Section 6.1). Let $A_\tau = \{x \in A, \tau(x^*x) < \infty\}$, then A_τ is a two sided ideal of A and we shall assume that A_τ is dense in A (which, together with the lower semicontinuity implies semifiniteness [11], 6.1.3). The trace τ, first defined for

positive elements of A, is extended by linearity to the (non closed) linear span of $A_\tau^* A_\tau$, where the value is finite (and complex). A_τ with the sesquilinear form $\tau(y^*x)$ is a pre-Hilbert space and its completion will be denoted by $L^2(A, \tau)$. We then have that $A_\tau = A \cap L^2(A, \tau)$ is dense in A as well as in $L^2(A, \tau)$. Since $\tau(y^*x) = \tau(yx^*)^-$ (where $^-$ means complex conjugation) we see that $x \to x^*$ extends by continuity to an antiisometry of $L^2(A, \tau)$. For any $a \in A$ we define $x \to ax$, $x \in A_\tau$, and since $\tau(x^*a^*ax)$ $\leq \|a\|^2 \tau(x^*x)$ we see that the mapping $x \to ax$ extends by continuity to an element in $B(L^2)$ [the space of bounded linear operators on $L^2(A, \tau)$]. Hence $\pi(a)x = ax$ gives us a continuous mapping of A into $B(L^2)$ and since $\pi(a)^* = \pi(a^*)$ we see that π is a *-representation of A on L^2. Since τ is faithful and A_τ is dense in A it follows that π is faithful. Hence π is an injection of A in $B(L^2)$, and we may identify A with its image $\pi(A)$. Now any $x \in A$ defines a densely defined map of $L^2(A, \tau)$, namely $a \to xa$ with domain A_τ.

Since $\tau(a^*x^*xa) = \tau(xaa^*x^*) \leq \|a\|^2 \tau(xx^*) = \|a\|^2 \tau(x^*x)$, we see that, for any $a \in A$, $x \to xa$ is strongly continuous in $L^2(A, \tau)$, hence it extends by continuity to all of $L^2(A, \tau)$. We shall also denote this extended map by $x \to xa$. It is a bounded map from $L^2 \times A$ into L^2. Hence for any fixed element $x \in L^2$ we have a densely defined map $a \to xa$ with domain A_τ. We have that $\tau(b^*xa) = \tau((x^*b)^*a)$ for any a and b in A_τ. Hence $a \to xa$ has a densely defined adjoint $a \to x^*a$ and it is therefore closable. We denote its closure by $\pi(x)$. $\pi(x)$ is then an extension of the representation π on A_τ to all of $L^2(A, \tau)$ mapping the elements in $L^2(A, \tau)$ into closed (possibly unbounded) operators in $L^2(A, \tau)$, such that $\pi(x)^* = \pi(x^*)$ (see e.g. [11b], Chapter 1, § 5, Exercise 6.e) and one may verify that π is linear in the sense that $\pi(x) + \pi(y) \subseteq \pi(x + y)$ and $\pi(\lambda x) = \lambda \pi(x)$. From the fact that π on A is faithful it follows that π on $L^2(A, \tau)$ is one-to-one and therefore allows us to identify $L^2(A, \tau)$ with a subset of closed operators on $L^2(A, \tau)$. We have especially that if x is invariant under $*$, i.e. $x^* = x$, then $\pi(x)$ $= \pi(x)^*$, so that $\pi(x)$ is self adjoint. Hence if $x \in L^2(A, \tau)$ we say that x is self adjoint if $x^* = x$ and we also say that $x \geq 0$ iff $\pi(x) \geq 0$ and $0 \leq x \leq 1$ iff $0 \leq \pi(x) \leq 1$ and so on.

A strongly continuous contraction semigroup Φ_t, $t \in R^+$, on the Hilbert space $L^2(A, \tau)$ is said to be *symmetric* iff $\langle \Phi_t(x), y \rangle = \langle x, \Phi_t(y) \rangle$ where $\langle \ , \ \rangle$ is the scalar product in L^2 and it is said to be *Markov* iff $0 \leq x \leq 1$ implies that $0 \leq \Phi_t(x) \leq 1$. It is said to be *conservative* if for any $a \in A^+$ we have $\tau(\Phi_t(a)) = \tau(a)$. In general if $M \in B(L^2)$ such that $0 \leq x \leq 1$ implies that $0 \leq M(x) \leq 1$ we say that M is *Markov*. If Φ_t is a strongly continuous contraction semigroup on $L^2(A, \tau)$ then the corresponding resolvent is

$$G_u = \int_0^\infty e^{-ut} \Phi_t \, dt, \quad u \geq 0. \tag{2.1}$$

We have that G_u satisfies the resolvent equation

$$G_u - G_v = -(u - v) G_u G_v \tag{2.2}$$

for u and v positive. Moreover we see that G_u is symmetric iff Φ_t is symmetric and G_u is Markov iff Φ_t is Markov. The latter follows from

$$\Phi_t(x) = \lim_{n \to \infty} \left(\frac{n}{t} G_{n/t} \right)^n (x) \tag{2.3}$$

where the limit is taken in the strong L^2-sense. Let now $x = x^*$ in L_2 such that $\pi(x)$ is bounded. Thus $-\|\pi(x)\| \leq \pi(x) \leq \|\pi(x)\|$ so that if Φ_t is Markov we have that $-\|\pi(x)\| \leq \Phi_t(x) \leq \|\pi(x)\|$. Hence $\|\pi(\Phi_t(x))\| \leq \|\pi(x)\|$. Since π is one-to-one on $L^2(A, \tau)$ we have that $\|x\|_\infty = \|\pi(x)\|$ [operator norm of $\pi(x)$] is a, possibly unbounded, norm on L^2. We have proved that $\|\Phi_t(x)\|_\infty \leq \|x\|_\infty$ and in the same way we get that $\|uG_u(x)\|_\infty \leq \|x\|_\infty$. From this it follows that Φ_t is a strong contraction semigroup on $L^\infty(A, \tau)$ i.e. on the completion of the domain of $\| \ \|_\infty$ in $L^2(A, \tau)$ with respect to the $\| \ \|_\infty$-norm. Hence by the theory of strong contraction semigroups we have that (2.3) also holds in the strong L^∞-sense. Since on the other hand the L^∞-norm restricted to $A_\tau = A \cap L^2$ coincides with the A-norm we have that A is the L^∞-closure of A_τ. This together with (2.1) and (2.3) gives us that Φ_t leaves A invariant if and only if G_u leaves A invariant. We thus have the following lemma

