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# The Energy-Momentum Spectrum in the $P(\varphi)_2$ Quantum Field Theory

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**Abstract.** The $P(\varphi)_2$ interaction with the periodic boundary conditions is considered. It is shown that the energy-momentum spectrum lies in the forward light cone. As a consequence, this result implies that the $P(\varphi)_2$ theory in the infinite volume with the periodic boundary conditions is Lorentz invariant.

## 1. Introduction

In the present paper we prove the spectral condition for models with the $P(\varphi)_2$ interaction with the periodic boundary conditions.

The paper [1] by Glimm and Jaffe was devoted to the investigation of the energy-momentum spectrum. Their arguments were based on the uniqueness of the vacuum for the Lorentz rotated Hamiltonian $H_V(\beta)$ $(=\beta_0 H_V + \beta P_V)$ and on the fact that the uniqueness of the vacuum and the translation invariance of the Hamiltonian imply the translation invariance of the vacuum, that is, the vacuum has the zero momentum.

Unfortunately, the semigroup of operators $\exp(-tH_V(\beta))$ does not preserve the positivity even for the free Hamiltonian and so the uniqueness proof which is based on the positivity preservation and the hypercontractivity does not work.

In our proof we do not prove the uniqueness of the vacuum, instead of this we use, following Seiler and Simon [2], the Osterwalder-Schrader positivity in the spatial direction to show that the vacuum subspace of $H_V(\beta)$ contains the vector with the zero momentum. Thus, our arguments may be represented by the following diagram.

OS positivity
in the spatial direction
for $H_{0,V}(\beta), P_V$

$H_V + c(\varepsilon) \geqq (1-\varepsilon)|P_V|$ → → → → $\Downarrow$

Vacuum overlap for $H_V(\beta)$

$[H_V(\beta), P_V] = 0$ → → → → → $\Downarrow$

$H_V^2 \geqq P_V^2$

For the $P(\varphi)_2$ theory the spectral condition in a periodic box $V$ implies the spectrum condition and the Lorentz invariance of the $P(\varphi)_2$ theory in the infinite volume with periodic boundary conditions [3–5]. In addition, following Glimm and Jaffe [1], we state the uniform bounds on the vacuum expectation values of the certain products of derivatives of field operators.

The same results may be obtained for the theory with exponential type interaction [6] and for the $Y_2$ interaction [7].

## 2. The Osterwalder-Schrader Positivity in the Spatial Direction

Let $H_{0,V}$ be the free Hamiltonian and $P_V$ be the momentum operator with the periodic boundary conditions in a box of volume $V$. The operator $H_{0,V}$ commutes strongly with $P_V$. Let $H_{0,V}(\beta)=\beta_0 H_{0,V}+\beta P_V$, where $\beta_0^2-\beta^2=1$, $\beta_0>0$.

For further use we need the Osterwalder-Schrader positivity in the spatial direction for the theory with the operator $H_{0,V}(\beta)$ as the Hamiltonian and $P_V$ as the momentum.

Let $\mathscr{P}(\mathscr{P}_+)$ be the operator algebra generated by the algebra $\mathscr{M}(\mathscr{M}_+)$ and by the polynomials of the time zero fields $\varphi_V(h)=\int dx\varphi_V(x)h(x)$, $h(x)\in C_0^\infty([-V/2, V/2])$ ($h(x)\in C_0^\infty([0, V/2])$). Here $\mathscr{M}(\mathscr{M}_+)$ is the von Neumann algebra generated by the spectral projection of the time zero fields $\varphi_V(h)$, $h\in C_0^\infty([-V/2, V/2])$ ($h\in C_0^\infty([0, V/2])$). In addition, let $\mathscr{P}_{+,\mathrm{coh}}$ be the operator algebra generated by the bounded operators $\exp(i\varphi_V(h))$, where $h$ is real and $h(x)\in C_0^\infty([0, V/2])$.

We define $\mathfrak{A}$ as a free commutative and associative algebra over the complex field, which is generated by the set of 2-tuples $(t, F)$, $t\in\mathbb{R}$, $F\in\mathscr{P}$, see [8]. Here $\mathbb{R}$ is the set of reals.

Let $\mathfrak{A}_+$, $\mathfrak{A}_{+,\mathrm{coh}}$ be the subalgebras of $\mathfrak{A}$ the generating sets of which are $\mathbb{R}\times\mathscr{P}_+$ and $\mathbb{R}\times\mathscr{P}_{+,\mathrm{coh}}$, respectively.

Let $a\in\mathfrak{A}$, then $q$ may be written in the form

$$a=\sum_{k\in A}\alpha_k\prod_{j_k\in A_k}(t_{j_k}, F_{j_k}) \tag{2.1}$$

for some finite sets $A$ and $A_k$.