Lemma 2.1. *Any Markov semigroup Φ_t on $L^2(A, \tau)$ extends to a strongly continuous semigroup on $L^\infty(A, \tau)$. Moreover this extension leaves $A \subset L^\infty(A, \tau)$ invariant if and only if the corresponding Markov resolvent leaves A invariant.* $\quad\square$

Let now x and y be self-adjoint elements in A_τ. Then $\tau(x^2) = \|x\|_2^2$ and $\tau(y^2) = \|y\|_2^2$ are finite, where $\| \ \|_2$ is the $L^2(A, \tau)$-norm. For x and y such that $f(x)g(y) \in A_\tau^* A_\tau$ we have that $\tau(f(x)g(y))$ is non negative and finite and otherwise it is $+\infty$. Moreover $\tau(f(x)g(y))$ is linear in f and g and we have

$$\tau(f(x)g(y)) \leq \|f(x)\|_2 \|g(y)\|_2 \, .$$

Since τ is a trace we have that if $f(\alpha) \leq c|\alpha|$, for all real α and some constant c, then

$$\|f(x)\|_2^2 \leq c^2 \|x\|_2^2 < \infty \, .$$

For f and g of this form there exists then a positive measure $\mu_{x,y}$ on $(R - \{0\}) \times (R - \{0\})$ with support contained in $\text{Spec}(x) \times \text{Spec}(y)$ such that

$$\tau(f(x)g(y)) = \iint f(\alpha)g(\beta)d\mu_{x,y}(\alpha, \beta) \, . \tag{2.4}$$

Moreover since $\tau(x^2 + y^2) < \infty$ we have

$$\iint (\alpha^2 + \beta^2)d\mu_{x,y}(\alpha, \beta) < \infty \, .$$

So that any continuous function $h(\alpha, \beta)$ with $|h(\alpha, \beta)| \leq c(\alpha^2 + \beta^2)$ on $(R - \{0\}) \times (R - \{0\})$ is in $L_1(d\mu_{x,y})$. In particular if f and g are Lipschitz continuous functions on R with $f(0) = g(0) = 0$, then $f(\alpha)g(\beta)$ and $(f(\alpha) - g(\beta))^2$ are both in $L_1(d\mu_{x,y})$ and by (2.4) we have

$$\tau((f(x) - f(y))^2) = \iint (f(\alpha) - f(\beta))^2 d\mu_{x,y}(\alpha, \beta) \, . \tag{2.5}$$

Let now $\text{Lip}(R, 0)$ be the Banach space of Lipschitz continuous functions of R into R which leave zero invariant. $\text{Lip}(R, 0)$ is a Banach space in the natural norm

$$\|f\|_{\text{Lip}} = \inf\{m; |f(\alpha) - f(\beta)| \leq m|\alpha - \beta|, \forall(\alpha, \beta) \in R^2\} \, . \tag{2.6}$$

Let $\|x\|_2$ be the $L^2(A, \tau)$ norm then we get from (2.5) that, for x and y self adjoint in A and $f \in \text{Lip}(R, 0)$,

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\text{Lip}} \|x - y\|_2 \, . \tag{2.7}$$

But this tells us that the mapping $x \to f(x)$ is uniformly continuous in the L^2-norm and therefore extends to a mapping from $L_h^2(A, \tau)$ into $L_h^2(A, \tau)$ such that (2.7) still holds, where $L_h^2(A, \tau)$ is the real Hilbert space of Hermitian elements in $L^2(A, \tau)$ i.e. $x \in L_h^2(A, \tau)$ iff $x = x^*$, $x \in L^2(A, \tau)$.

Lemma 2.2. *Let $f \in \mathrm{Lip}(R, 0)$ then the mapping $x \to f(x)$ defined on the Hermitian part A_τ^h of A_τ is uniformly continuous in the strong L^2-norm topology and thus extends to $L_h^2(A, \tau)$ where it satisfies*

$$\| f(x) - f(y) \|_2 \leqq \| f \|_{\mathrm{Lip}} \| x - y \|_2 \,. \qquad \square$$

Let now M be a bounded operator on $L^2(A, \tau)$ which is symmetric and Markov.

We shall proceed similarly as we did for the derivation of (2.5). Let x and y be in A_τ^h, then we have that, for f and g positive and continuous functions on R, $\tau(f(x)M(g(x)))$ is positive and linear in f and g. Moreover for f and g such that $f(\alpha) \leqq c|\alpha|$ and $g(\alpha) \leqq c'|\alpha|$, where c, c' are constants, we see as before that there exists a positive measure μ_x on $(R - \{0\}) \times (R - \{0\})$ with support contained in $\mathrm{Spec}(x) \times \mathrm{Spec}(x)$ such that $\mu_x(\alpha, \beta) = \mu_x(\beta, \alpha)$ and

$$\tau(f(x)M(g(x))) = \iint f(\alpha)g(\beta)d\mu_x(\alpha, \beta) \,. \tag{2.8}$$

Since also $1 - M(1)$ is positive, we have also that there is a positive measure v_x on $R - \{0\}$ with support contained in $\mathrm{Spec}(x)$ such that

$$\tau(f(x)(1 - M(1))) = \int f(\alpha)dv_x(\alpha) \,. \tag{2.9}$$

Consider now the quadratic form $\tau(x(1 - M)x)$. We then have

$$\tau(f(x)(1 - M)f(x)) = \tau(f(x)^2(1 - M(1))) + \tau(f(x)^2M(1) - f(x)Mf(x)) \,. \tag{2.10}$$

From (2.8) and (2.9) we therefore have

$$\tau(f(x)(1 - M)f(x)) = \int f(\alpha)^2 dv_x(\alpha) + \tfrac{1}{2} \iint (f(\alpha) - f(\beta))^2 d\mu_x(\alpha, \beta) \,. \tag{2.11}$$

But this immediately gives us that $\tau(x(1 - M)x)$ is a positive form on $A_\tau^h \times A_\tau^h$, hence on $L_h^2 \times L_h^2$, and that for any $f \in \mathrm{Lip}(R, 0)$ and $x \in A_\tau^h$ we have

$$\tau(f(x)(1 - M)f(x)) \leqq \| f \|_{\mathrm{Lip}}^2 \tau(x(1 - M)x) \,, \tag{2.12}$$

and by continuity (2.12) also holds for $x \in L_h^2(A, \tau)$.