We set

$$S(a)=\sum_{k\in A}\alpha_k\left(\Omega_{0,V}, \bar{T}\prod_{j_k\in A_k}\hat{F}_{j_k}(t_{j_k})\Omega_{0,V}\right). \tag{2.2}$$

Here $\bar{T}$ is the operation of the antichronological ordering over the (time) variables $t$ and $\hat{F}(t)=e^{-tH_{0,V}(\beta)}Fe^{tH_{0,V}(\beta)}$.

Since $\exp(-\tau H_{0,V}(\beta))f\in\mathscr{D}(\varphi_V(h)^n)$ and $\mathscr{P}\Omega_{0,V}\subset\mathscr{D}(\varphi_V(h)^n)$ for $\tau>0$ and for each vector $f\in\mathscr{F}_V$ (= the Fock Hilbert space for the box of volume $V$) and for each $n$ and $h\in C_0^\infty([-V/2, V/2])$, so the expression for $S(a)$ is well defined.

Since $\mathscr{P}$ is a commutative and associative algebra, so $S(a)$ is a linear functional on $\mathfrak{A}$.

This proves the following lemma:

**Lemma 2.1.** *The formula (2.2) defines the linear functional on the algebra* $\mathfrak{A}$.

Let us define on the algebra $\mathfrak{A}$ the involution $\Theta$. If $a$ is given by (2.1), then

$$\Theta(a)=\sum_{k\in A}\alpha_k^*\prod_{j_k\in A_k}(t_{j_k},\vartheta F_{j_k}^*\vartheta^{-1})$$

where * denotes the complex (hermitian) conjugate and $\vartheta$ is the unitary operator of the space reflection, $\vartheta\varphi_V(x)\vartheta^{-1}=\varphi_V(-x)$, in the Fock space $\mathscr{F}_V$.

$\Theta(\cdot)$ is an antilinear operator on the algebra $\mathfrak{A}$.

**Theorem 2.2.** *$S(\Theta(a)b)$ is an hermitian positive bilinear form on $\mathfrak{A}_+$, antilinear in the first argument.*

*Proof of Theorem 2.2.* The bilinearity of the $S(\Theta(a)b)$ follows from the linearity of the $S(\cdot)$, antilinearity of the $\Theta(\cdot)$ and from the fact that $\mathfrak{A}$ is an algebra.

To prove the hermiticity and the positivity, we first demonstrate that two-point function (2.3) satisfies the Osterwalder-Schrader positivity condition in the spatial direction.

The (anti-time-ordered) two-point function is defined by

$$G_{\beta V}(t_1,x_1;t_2,x_2)=\begin{cases}(\Omega_{0,V},\varphi_V(x_1)e^{-(t_2-t_1)H_{0,V}(\beta)}\varphi_V(x_2)\Omega_{0,V}) & \text{for} \quad t_1\leqq t_2\\ (\Omega_{0,V},\varphi_V(x_2)e^{-(t_1-t_2)H_{0,V}(\beta)}\varphi_V(x_1)\Omega_{0,V}) & \text{for} \quad t_1>t_2\end{cases}. \tag{2.3}$$

Let

$$G_\beta(t_1,x_1;t_2,x_2)=\begin{cases}(\Omega_0,\varphi(x_1)e^{-(t_2-t_1)H_0(\beta)}\varphi(x_2)\Omega_0) & \text{for} \quad t_1\leqq t_2,\\ (\Omega_0,\varphi(x_2)e^{-(t_1-t_2)H_0(\beta)}\varphi(x_1)\Omega_0) & \text{for} \quad t_1>t_2,\end{cases}$$

where $\varphi(x)$, $H_0(\beta)$, $\Omega_0$ are the free field, the free Lorentz rotated Hamiltonian and the free vacuum, respectively, in the full Fock space $\mathscr{F}$, that is, in the Fock space for the free boundary conditions.

The functions $G_{\beta V}$, $G_\beta$ are translation invariant. We have

$$G_{\beta V}(t_1-t_2,x_1-x_2)=\sum_{n=-\infty}^{\infty}G_\beta(t_1-t_2,x_1-x_2+nV) \tag{2.4}$$

and the series converges in the sense of distributions.