Lemma 2.3. *Let M be a bounded operator on $L_h^2(A, \tau)$ such that M is symmetric and Markov. Then the form $\langle x, (1 - M)x \rangle$ on L_h^2 is symmetric and positive. Moreover for any $f \in \mathrm{Lip}(R, 0)$ we have that*

$$\langle f(x), (1 - M)f(x) \rangle \leqq \| f \|_{\mathrm{Lip}}^2 \langle x, (1 - M)x \rangle$$

where $\langle x, x \rangle$ is the square norm in L_h^2. $\quad \square$

Let $E(x, x)$ be a positive closed quadratic form on $L_h^2(A, \tau)$ (not necessarily bounded) with dense domain $D(E)$. We say that E is a *Dirichlet form* if in addition to being densely defined, positive and closed it satisfied the condition that $D(E)$ is invariant under the mapping $x \to f(x)$ for any $f \in \mathrm{Lip}(R, 0)$ and

$$E(f(x), f(x)) \leqq \| f \|_{\mathrm{Lip}}^2 E(x, x) \,. \tag{2.13}$$

Corollary 2.4. *If M is a bounded operator on $L^2_h(A, \tau)$ which is symmetric and Markov then $\langle x, (1 - M)x \rangle$ is a regular Dirichlet form.*

[*We say that E is a regular Dirichlet form if in addition $A^h_\tau \cap D(E)$ is norm dense in A^h and is also dense in $D(E)$ where $D(E)$ is equipped with its natural norm $\|x\|^2_E = E(x, x) + \tau(x^2)$*]. ☐

Let $\varepsilon(x, x)$ be a positive quadratic form defined on a domain $D(\varepsilon)$ which is a linear subspace of A^h (the real Banach space of Hermitian elements in A). We say that $\varepsilon(x, x)$ is a *Markov form on A* if, for any $\delta > 0$, there exists a non-decreasing real function $\varphi_\delta(t)$, $t \in R$, satisfying the following conditions

$$\varphi_\delta(t) = t \quad \text{for} \quad 0 \leq t \leq 1$$

$$|\varphi_\delta(t)| \leq |t| \quad \text{and} \quad -\delta \leq \varphi_\delta(t) \leq 1 + \delta \quad \text{for all} \quad t \tag{2.14}$$

such that if $x \in D(\varepsilon)$ then $\varphi_\delta(x) \in D(\varepsilon)$ and

$$\varepsilon(\varphi_\delta(x), \varphi_\delta(x)) \leq \varepsilon(x, x). \tag{2.15}$$

(The definition of a Markov form for a commutative C^*-algebra was given by Fukushima [20].) We say that a positive quadratic form $\varepsilon(x, x)$ on A^h is *compatible with the trace τ* if $D(\varepsilon) \cap A^h_\tau$ is dense in $L^2(A, \tau)$ and the restriction of ε to $D(\varepsilon) \cap A^h_\tau$ is a closable form in $L^2_h(A, \tau)$. If ε is compatible with τ then its closure E defines a unique non negative self adjoint operator H on $L^2_h(A, \tau)$ such that $D(E)$ is the same as $D(H^{1/2})$, the domain of $H^{1/2}$, and $E(x, x) = \langle x, Hx \rangle$. H is then of course the infinitesimal generator of a strongly continuous symmetric semigroup on $L^2(A, \tau)$. We shall return to this point later. We have now the following theorem connecting Markov and Dirichlet forms. The proof of this theorem is the same as in the commutative case where it was given by Fukushima [20].

Theorem 2.5. *Let ε be a Markov form on a C^*-algebra A. It τ is a semifinite and lower semicontinuous faithful trace such that A_τ is norm dense in A and ε is compatible with τ, then the closure E of ε on $L^2_h(A, \tau)$ is a Dirichlet form. E is a regular Dirichlet form if $D(\varepsilon) \cap A^h_\tau$ is norm dense in A^h.* ☐

Let now Φ_t be a symmetric Markov semigroup and let G_u be the corresponding resolvent. Let H be the infinitesimal generator of Φ_t. We know that H is a positive self adjoint operator on $L^2(A, \tau)$ such that $\Phi_t = e^{-tH}$ and

$$Hx = \lim_{t \to 0} \frac{1}{t} (1 - \Phi_t)x \tag{2.16}$$

or

$$Hx = \lim_{u \to \infty} u(1 - uG_u)x \tag{2.17}$$

in the sense that $x \in D(H)$ iff any one of the strong limits above exists and in which case Hx is given by the right hand side of (2.16) or (2.17).

The following lemma is an immediate consequence of the spectral decomposition of strongly continuous symmetric contraction semigroups on Hilbert spaces.