We note that

$$G_\beta(t_1-t_2,x_1-x_2)=W_2(i\beta_0|t_1-t_2|,i\beta(t_1-t_2)+x_1-x_2)$$

where $W_2(t,x)$ is the two-point Wightman function for the free scalar field. The Hall-Wightman Theorem [9] (or the explicit form of $W_2$) implies that the two-point function $W_2$ is invariant under the complex Lorentz group. We use the complex Lorentz rotation[1] $(t,x)\to(ix,it)$ to obtain

$$W_2(i\beta_0|t|,i\beta t+x)=W_2(ix-\beta t,\beta_0|t|)=W_2(ix-\beta t,\beta_0 t).$$

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[1] We note that the original proof used the boost with the velocity $-\beta\beta_0^{-1}$. We are indebted to A. Jaffe for pointing out referee's suggestion to use the compleb Lorents invariance

Let $f_{1,2} \in C_0^\infty(\mathbb{R} \times [0, V/2])$, $\vartheta(f(t,x)) = f(t, -x)$. Using the above equality and the relation

$$|nV - x_1 - x_2| = |nV/ - x_1| + |nV/2 - x_2|$$

for $0 \leqq x_1, x_2 \leqq V/2$, we obtain that the $n$-th term of the sum (2.4) is the scalar product

$$\langle G_\beta(t_1 - t_2, x_1 - x_2 + nV), \vartheta(f_1^*) f_2 \rangle = (g_1, g_2)_{\mathscr{F}}$$

where $g_{1,2}$ are the vectors in the Fock space $\mathscr{F}$,

$$g_i = \int dt dx f_i(t,x)^{\cdot} \cdot \exp(-|nV/2 - x| H_0 + i\beta t H_0) \varphi(\beta_0 t) \Omega_0 .$$

Thus, the two-point function $G_\beta$ and, as the consequence, $G_{\beta V}$ satisfies the Osterwalder-Schrader positivity in the spatial direction.

Now a positive semidefinite bilinear form is defined on $C_0^\infty\left(\mathbb{R} \times \left[0, \frac{V}{2}\right]\right)$. We form a Hilbert space $\mathscr{F}_{G1}$ by dividing out by the vectors of norm 0 and completing. Let $\mathscr{F}_G$ be the symmetric Fock space over $\mathscr{F}_{G1}$.

Now we shall prove the hermiticity and the positivity of the bilinear form $S(\Theta(a)b)$.

First of all we assert that it is sufficient to prove the theorem for $a, b, \in \mathfrak{A}_{+,\mathrm{coh}}$.

Indeed, * algebra of bounded operators $\mathscr{P}_{+,\mathrm{coh}}$ is strongly dense in the von Neumann algebra $\mathscr{M}_+$. The Kaplansky density theorem [10, p. 46] implies that each $F \in \mathscr{M}_+$ can be approximated strongly by $F(m) \in \mathscr{P}_{+,\mathrm{coh}}$, in such a way that $F(m) \xrightarrow{st} F$ and $\|F(m)\| \leqq \|F\|$.

Moreover, on $\{\mathscr{P}\Omega_{0,V}\} \cup \{\exp(-\tau H_{0,V}(\beta)) \mathscr{F}_V, \tau > 0\}$

$$(-im)^n (\exp(i\varphi_V(h)/m) - 1)^n \xrightarrow[m \to \infty]{st} \varphi_V(h)^n$$

for real $h \in C_0^\infty\left(\left[0, \frac{V}{2}\right]\right)$.

If now $a$, $b \in \mathfrak{A}_+$ and

$$a = \sum_{k \in A} \alpha_k \prod_{j_k \in A_k} (t_{j_k}, F_{j_k})$$

then let

$$a_m = \sum_{k \in A} \alpha_k \prod_{j_k \in A_k} (t_{j_k}, F_{j_k}(m)),$$

where $F_{j_k}(m)$ are defined in the following way. $F_{j_k} \in \mathscr{P}_+$ and so

$$F_{j_k} = \sum_l G(l, j_k) P(l, j_k),$$

where $G(l, j_k) \in \mathscr{M}_+$ and $P(l, j_k)$ are the polynomials of the fields $\varphi_V(h)$, $h \in C_0^\infty\left(\left[0, \frac{V}{2}\right]\right)$.

Using Kaplansky's density theorem we replace each $G(l,j_k)$ by $G(l,j_k,m)$ with $\|G(l,j_k,m)\| \leqq \|G(l,j_k)\|$, $G(l,j_k,m) \in \mathscr{P}_{+,\mathrm{coh}}$ and

$$G(l,j_k,m) \xrightarrow{st} G(l,j_k)$$

and we replace each $\varphi_V(h)$ in the polynomials by

$$\{-im(\exp(i\varphi_V(\mathrm{Re}\, h)/m)-1)+m(\exp(i\varphi_V(\mathrm{Im}\, h)/m)-1)\}.$$

We define $b_m$ in the same way.

Now the operator equality

$$A_1 \ldots A_n - B_1 \ldots B_n = (A_1 - B_1)A_2 \ldots A_n + \ldots + A_1 \ldots A_{n-1}(A_n - B_n)$$

and simple arguments imply that

$$S(\Theta(a_m)b_m) \xrightarrow[m\to\infty]{} S(\Theta(a)b).$$

Hence it is sufficient to prove the theorem for $a, b \in \mathfrak{A}_{+,\mathrm{coh}}$.