Lemma 2.6. *For any* $x \in L_h^2(A, \tau)$ *we have that*

$$\frac{1}{t} \langle x, (1 - \Phi_t) x \rangle \quad and \quad u \langle x, (1 - u G_u) x \rangle$$

increase as $t \downarrow 0$ *and* $u \uparrow \infty$. *Each of these expressions remains bounded if and only if* $x \in D(H^{1/2})$, *where* H *is the infinitesimal generator of* Φ_t *and in this case we have, with* $E(x, x) = \|H^{1/2} x\|_2^2$, *that*

$$E(x, x) = \lim_{t \downarrow 0} \frac{1}{t} \langle x, (1 - \Phi_t) x \rangle = \lim_{u \uparrow \infty} u \langle x, (1 - u G_u) x \rangle. \qquad \square$$

Now by Corollary 2.4 $\frac{1}{t} \langle x, (1 - \Phi_t) x \rangle$ is a regular Dirichlet form so that for any $f \in \mathrm{Lip}(R, 0)$ we have

$$\frac{1}{t} \langle f(x), (1 - \Phi_t) f(x) \rangle \leq \|f\|_{\mathrm{Lip}}^2 \frac{1}{t} \langle x, (1 - \Phi_t) x \rangle. \qquad (2.18)$$

Since by Lemma 2.6 $D(E)$ consists exactly of those elements x for which $\frac{1}{t} \langle x, (1 - \Phi_t) x \rangle$ remains finite as $t \downarrow 0$ we get by (2.18) that $D(E)$ is invariant under $x \to f(x)$. Moreover by taking the limit $t \downarrow 0$ in (2.18) we get that $E(f(x), f(x)) \leq \|f\|_{\mathrm{Lip}}^2 E(x, x)$. Hence we have proved that $E(x, x) = \|H^{1/2} x\|_2$ is a Dirichlet form. We summarize these results in the following theorem.

Theorem 2.7. *Let* $E(x, x) = \|H^{1/2} x\|_2$ *be the positive quadratic form given by a symmetric Markov semigroup* Φ_t *on* $L^2(A, \tau)$, *then* $E(x, x)$ *is a Dirichlet form.* $\square$

Consider now an arbitrary Dirichlet form $E(x, x)$ and let H be the corresponding positive self-adjoint operator on $L_h^2(A, \tau)$ so that $E(x, x) = \|H^{1/2} x\|_2$. Let $G_u = (u + H)^{-1}$, $u > 0$ be the corresponding resolvent. Set

$$E_u(x, x) = E(x, x) + u \langle x, x \rangle. \qquad (2.19)$$

Then we have for any x and y in $D(E)$ that

$$E_u(G_u y, x) = \langle y, x \rangle \qquad (2.20)$$

and

$$E_u(x - u G_u y, x - u G_u y) = E(x, x) + u [\langle x, x \rangle + u \langle y, G_u y \rangle - 2 \langle x, y \rangle]$$
$$= E(x, x) + u \langle x - y, x - y \rangle - u \langle y, (1 - u G_u) y \rangle. \qquad (2.21)$$

Thus $E(x, x) + u \|x - y\|_2^2$ has a unique minimum for $x = u G_u y$. Let now y be such that $y = f(y)$ where f is a contraction of the real line leaving zero fixed, i.e. $f \in \mathrm{Lip}(R, 0)$ with $\|f\|_{\mathrm{Lip}} \leq 1$. Then by the assumption that $E(x, x)$ is a Dirichlet form together with Lemma 2.2 we get, since $y = f(y)$, that

$$E(f(x), f(x)) + u \|f(x) - y\|_2^2 = E(f(x), f(x)) + u \|f(x) - f(y)\|_2^2$$
$$\leq E(x, x) + u \|x - y\|_2^2. \qquad (2.22)$$

If we take now for x the minimal point $x = u G_u y$ then we have by (2.22) that $f(x)$ also gives a minimal point and by uniqueness of this minimal point we get $f(x) = x$.

Hence we have proved that if $y=f(y)$ then $uG_u y=f(uG_u y)$. Take now $f(\alpha)$ $=(0 \vee \alpha) \wedge 1$ which is obviously a contraction of the real line leaving zero fixed, then we get that uG_u is Markov for all u. Since this implies that Φ_t is Markov we have the following

Theorem 2.8. *Let $E(x, x)$ be a Dirichlet form on $L_h^2(A, \tau)$, and let H be the positive self adjoint operator given by E. Then the symmetric contraction semigroup generated by H is a Markov semigroup on $L^2(A, \tau)$.*

3. Completely Markov Semigroups

Let now Φ_t be a strongly continuous contraction semigroup on the Hilbert space $L^2(A, \tau)$. We said that Φ_t is Markov if $x \in L^2(A, \tau)$ such that $0 \leq x \leq 1$ implies that $0 \leq \Phi_t(x) \leq 1$. Recall that $0 \leq x \leq 1$ was defined by considering the closure L_x of the operator $a \to xa$ defined for $x \in L^2(A, \tau)$ with domain $A_\tau \subset L^2(A, \tau)$ [the notation $\pi(x)$ was used for L_x in the previous section] and then $0 \leq x \leq 1$ was equivalent to $0 \leq L_x \leq 1$. $L^\infty(A, \tau)$ was then defined as the completion in the norm $\|x\|_\infty = \|L_x\|$ of the linear subspace of $L^2(A, \tau)$ consisting of elements x such that L_x is a bounded operator on $L^2(A, \tau)$, and hence the map $x \to L_x$ extends to an isomorphism of $L^\infty(A, \tau)$ with the weakly closed subalgebra $L \subset B(L^2(A, \tau))$ obtained by taking the weak closure of the set L_x with $x \in A \subset L^\infty(A, \tau)$. Hence if π is the representation of A by left translation on $L^2(A, \tau)$ considered in the previous section we have $L = \pi(A)''$ [the weak closure of $\pi(A)$]. Therefore we see that Φ_t is Markov if and only if it extends to $L^\infty(A, \tau) = L$ and defines a positivity preserving semigroup Φ_t on the W^*-algebra L such that $\Phi_t(1) \leq 1$.