Now let $a, b \in \mathfrak{A}_{+,\mathrm{coh}}$. We shall show that

$$S(\Theta(a)b) = (N(a), N(b))_{\mathscr{F}_G}$$

where $N$ is a normal ordering, i.e., $N : \mathfrak{A}_{+,\mathrm{coh}} \to \mathscr{F}_G$ is the linear mapping from the subalgebra $\mathfrak{A}_{+,\mathrm{coh}}$ into $\mathscr{F}_G$.

If $a \in \mathfrak{A}_{+,\mathrm{coh}}$, then

$$a = \sum_{k\in A} \alpha_k \prod_{j_k \in A_k} \left(t_{j_k}, \sum_{r\in R(j_k)} \alpha(j_k, r)\exp(i\varphi_V h_{j_k,r})\right)$$

for some finite sets $A$, $A_k$, and $R(j_k)$.

We define

$$\begin{aligned} N(a) = \sum_{k\in A} \alpha_k \sum_{r(\cdot)\in\mathscr{R}_k} \Bigg\{ &\left(\prod_{j_k\in A_k} \alpha(j_k, r(j_k))\right) \exp\Bigg(-\frac{1}{2}\int dx'dx''dt'dt'' \\ &\cdot\left(\sum_{j_k\in A_k} h_{j_k,r(j_k)}(x')\delta(t_{j_k}-t')\right) G_{\beta V}(t'-t'', x'-x'') \\ &\cdot\left(\sum_{j_k\in A_k} h_{j_k,r(j_k)}(x'')\delta(t_{j_k}-t'')\right)\Bigg) \\ &\cdot \mathrm{Exp}\left(i \sum_{j_k\in A_k} (h_{j_k,r(j_k)} \otimes \delta_{t_{j_k}})\right)\Bigg\} \end{aligned}$$

where $\delta_t = \delta(\cdot - t)$ is the translated $\delta$-function and $\mathscr{R}_k$ is the set of all functions from $A_k$ to $\bigcup_{j_k\in A_k} A(j_k)$ with the following properties, if $r(\cdot)\in\mathscr{R}_k$, then $r(j_k)\in A(j_k)$.

And we denote by $\mathrm{Exp}(if)$ a coherent vector in the space $\mathscr{F}_G$

$$\mathrm{Exp}(if) = 1 \oplus \bigoplus_{n=1}^{\infty} \frac{i^n}{n!} f(x_1,t_1) \otimes_s f(x_2,t_2) \otimes_s \ldots \otimes_s f(x_n,t_n).$$

Now it is easy to see that $N(\cdot)$ is a linear mapping from $\mathfrak{A}_{+,\mathrm{coh}}$ into $\mathscr{F}_G$.

Let us calculate $(N(a), N(b))_{\mathscr{F}_G}$ for $a, b \in \mathfrak{A}_{+,\mathrm{coh}}$

$$
\begin{aligned}
(N(a), N(b))_{\mathscr{F}_G} = \sum_{k\in A} \sum_{l\in B} \sum_{r(\cdot)\in\mathscr{R}_k} \sum_{s(\cdot)\in\mathscr{S}_l} & \\
\Bigg\{ \alpha_k^* \beta_l \Bigg( \prod_{i_k\in A_k} \alpha(i_k, r(i_k))^* \Bigg) & \\
\cdot \Bigg( \prod_{j_l\in B_l} \beta(j_l, s(j_l)) \Bigg) \exp \Bigg( \int dx' dx'' dt' dt'' & \\
\cdot \Bigg( \sum_{i_k\in A_k} h_{i_k, r(i_k)}(-x') \delta(t_{i_k} - t') \Bigg) & \\
\cdot G_{\beta V}(t' - t'', x' - x'') \Bigg( \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x'') \delta(t_{j_l} - t'') \Bigg) \Bigg) & \\
\cdot \exp \Bigg( -\frac{1}{2} \int dx' dx'' dt' dt'' & \\
\cdot \Bigg( \sum_{i_k\in A_k} h_{i_k, r(i_k)}(x') \delta(t_{i_k} - t') \Bigg) & \\
\cdot G_{\beta V}(t' - t'', x' - x'')^* \Bigg( \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x'') \delta(t_{j_l} - t'') \Bigg) \Bigg) & \\
\cdot \exp \Bigg( -\frac{1}{2} \int dx' dx'' dt' dt'' \Bigg( \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x') \delta(t_{j_l} - t') \Bigg) & \\
\cdot G_{\beta V}(t' - t'', x' - x'') \Bigg( \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x'') \delta(t_{j_l} - t'') \Bigg) \Bigg) \Bigg\} & \\
= \sum_{k\in A} \sum_{l\in B} \sum_{r(\cdot)\in\mathscr{R}_k} \sum_{s(\cdot)\in\mathscr{S}_l} & \\
\Bigg\{ \alpha_k^* \beta_l \Bigg( \prod_{i_k\in A_k} \alpha(i_k, r(i_k))^* \Bigg) & \\
\cdot \Bigg( \prod_{j_l\in B_l} \beta(j_l, s(j_l)) \Bigg) \exp \Bigg( -\frac{1}{2} \int dx' dx'' dt' dt'' & \\
\cdot \Bigg( - \sum_{i_k\in A_k} h_{i_k, r(i_k)}(-x') \delta(t_{i_k} - t') & \\
+ \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x') \delta(t_{j_l} - t') \Bigg) G_{\beta V}(t' - t'', x' - x'') & \\
\cdot \Bigg( - \sum_{i_k\in A_k} h_{i_k, r(i_k)}(-x'') \delta(t_{i_k} - t'') & \\
+ \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x'') \delta(t_{j_l} - t'') \Bigg) \Bigg) \Bigg\} . &
\end{aligned}
$$

We take into account that

$G_{\beta V}(t, x)^* = G_{\beta V}(t, -x)$.

Now we shall calculate $S(\Theta(a)b)$.

Since

$$\langle \bar{T} \exp(\hat{i}\varphi_V(h_1)(t_1) \dots \exp(\hat{i}\varphi_V(h_n))(t_n)\rangle$$
$$= \exp\left(-\sum_{i<j} \int dx'dx''h_i(x')G_{\beta V}(t_i - t_j, x' - x'')h_j(x'')\right)$$
$$\cdot \exp\left(-\frac{1}{2}\sum_{i=j} \int dx'dx''h_i(x')G_{\beta V}(0, x' - x'')h_j(x'')\right),$$

so

$$S(\Theta(a)b) = \sum_{k\in A}\sum_{l\in B}\sum_{r(\cdot)\in\mathscr{R}_k}\sum_{s(\cdot)\in\mathscr{S}_l}$$
$$\left\{\alpha_k^*\beta_l\left(\prod_{i_k\in A_k}\alpha(i_k, r(i_k))^*\right)\left(\prod_{j_l\in B_l}\beta(j_l, s(j_l))\right)\right.$$
$$\cdot \exp\left(-\frac{1}{2}\int dx'dx''dt'dt''\left(-\sum_{i_k\in A_k} h_{i_k, r(i_k)}(-x')\delta(t_{i_k} - t')\right.\right.$$
$$\left. + \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x')\delta(t_{j_l} - t')\right) G_{\beta V}(t' - t'', x' - x'')$$
$$\cdot\left(-\sum_{i_k\in A_k} h_{i_k, r(i_k)}(-x'')\delta(t_{i_k} - t'')\right.$$
$$\left.\left.\left. + \sum_{j_l\in B_l} h_{j_l, s(j_l)}(x'')\delta(t_{j_l} - t'')\right)\right)\right\}.$$

Here we use the fact that $G_{\beta V}(t, x) = G_{\beta V}(-t, -x)$ and transform the sum

$$\sum_{i<j} = \frac{1}{2}\left(\sum_{i<j} + \sum_{i>j}\right) = \frac{1}{2}\sum_{i,j} - \frac{1}{2}\sum_{\substack{i,j\\ i=j}}.$$

Hence,

$$S(\Theta(a)b) = (N(a), N(b))_{\mathscr{F}_G}.$$

Since $(\ ,\ )_{\mathscr{F}_G}$ is the scalar product in the Hilbert space $\mathscr{F}_G$ and $N$ is a linear operator, it follows that Theorem 2.2 is proved.

## 3. Vacuum Overlap and the Energy-Momentum Spectrum

In the connection with Theorem 2.2 we shall define Jost states [2]. Since $H_{0,V}^2 \geqq P_V^2$, the vector valued distribution $\varphi_V(x_1) \dots \varphi_V(x_n)\Omega_{0,V}$ is the boundary value of a vector-valued function (i.e., Jost state), analytic in the region $\operatorname{Im} z_1, \operatorname{Im}(z_2 - z_1), \dots, \operatorname{Im}(z_n - z_{n-1}) \in V_+$, where $V_+$ is the forward light cone. By cyclicity of the vacuum, the set of linear combinations of Jost states is clearly dense in $\mathscr{F}_V$. We call a Jost state $\beta$ Euclidean iff each $z_j$ is $z_j = (x_j + i\beta t_j, i\beta_0 t_j)$ with $x_j, t_j$ real, $\beta_0 = (1 + \beta^2)^{1/2}$ and, moreover, the $t_j$'s are non-coincident.