Let A be a C^*-algebra and M_n the C^*-algebra of $n \times n$ complex matrices. The elements $X \in A \otimes M_n$ may be represented by $X = \{x_{ij}\}$, a $n \times n$ matrix with elements $x_{ij} \in A$, and if $Y = \{y_{ij}\}$ then $XY = \left\{ \sum_k x_{ik} y_{kj} \right\}$. If Φ is a map of A we define Φ_n as the map of $A \otimes M_n$ given by

$$\Phi_n(X) = \{\Phi(x_{ij})\} \quad \text{for} \quad X = \{x_{ij}\} \tag{3.1}$$

i.e. $\Phi_n(x) = \Phi \otimes 1_n$, where 1_n is the identity of M_n. A linear map Φ of A is said to be *completely positive* iff Φ_n is a positive map of the C^*-algebra $A \otimes M_n$ for any n. Especially we have that a completely positive map is positive. Similarly we say that a map Φ is *completely Markov* iff Φ_n is Markov for any n, and we say that a semigroup Φ_t is completely Markov iff Φ_t is completely Markov for any $t \geq 0$.

Let us now briefly recall some known constructions of the theory of Hilbert algebras (see e.g. [11b], Chapter 1, § 5). For any $x \in L^2(A, \tau)$ denote by R_x the closure of the mapping given by $a \to ax$ with domain A and let R be the weak closure of the set of elements in $B(L^2(A, \tau))$ of the form R_x.

It is easy to see that $R = \pi(A)'$ i.e. R is equal to the commutant of $\pi(A)$, so that $R' = L$ and $L' = R$. We also remark that, while the restriction of L_x to $x \in A$ is a faithful *-representation of A on $L^2(A, \tau)$, we have that the restriction of R_x to $x \in A$ is a faithful anti *-representation of A on $L^2(A, \tau)$. Let S be the anti isometry of $L^2(A, \tau)$ given by $S_x = x^*$, then it follows immediately that $R_x = SL_{x^*}S = SL_x^*S$ and therefore $\|R_x\| = \|L_x^*\| = \|L_x\|$. Hence R_x extends by continuity to an isometry of $L^\infty(A, \tau)$ onto $R \subset B(L^2(A, \tau))$.

For any C^*-algebra A we define the conjugate algebra A which is identical with A apart from the scalar multiplication which in $\bar{A}$ is defined by $(\lambda, a) \to \bar{\lambda}a$, $\lambda \in C$ and $a \in \bar{A}$, where $\bar{\lambda}$ is the complex conjugate of λ. Let now $\Phi \in B(L^2(A, \tau))$ be Markov i.e. $0 \leq x \leq 1$ implies $0 \leq \Phi(x) \leq 1$. If Φ is completely Markov then $\Phi_n = \Phi \otimes 1_n$ is Markov on $A \otimes M_n$. Let now τ_n be the natural trace on M_n, then $\tau \otimes \tau_n$ is a trace on $A \otimes M_n$ so that $\tau \otimes \tau_n(XY) \geq 0$ if $X \geq 0$ and $Y \geq 0$ in $A \otimes M_n$. Hence

$$\tau \otimes \tau_n(\Phi_n(X)Y) = \sum_{ij} \tau(\Phi(x_{ij})y_{ji})$$

$$= \sum_{ij} \tau(\Phi(x_{ij})y_{ij}^*) \geq 0 \qquad (3.2)$$

whenever $X = \{x_{ij}\}$ and $Y = \{y_{ij}\}$ are positive elements in $A \otimes M_n$. Remark that since $Y = Y^*$ we have $y_{ij} = y_{ji}^*$.

Let now $x_1, \ldots, x_n$ be in A_τ then $u = \left(\sum_i x_i \otimes y_i\right)^* \sum_i (x_i \otimes y_i) = \sum_{ij} x_i^* x_j \otimes y_i^* y_j$ is a positive element in $A \otimes \bar{A}$ (we have considered $y_1, \ldots, y_n$ to be in $\bar{A}$), where $A \otimes \bar{A}$ is the algebraic tensor product of A and $\bar{A}$. We now define a linear functional ϱ on the algebraic tensor product $A \otimes \bar{A}$ with domain of definition $A_\tau \otimes \bar{A}_\tau$, where $\bar{A}_\tau$ is the image of A_τ in $\bar{A}$, by

$$\varrho(x \otimes y) = \tau(\Phi(x)y^*), \qquad (3.3)$$

then

$$\varrho(u) = \sum_{ij} \varrho(x_i^* x_j \otimes y_i y_j^*) = \sum_{ij} \tau(\Phi(x_i^* x_j)y_j y_i^*). \qquad (3.4)$$

Now $X = \{x_i^* x_j\}$ and $Y = \{y_i y_j^*\}$ are obviously positive elements in $A \otimes M_n$ so by (3.2) we have that $\varrho(u) \geq 0$. Hence ϱ is a positive linear functional on $A \otimes \bar{A}$ or a weight on $A \otimes \bar{A}$ with domain $A_\tau \otimes \bar{A}_\tau$.

Let now conversely $\Phi \in B(L^2(A, \tau))$ be Markov and let us assume that $\tau(\Phi(x)y^*) = \varrho(x \otimes y)$, where ϱ is a weight on the algebraic tensor product $A \otimes \bar{A}$ with domain $A_\tau \otimes \bar{A}_\tau$. The densely defined weight ϱ gives rise to a representation of $A \otimes \bar{A}$ on a Hilbert space K by the GNS construction and let η be the corresponding mapping from $A \otimes \bar{A}$ into K. Consider now the linear mapping from $L^2(A, \tau)$ into K with dense domain A_τ given by

$$Vy = \eta(1 \otimes y^*). \qquad (3.5)$$

Since we have

$$(Vy, Vz) = (\eta(1 \otimes y^*), \eta(1 \otimes z^*))$$

$$= \varrho((1 \otimes z^*)^*(1 \otimes y^*)) = \varrho(1 \otimes zy^*)$$

$$= \tau(\Phi(1)yz^*) = \langle \Phi(1)y, z \rangle, \qquad (3.6)$$

where $(\ ,\ )$ is the inner product in K and $\langle\ ,\ \rangle$ is the inner product in $L^2(A, \tau)$. It follows from (3.6) that $V^*V = \Phi(1) \leq 1$ so that V is bounded and extends by continuity to a bounded linear map of $L^2(A, \tau)$ into K. We observe that if $\Phi(1) = 1$ then V is an isometric imbedding of $L^2(A, \tau)$ into K. Let π_ϱ be the representation of $A \otimes \bar{A}$ given by ϱ, by the GNS construction, and set $\pi(x) = \pi_\varrho(x \otimes 1)$. π is then a *-