We call a vector a $\beta$ "good" Jost state if it is an integral over space variables $x$ of $\beta$ Euclidean Jost states with a function

$$\prod_{i=1}^n f_i(x_i), \qquad f_i(x_i) \in C_0^\infty([-V/2, V/2])$$

and $t_1, t_1 - t_2, t_2 - t_3, \dots, t_n - t_{n-1}$ are all positive.

We say the state is supported in $(a, b)\times(c, d)$ if $\operatorname{supp} f_i \subset (a, b)$ and $c<t_i<d$.

**Lemma 3.1.** *Fix $a, b, c, d$ with $-V/2<a<b<V/2$, $0<c<d$. The linear combination of $\beta$ good Jost states with support in $(a, b)\times(c, d)$ are dense in Fock space $\mathscr{F}_V$.*

*Proof.* The proof is the same as the one of Lemma 5.2 [2]. Suppose, $\eta$ is orthogonal to all $\beta$ good Jost states with the support property. By taking the smearing functions to delta functions, $\eta$ is orthogonal to all $\beta$ Euclidean Jost states with the support property. By analyticity, it is then orthogonal to all Jost states and hence is zero. The lemma is proved.

Now let $H_{I,V}$ be the interaction Hamiltonian of the $P(\varphi)_2$ theory with the periodic boundary conditions in the box of volume $V$. Let

$$\begin{aligned} &E_V=\inf \operatorname{spectrum}(H_{0,V}+H_{I,V}),\\ &H_V=H_{0,V}+H_{I,V}-E_V. \end{aligned} \tag{3.1}$$

The operator (3.1) is essentially self-adjoint on the domain $\mathscr{D}=\mathscr{D}(H_{0,V})\cap\mathscr{D}(H_{I,V})$ [11]. The operator $H_V$ commutes strongly with $P_V$.

With the help of spectral theorem we introduce the following self-adjoint operator

$$H_V(\beta)=\int(\beta_0\lambda_0+\beta\lambda_1)dE(\lambda).$$

Here $dE(\lambda)$ is the common spectral measure of operators $H_V$ and $P_V$ and $\beta_0=(\beta^2+1)^{1/2}$.

**Lemma 3.2.** *The operator $H_V(\beta)$ is bounded below, has a discrete spectrum and*

$$H_V(\beta)=\beta_0 H_V+\beta P_V.$$

*Proof of Lemma 3.2.* Since $H_V$ is essentially self-adjoint on $\mathscr{D}$ and $\varepsilon H_{0,V}+H_{I,V}$ is bounded below for $\varepsilon>0$, so

$$H_V+c(\varepsilon)\geqq(1-\varepsilon)H_{0,V}\geqq(1-\varepsilon)|P_V|$$

for sufficiently large positive $c(\varepsilon)$.

Then the spectral theorem and the commutativity of $H_V$ and $P_V$ imply that

$$(H_V+c(\varepsilon))^2\geqq(1-\varepsilon)^2P_V^2.$$

Thus, the operator $\beta\int\lambda_1 dE(\lambda)$ is bounded relatively to the operator $\beta_0 H_V$ with the relative bound smaller than $(\beta_0^2-1)^{1/2}\beta_0^{-1}(1-\varepsilon)^{-1}<1$ for sufficiently small $\varepsilon$. Then, by the Rellich theorem [12, p. 287],

$$H_V(\beta)=\beta_0 H_V+\beta P_V.$$

Moreover,

$$H_V(\beta)+\beta_0 c(\varepsilon)\geqq((1-\varepsilon)\beta_0-(\beta_0^2-1)^{1/2})H_{0,V},$$

so the discreteness of the spectrum of $H_{0,V}$ and the Rellich theorem [13, p. 386] lead to the discreteness of the spectrum of the Lorentz rotated Hamiltonian $H_V(\beta)$. Lemma 3.2 is proved.

**Theorem 3.3.** *The Fock vacuum $\Omega_{0,V}$ overlaps the vacuum for $H_V(\beta)$.*

*Proof of Theorem 3.3.* The idea of the proof is the same as in the proof of Theorem 5.3 in [2]. We apply the Osterwalder-Schrader positivity in the spatial direction to the expression

$$(\eta, \exp(-tH_V(\beta))\eta)$$

for the appropriate $\eta$.

For this purpose, we shall approximate $\exp(-tH_V(\beta))$ by some expression and then we shall use Theorem 2.2.