representation of A on K. Then for y and z in A_τ we have

$$\langle V^*\pi(x)Vy, z\rangle = (\pi_\varrho(x\otimes 1)Vy, Vz)$$
$$= (\pi_\varrho(x\otimes 1)\eta(1\otimes y^*), \eta(1\otimes z^*))$$
$$= (\eta(x\otimes y^*), \eta(1\otimes z^*)) = \varrho((1\otimes z^*)^*(x\otimes y^*))$$
$$= \varrho(x\otimes zy^*) = \tau(\Phi(x)yz^*) = \langle \Phi(x)y, z\rangle . \tag{3.7}$$

Hence we get

$$\Phi(x) = V^*\pi(x)V \tag{3.8}$$

which is a completely positive map since it is the composition of two completely positive maps, namely $x\to\pi(x)$ and $y\to V^*yV$. We summarize these results in the following theorem

Theorem 3.1. *Let Φ be a bounded map of $L^2(A, \tau)$ into $L^2(A, \tau)$ which is Markov. Then Φ is completely Markov if and only if there is a weight ϱ on the algebraic tensor product $A\otimes\bar{A}$ with domain containing $A_\tau\otimes\bar{A}_\tau$ such that*

$$\tau(\Phi(x)y^*) = \varrho(x\otimes y). \quad \square$$

Let now Φ_t be a strongly continuous one parameter contraction semigroup on $L^2(A, \tau)$ which is symmetric and completely Markov. By the previous theorem we have a one parametric family of weights ϱ_t on $A\otimes\bar{A}$ such that $\dfrac{1}{2t}\langle\Phi_t(x), y\rangle$ $= \varrho_t(x\otimes y)$ and since for $x\in L^2(A, \tau)$ we have

$$\langle(1-\Phi_t)x, x\rangle = \langle(1-\Phi_t(1))x, x\rangle + \langle\Phi(1)x, x\rangle - \langle\Phi_t(x), x\rangle \tag{3.9}$$

then

$$\frac{1}{t}\langle(1-\Phi_t)x, x\rangle = w_t(x^2) + \varrho_t((x\otimes 1 - 1\otimes x)^2) \tag{3.10}$$

where $t\cdot w_t(x) = \tau((1-\Phi_t(1))x)$ is a weight on A, since $0\leqq\Phi_t(1)\leqq 1$. By Lemma 2.6 we have that $\dfrac{1}{t}\langle(1-\Phi_t)x, x\rangle\uparrow E(x, x)$ as $t\downarrow 0$, where $E(x, x)$ is the Dirichlet form corresponding to Φ_t. Observe that $\varrho(x\otimes y) = \overline{\varrho(y\otimes x)}$ since Φ_t is symmetric. We say that ϱ is a *symmetric weight* on $A\otimes\bar{A}$. Hence we have that the Dirichlet form $E(x, x)$ corresponding to a symmetric and completely Markov semigroup is the increasing limit of bounded Dirichlet forms of the type $E_t(x, x) = w_t(x^2) + \varrho_t((x\otimes 1 - 1\otimes x)^2)$ where w and ϱ are weights on A and $A\otimes\bar{A}$ respectively.

On the other hand assume that w_γ is a weight on A and ϱ_γ is a weight on the algebraic tensor product $A\otimes\bar{A}$, such that

$$E_\gamma(x, x) = w_\gamma(x^2) + \varrho_\gamma((x\otimes 1 - 1\otimes x)^2) \tag{3.11}$$

is a bounded positive bilinear form on $L_h^2(A, \tau)$ such that $0\leqq E_{\gamma_1}(x, x)\leqq E_{\gamma_2}(x, x)$ for $\gamma_1\leqq\gamma_2$ and let us assume that there is a closed bilinear form F such that $E_\gamma(x, x)$ $\leqq F(x, x)$ for any γ in R^+. Let now $x\in A_\tau$ such that $x = x^*$. From the spectral representation theorem and the fact that w_γ and ϱ_γ are weights we get that

$$w_\gamma(f(x)) = \int f(\alpha)dv_\gamma(\alpha) \tag{3.12}$$

and

$$\varrho_\gamma((f(x)\otimes 1 - 1\otimes f(x))^2) = \int\int (f(\alpha)-f(\beta))^2 d\mu_\gamma(\alpha,\beta),$$

where ν_γ and μ_γ are positive Radon measures on R and $R\times R$ respectively, depending on x, and with support on $\mathrm{Spec}(x)$ and $\mathrm{Spec}(x)\times\mathrm{Spec}(x)$ respectively. Hence we have that for $x\in A_\tau\cap L^2_h(A,\tau)$

$$E_\gamma(f(x),f(x)) = \int f(\alpha)^2 d\nu_\gamma(\alpha) + \int\int (f(\alpha)-f(\beta))^2 d\mu_\gamma(\alpha,\beta).$$

(3.13)

For $f\in\mathrm{Lip}(R,0)$ we have, for $x\in A_\tau\cap L^2_h(A,\tau)$, that

$$E_\gamma(f(x),f(x)) \leqq \|f\|^2_{\mathrm{Lip}} E_\gamma(x,x),$$ (3.14)

which obviously implies that E_γ with domain $A_\tau\cap L^2_h$ is Markov and since E_γ is bounded it is a Dirichlet form.

If $E(x,x)$ is a sesquilinear form on $L^2(A,\tau)$ we set

$$E_n(X,X) = \sum_{ij} E(x_{ij}, x_{ij})$$ (3.15)

for $X=\{x_{ij}\}\in L^2_h(A\otimes M_n, \tau\otimes\tau_n)$.   Since $L^2(A\otimes M_n, \tau\otimes\tau_n)=L^2(A,\tau)$ $\otimes L^2(M_n,\tau_n)$ and $E_n=E\otimes 1_n$ where 1_n is the form given by the identity in $L^2(M_n,\tau_n)$ we see that E_n is closable if and only if E is closable and if E is closed then $D(E_n)=D(E)\otimes M_n$. We say that E is a *completely Markov form* if E_n is a Markov form for each $n\geqq 1$, and we say that E is a *completely Dirichlet form* iff E_n is a Dirichlet form for each n. By what is above we see that a Dirichlet form which is completely Markov is completely Dirichlet.

Consider now $E_\gamma(x,x)$ given by (3.11). Since $E_{\gamma,n}$ is given in the way of (3.11) by $w_\gamma\otimes\tau_n$ and $\varrho_\gamma\otimes(\tau_n\otimes\tau_n)$, where τ_n is the natural trace in M_n, it follows as above that $E_{\gamma,n}(X,X)$ is a Dirichlet form. Therefore $E_\gamma(x,x)$ is a completely Dirichlet form.

Let now $E_\gamma(x,x)$ be an increasing sequence of bounded Dirichlet forms i.e. $0\leqq E_{\gamma_1}(x,x)\leqq E_{\gamma_2}(x,x)$ for $\gamma_1\leqq\gamma_2$ and let us also assume that $E_\gamma(x,x)\leqq F(x,x)$ where F is a closed form. It is then well known that $E(x,x)=\lim_{\gamma\uparrow\infty} E_\gamma(x,x)$, with domain $D(E)$ consisting of those x for which the limit is finite, is a closed form. From (3.14) we have that $x=x^*\in D(E)$ and $f\in\mathrm{Lip}(R,0)$ implies that $f(x)\in D(E)$ and $E(f(x), f(x))\leqq \|f\|^2_{\mathrm{Lip}} E(x,x)$ so that $E(x,x)$ is a Dirichlet form. If moreover E_γ is of the form (3.11) then we have that $E_{\gamma,n}$ are Dirichlet forms, and since $E_{\gamma,n}=E_\gamma\otimes 1_n\uparrow E\otimes 1_n=E_n$ it follows that E_n is a Dirichlet form so that E_γ is a completely Dirichlet form. We summarize these results in the following theorem.

Theorem 3.2. *Let E be a Dirichlet form on $L^2(A,\tau)$ and Φ_t the corresponding symmetric Markov semigroup. Then Φ_t is completely Markov if and only if E is completely Dirichlet. Moreover E is completely Dirichlet if and only if there exists an increasing sequence of positive bounded forms E_γ,*

$$0\leqq E_{\gamma_1}(x,x)\leqq E_{\gamma_2}(x,x) \quad for \quad \gamma_1\leqq\gamma_2 \quad and$$

$$E(x,x) = \lim_{\gamma\uparrow\infty} E_\gamma(x,x)$$

with domain $D(E)$ equal to the set of x for which this limit is finite, and E_γ has the form

$$E_\gamma(x, x) = w_\gamma(x^2) + \varrho_\gamma((x \otimes 1 - 1 \otimes x)^2)$$

where w_γ and ϱ_γ are weights on A and on the algebraic tensor product $A \otimes \bar{A}$ respectively, where $\bar{A}$ is the conjugate C^*-algebra corresponding to A.

4. Normal Contractions on C^*-Algebras

For the commutative C^*-algebra $C(X)$, X a locally compact space one says that $v \in C(X)$ is a *normal contraction* of $u \in C(X)$ iff $|v(\alpha)| \leq |u(\alpha)|$ and $|v(\alpha) - v(\beta)| \leq |u(\alpha) - u(\beta)|$ for any α and β in X. We see that if $v(\alpha)$ is a normal contraction of $u(\alpha)$ then $v(\alpha)$ is continuous in the topology generated by $u(\alpha)$ hence there is a continuous mapping f of the real line R into itself such that $v(\alpha) = f(u(\alpha))$, and from the two inequalities $|v(\alpha)| \leq |u(\alpha)|$ and $|v(\alpha) - v(\beta)| \leq |u(\alpha) - u(\beta)|$ it follows that $f(\alpha)$ may be taken as a contraction of R leaving zero fixed i.e. $f(0) = 0$ and $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$.

Let now $C_h(X)$ be the real (self adjoint) part of $C(X)$, then $v(\alpha)$ is a normal contraction of $u(\alpha)$ iff $v^2 \leq u^2$ and $(v(\alpha) - v(\beta))^2 \leq (u(\alpha) - u(\beta))^2$. Now $u(\alpha) - u(\beta)$ may be considered as an element in $C_h(X) \otimes C_h(X) \subseteq C_h(X \times X)$ where the tensor product is the algebraic tensor product, namely $u(\alpha) - u(\beta) \equiv u(\alpha) \otimes 1 - 1 \otimes u(\beta) \in C_h(X) \otimes C_h(X)$. Hence we may write the condition for normal contraction as

$$v^2 \leq u^2 \quad \text{and} \quad (v \otimes 1 - 1 \otimes v)^2 \leq (u \otimes 1 - 1 \otimes u)^2 \tag{4.1}$$

where the first inequality is in $C(X)$ and the second inequality is in $C(X) \otimes C(X)$. Let now A be an arbitrary C^*-algebra with a unit. If x and y are in A_h (the self adjoint part of A) then we say that x is a *normal contraction* of y if

$$x^2 \leq y^2 \quad \text{and} \quad (x \otimes 1 - 1 \otimes x)^2 \leq (y \otimes 1 - 1 \otimes y)^2 \tag{4.2}$$

where the first inequality is in A and the second inequality is in $A \otimes A$, the algebraic tensor product of A with itself. Since x and y are self-adjoint, $x = x^*$ and $y = y^*$, we may also consider the second inequality to be in $A \otimes \bar{A}$, where $\bar{A}$ is the conjugate algebra. If $y \in A_h$ and $x = f(y)$ where $f(0) = 0$ and $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$ it follows easily from what is said before that x is a normal contraction of y because in this case x and y are in the same commutative subalgebra. We shall now see that if x is a normal contraction of y then $x = f(y)$ where $f(0) = 0$ and $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$.