Let

$$W_+ = \int_{[0,V/2]} dx H_{I,V}(x)$$

$$W_- = \int_{[-V/2,0]} dx H_{I,V}(x)$$

then

$$H_{I,V} = W_+ + W_-$$

and $W_\pm \in L_p$ for some $p>2$ [11].

Let

$$W_+(n) = \begin{cases} W_+ & \text{for} \quad |W_+| \leqq n \\ n & \text{for} \quad |W_+| > n \end{cases}$$

then $W_+(n) \in \mathcal{M}_+$. Since $W_+ \in L_p$, so $W_+(n) \xrightarrow[n\to\infty]{} W_+$ in any $L_q$ norm with $q<p$ (Lemma 3.5 [2]). Let $W_-(n) = \exp(-iV/2P_V)W_+(n)\exp(iV/2P_V)$ and $W(n) = W_+(n) + W_-(n)$.

Since $\|\exp(-tW(n))\|_1$ is bounded uniformly in $n$ for each $t$, so the Corollary 2.14 [11, p. 133] implies that

$$\beta_0(H_{0,V} + W(n)) + \beta P_V \geqq -c(\beta)$$

uniformly in $n$.

Since $W(n) \to W$ in any $L_q$ norm with $q<p$ and $p>2$, so on the domain $F \cap \mathcal{D}(H_{0,V})$, where $F$ is the set of vectors with finite number of particles,

$$\beta_0(H_{0,V} + W(n) - E_V) + \beta P_V \xrightarrow[n\to\infty]{st} \beta_0 H_V + \beta P_V .$$

Since the domain $F \cap \mathcal{D}(H_{0,V})$ is a core for $\beta_0 H_V$ [11, Theorem 4.2d], and hence, for $\beta_0 H_V + \beta P_V$, so the corollary and the theorem [12, pp. 429, 502] imply that

$$\exp(-t(\beta_0(H_{0,V} + W(n) - E_V) + \beta P_V)) \xrightarrow[n\to\infty]{st} \exp(-t(\beta_0 H_V + \beta P_V)) .$$

Since $W(n)$ are bounded functions, so

$$\exp(-t(\beta_0(H_{0,V} + W(n)) + \beta P_V))$$
$$= \operatorname*{s-lim}_{m\to\infty} \left(\exp\left(-\frac{t}{m} H_{0,V}(\beta)\right) \exp\left(-\frac{t\beta_0}{m} W(n)\right)\right)^m .$$

Let now $\eta$ be a linear combination of $\beta$ good Jost states supported in the domain $(t, x)=[1,2]\times[V/8, V/4]$. Then

$$\left(\eta, \left(\exp\left(-\frac{t}{m} H_{0,V}(\beta)\right)\exp\left(-\frac{t\beta_0}{m} W(n)\right)\right)^m \eta\right)$$
$$=S\left(\bar{\eta} \prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_-(n)\right)\right)\eta_t\right) \tag{3.2}$$

where we have used the same notation for $\eta$ as the element of the Fock space $\mathscr{F}_V$ and as the element of the algebra $\mathfrak{A}$.

Here the operations $(\cdot)_t$, $\overline{(\cdot)}$ are given by the following expressions:

$$a_t=\sum_{k\in A}\alpha_k \prod_{i_k\in A_k}(t_{i_k}+t, F_{i_k})$$

$$\bar{a}=\sum_{k\in A}\alpha_k^* \prod_{i_k\in A_k}(-t_{i_k}, F_{i_k}^*)$$

and where $a$ is given by (2.1).

Thus, we have

$$(3.2)=S\left(\Theta\left(\Theta(\bar{\eta}\eta_t)\prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\right)\right.$$
$$\left.\cdot\prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\right)$$

and Theorem 2.2 implies that this expression is bounded by

$$S\left(\Theta\left(\Theta(\bar{\eta}\eta_t)\prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\right)\right.$$
$$\left.\cdot\,\Theta(\bar{\eta}\eta_t)\prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\right)^{1/2}$$
$$\cdot S\left(\Theta\left(\prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\right)\right.$$
$$\left.\cdot\prod_{k=1}^{m}\left(tk/m, \exp\left(-\frac{t\beta_0}{m} W_+(n)\right)\right)\right)^{1/2}$$
$$=\left(\eta', \left(\exp\left(-\frac{t}{m} H_{0,V}(\beta)\right)\exp\left(-\frac{t\beta_0}{m} W(n)\right)\right)^m \eta'\right)^{1/2}$$
$$\cdot\left(\Omega_{0,V}\left(\exp\left(-\frac{t}{m} H_{0,V}(\beta)\right)\exp\left(-\frac{t\beta_0}{m} W(n)\right)\right)^m \Omega_{0,V}\right)^{1/2}$$

where $\eta'$ is some state supported in the domain $[1,2]\times[-V/4, -V/8]\cup[V/8, V/4]\}$ and $\eta'$ does not depend on $n$, $m$.

Then, taking $m\to\infty$, $n\to\infty$ we obtain

$$(\eta, \exp(-tH_V(\beta))\eta)\leqq(\eta', \exp(-tH_V(\beta))\eta')^{1/2}$$
$$\cdot(\Omega_{0,V}\exp(-tH_V(\beta))\Omega_{0,V})^{1/2}.$$

This inequality, Lemma 3.1 and Lemma 5.1 [2] prove Theorem 2.3.

In other words, the vacuum subspace of $H_V(\beta)$ has a vector with a zero momentum.

**Theorem 3.4.** *The joint spectrum of $H_V$ and $P_V$ is contained in the forward light cone. That is,*

$$H_V(\beta) \geqq 0, \quad H_V^2 - P_V^2 \geqq 0$$

$$\inf \operatorname{spectrum} H_V(\beta) = 0, \quad \pi(\beta) = \pi(\beta = 0).$$

*Here $\pi(\beta)$ denotes the projection onto the vacuum subspace of $H_V(\beta)$.*

*Proof of Theorem 3.4.* By Lemma 5.1 [2] and by the inequality (3.2)

$$\begin{aligned}\inf \operatorname{spectrum} H_V(\beta) &= -\lim_{t\to\infty} \frac{1}{t} \ln(\Omega_{0,V}, \exp(-tH_V(\beta))\Omega_{0,V}) \\ &= -\lim_{t\to\infty} \frac{1}{t} \ln(\Omega_{0,V}, \exp(-tH_V)\Omega_{0,V}) = 0.\end{aligned}$$

Hence,

$$H_V(\beta) \geqq 0.$$

The commutativity of $H_V$ and $P_V$ implies the commutativity of $\pi(\beta)$ and $P_V$. Thus,

$$\pm P_V \pi(\beta) \leqq c(\beta) H_V(\beta) \pi(\beta) = 0$$

hence

$$H_V \pi(\beta) = 0, \quad H_V(\beta) \pi(\beta = 0) = 0$$

and so

$$\pi(\beta) = \pi(\beta = 0).$$

Furthermore, the commutativity of $H_V$ and $P_V$ implies that

$$H_V^2 - \beta_0^{-2} \beta^2 P_V^2 \geqq 0$$

and so, by limits as $\beta \to \infty$,

$$H_V^2 - P_V^2 \geqq 0.$$

This completes the proof of the theorem.

**Corollary 3.5.** *For the $P(\varphi)_2$ theory in a periodic box we have*

a) $0 \leqq H_V \pm \pi_V(f) + \frac{1}{2} \|f\|_2^2$

b) $0 \leqq H_V + \beta_0^{-1} \beta P_V \pm (\pi_V(f) - \beta_0^{-1} \beta \nabla \varphi_V(f)) + \frac{1}{2} \|f\|_2^2$

c) *For $u \in C^\infty(H_V)$*

$$\|H_V^j \pi_V(f) u\| + \|H_V^j \nabla \varphi_V(f) u\| \leqq |f|_j \|(H_V + 1)^{j+4} u\|.$$

*The Schwartz space norms $|\cdot|_j$ are independent of the box cutoff $V$ and they are independent of the polynomial $P$ and mass $m_0$ which define $H_V$.*

d) *There is a Schwartz space norm $|\cdot|_n$ defined on $\mathscr{S}(\mathbb{R}^{2n})$, such that for $f\in\mathscr{S}(\mathbb{R}^{2n})$ we have*

$$\left|\int f(x,t)(\Omega_V, \Phi_{v_1}(x_1,t_1)\dots\Phi_{v_n}(x_n,t_n)\Omega_V)dxdt\right|\leqq |f|_n$$

*where each $\Phi_v$ is a $\pi_V$ or a $\nabla\varphi_V$, and $|\cdot|_n$ is independent of $P$, $m_0$, $V$.*

Here $\nabla\varphi_V(f)=-\int\varphi_V(x)\nabla f(x)dx$, $\pi_V(f)=\int\pi_V(x)f(x)dx$, and $\pi_V(x)$ is the conjugate time zero field and $f(x)$ is a smooth function on $V$ with periodic boundary condition.

*Proof.* These statements can be proved in the same way as given in the paper [1] by Glimm and Jaffe.

**Corollary 3.6.** *The $P(\varphi)_2$ theory with the periodic boundary conditions in the infinite volume satisfies the spectral condition and is Lorentz invariant. Furthermore, the corresponding estimates of Corollary 3.5 are valid.*

*Proof.* Proof of the spectral condition and of the estimates for the infinite volume is given in [3] and the proof of Lorentz invariance is given in [4, 5].

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