We may assume that $A \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and as we are only interested in the subalgebra of A generated by x and y we may also assume that $\mathcal{H}$ is separable. Let $L^2(\mathcal{H})$ be the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$. Then $L^2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$ and $B(\mathcal{H}) \otimes \overline{B(\mathcal{H})}$ is naturally imbedded in $B(L^2(\mathcal{H}))$ by the correspondence $a \otimes b \in B(\mathcal{H}) \otimes \overline{B(\mathcal{H})}$ goes to the mapping $m \to amb^*$ contained in $B(L^2(\mathcal{H}))$, ($\overline{B(\mathcal{H})}$ is the conjugate algebra of $B(\mathcal{H})$). If we denote $m \to am$ by $L(a)$ and $m \to mb$ by $R(a)$ we have the imbedding of $B(\mathcal{H}) \otimes \overline{B(\mathcal{H})}$ into $B(L^2(\mathcal{H}))$ is given by $a \otimes b \to L(a)R(b^*)$. It is easy to see that this imbedding is a faithful $*$-representation. Hence we have that $(x \otimes 1 - 1 \otimes x)^2 \leq (y \otimes 1 - 1 \otimes y)^2$ if and only if

$$(L(x) - R(x))^2 \leq (L(y) - R(y))^2 \tag{4.3}$$

in $B(L^2(\mathscr{H}))$. Hence $(x\otimes 1 - 1\otimes x)^2 \leqq (y\otimes 1 - 1\otimes y)^2$ is equivalent with the statement that for any $m\in L^2(\mathscr{H})$, i.e. for any $m\in B(\mathscr{H})$ with $\mathrm{tr}(m^*m)<\infty$ we have that

$$(m,(L(x)-R(x))^2m)\leqq(m,(L(y)-R(y))^2m) \tag{4.4}$$

where $(\ ,\ )$ is the inner product in $L^2(\mathscr{H})$ so that (4.4) is equivalent with

$$\mathrm{tr}(m^*[x[x,m]])\leqq\mathrm{tr}(m^*[y,[y,m]]). \tag{4.5}$$

Here $[x,m]=(L(x)-R(x))m$ i.e. the commutator of x and m. From (4.5) we get that

$$\mathrm{tr}([x,m]^*[x,m])\leqq\mathrm{tr}([y,m]^*[y,m]). \tag{4.6}$$

Take now m to commute with y then by (4.6) the Hilbert-Schmidt norm of $[x,m]$ is zero so that x commutes with m. Hence the commutant of x contains the commutant of y, and therefore x is in the commutative algebra generated by y. Since x and y are in the same commutative subalgebra we have the argument above that $x=f(y)$ where f is a contraction of the real line i.e. $|f(\alpha)-f(\beta)|\leqq|\alpha-\beta|$ for any real α and β. We summarize these results in the following theorem.

Theorem 4.1. *Let xa and y be self adjoint elements in a C^*-algebra A with unit. Then*

$$(x\otimes 1 - 1\otimes x)^2 \leqq (y\otimes 1 - 1\otimes y)^2$$

in the algebraic tensor product $A\otimes A$ if and only if there is a contraction f of the real line $(|f(\alpha)-f(\beta)|\leqq|\alpha-\beta|)$ such that $x=f(y)$.

Moreover x is a normal contraction of y if and only if there is a contraction f of the real line such that $f(0)=0$ and $x=f(y)$.

Combining this theorem with the results of Section 2 we get the following theorem.

Theorem 4.2. *Let $E(x,x)$ be a closed positive form on $L^2(A,\tau)$ where τ is a lower semicontinuous faithful trace on the C^*-algebra A. Then $E(x,x)$ is a Dirichlet form if and only if for $y\in D(E)$ and x a normal contraction of y, $x\in D(E)$ and $E(x,x)\leqq E(y,y)$.* $\square$

From the proof of Theorem 4.1 we have the following lemma.

Lemma 4.3. *If $A\subset B(\mathscr{H})$ and x and y are in A, then x is a normal contraction of y if and only if*

$$x^2\leqq y^2$$

and for any $m\in B(\mathscr{H})$ such that $\mathrm{tr}(m^*m)<\infty$ we have that

$$\mathrm{tr}([x,m]^*[x,m])\leqq\mathrm{tr}([y,m]^*[y,m]). \quad \square$$

Let now τ be a lower semicontinuous faithful trace on the C^*-algebra A and assume that $A\subset B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. If M is a positive selfadjoint operator on $\mathscr{H}$ (not necessarily bounded) and $m_i\in B(\mathscr{H})$ with $\mathrm{tr}(m_i^*m_i)<\infty$ we consider the form on $L^2(A,\tau)$ given by

$$E(x,x)=\mathrm{tr}(x^2M)+\sum_{i=1}^{\infty}\mathrm{tr}([x,m_i]^*[x,m_i]). \tag{4.7}$$

Then if $E(x, x)$ is closable on $L^2(A, \tau)$ then by Theorem 4.2 and Lemma 4.3 we have that $E(x, x)$ is Dirichlet. It follows easily that it is completely Dirichlet because the form $E^{(n)}(x, x)$ on $A \otimes M_n$ is obtained by replacing M by $M \otimes 1_n$ and m_i by $m_i \otimes 1_n$, and therefore $E^{(n)}(x, x)$ is again Dirichlet. Hence we have

Corollary 4.4. *If $A \subset B(\mathcal{H})$ and $M \geq 0$ is a self adjoint operator (not necessarily bounded) and $m_i \in B(\mathcal{H})$ satisfy $\mathrm{tr}(m_i^* m_i) < \infty$ then if*

$$E(x, x) = \mathrm{tr}(x^2 M) + \sum_{i=1}^{\infty} \mathrm{tr}([x, m_i]^* [x, m_i])$$

is closable on $L^2(A, \tau)$ then the closure of $E(x, x)$ is completely Dirichlet.

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