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*Note*

# Exact Two-Particle S-Matrix of Quantum Sine-Gordon Solitons

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**Abstract.** The exact and explicit formulas for the quantum $S$-matrix elements of the soliton-antisoliton scattering which satisfy unitarity and crossing conditions and have correct analytical properties are constructed. This $S$-matrix is in agreement with the massive Thirring model perturbation theory and with the semiclassical sine-Gordon results.

As it is known the sine-Gordon model, i.e. the model of the field $\phi(x)$ in $(1+1)$ space-time, described by Lagrangian density

$$L=\frac{1}{2}(\partial\mu\phi)^2+\frac{m_0^2}{\beta^2}\operatorname{Cos}(\beta\phi) \tag{1}$$

has an infinite number of conservation lows both on the classical and on the quantum levels [1, 2]. This gives hard restrictions on particle scattering processes in this model. Namely, the set of particles constituting the final state of scattering and the set of their momenta coincide with the sets of the particles and momenta of the initial state, i.e. the particles may only exchange their momenta in the process of collision [2, 3, 9, 10].

The particle spectrum in model (1) consists of a soliton, an antisoliton (this particles we shall denote as $A$ and $\bar{A}$) and a certain number of soliton-antisoliton bound states. The masses of the latter are described by the formula:

$$m_k=2m\operatorname{Sin}\frac{k\gamma}{16}; \quad k=1,2,\ldots<\frac{8\pi}{\gamma}, \tag{2}$$

where $m$ is a soliton mass and $\gamma=\beta^2\left[1-\frac{\beta^2}{8\pi}\right]^{-1}$. This expression was obtained by semiclassical quantization of double-soliton solution of the classical sine-Gordon equation but seems to be exact [5, 3]. In the following consideration we shall always treat (2) as exact values of particle masses. Let us consider two-particle soliton-antisoliton scattering. Corresponding $S$-matrix element consists of two components $S_1(s)$ and $S_2(s)$ only [$s=(p_1+p_2)^2$ where $p_1$ and $p_2$ are the momenta of initial particles], describing two possible direct channels of reaction: the forward scattering (penetration) and the backward scattering (reflection) respectively. $S_1(s)$ and $S_2(s)$

are the analytical functions of complex variable $s$ in the complex $s$-plane with two cuts along the real axis $s \leqq 0$ and $s \geqq 4m^2$.

Since there are only two-particle unitarity conditions in $A+\bar{A}$ and $A+A$ scattering, the thresholds $s=0$ and $s=4m^2$ are the square root branching points of $S_1(s)$ and $S_2(s)$ (in general there is a more complicated branching point at $s=\infty$). So, it is convenient to use the variable[1]:

$$\theta = \ln\left[\frac{s-2m^2+\sqrt{s(s-4m^2)}}{2m^2}\right]. \tag{3}$$

After substitution (3) the functions $S_1(\theta)$ and $S_2(\theta)$ should become the meromorphic functions of $\theta$. Transformation (3) maps the physical sheet of complex $s$-plane on the strip $0<\operatorname{Im}\theta<\pi$, the edges of the right and the left cuts of the physical sheet of the $s$-plane being mapped on the axes $\operatorname{Im}\theta=0$ and $\operatorname{Im}\theta=\pi$ respectively. Therefore the $A+\bar{A}$ scattering ($s$-channel) is described by the values of $S_1(\theta)$ and $S_2(\theta)$ on the positive half-line of the axis $\operatorname{Im}\theta=0$, and the $A+A$ scattering ($u$-channel for penetration) is described by the values of $S_1(\theta)$ on the negative half-line of the axis $\operatorname{Im}\theta=\pi$. The $u$-channel for reflection coïncides with the $s$-channel. Hence the crossing-symmetry relation for $S_2(\theta)$ is:

$$S_2(\theta)=S_2(i\pi-\theta). \tag{4}$$

Unitarity conditions for $A+\bar{A}$ and $A+A$ reactions may be represented in the following analytical form:

$$\begin{aligned} &S_1(\theta)S_1(-\theta)+S_2(\theta)S_2(-\theta)=1\\ &S_1(\theta)S_2(-\theta)+S_2(\theta)S_1(-\theta)=0 \end{aligned} \tag{5}$$

and

$$S_1(\theta)S_1(2\pi i-\theta)=1. \tag{6}$$

Formula (2) means that on the segment $0<\operatorname{Im}\theta<\pi$ of the imaginary axis of the $\theta$-plane at the points

$$\theta_k = i\pi - ik\frac{\gamma}{8}; \qquad k=1,2,\ldots<\frac{8\pi}{\gamma}, \tag{7}$$

there are poles corresponding to soliton-antisoliton bound states. There are special values of $\gamma$ $\left(\text{namely, } \gamma=\frac{8\pi}{n}, n \text{ is a positive integer}\right)$ for which the $n$-th bound state (7) becomes unbound, i.e. the corresponding pole leaves the stripe $0<\operatorname{Im}\theta<\pi$.

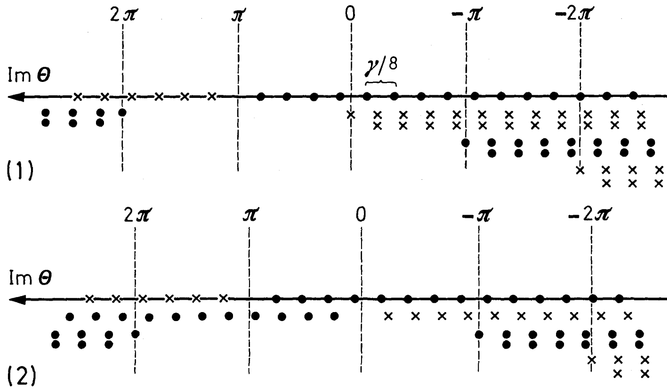
Korepin and Faddeev suggested [3], that for these values of $\gamma$ the exact expression for $S_1(\theta)$ is

$$S_1(\theta)=e^{in\pi}\prod_{k=1}^{n}\frac{e^{\theta-i\frac{\pi k}{n}}+1}{e^{\theta}+e^{-i\frac{\pi k}{n}}}. \tag{8}$$

To some extent this suggestion is confirmed by the semiclassical calculation of $S_2(\theta)$ [8], which turns out to vanish at $\gamma=\frac{8\pi}{n}$[2].

[1] The variable $\theta$ is "hyperbolic angle" between the momenta $p_1$ and $p_2$: $p_i^0 = m\operatorname{Ch}\theta_i$; $p_i^1 = m\operatorname{Sh}\theta_i$; $i=1,2$; $\theta=|\theta_1-\theta_2|$

[2] If one supposes that at $\gamma=\frac{8\pi}{n}$ the reflection is absent, then (8) will be dictated by (5)–(7) with the arbitraryness in sign only




**Figs. 1 and 2.** Crosses × mean zeroes, points ● mean poles. In Figure 1 and Figure 2 some of the poles and zeroes are displaced from the imaginary axis for the sake of clearness. In fact, all the singularities are at $\mathrm{Re}\,\theta=0$

Adopting (8) to be exact, we shall try to derive the explicit expressions for $S_1(\theta)$ and $S_2(\theta)$ for arbitrary $\gamma$. At first consider the function $S_1(\theta)$. Since there are no resonance states in $A+\bar{A}$ scattering [5], formula (7) implies that $S_1(\theta)$ has the set of poles at the points $\theta_k=i\pi-i\frac{k\gamma}{8}$; $k=1,2,\ldots$ ad inf. For $\gamma=\frac{8\pi}{n}$ there are zeroes of $S_1(\theta)$ in the strip $-\pi<\mathrm{Im}\,\theta<0$ at the points $-i\frac{k\pi}{n}$; $k=1,2,\ldots,n-1$ [see (8)]. Hence, for the arbitrary $\gamma$ the function $S_1(\theta)$ should have zeroes at the points $\theta'_k=-ik\frac{\gamma}{8}$; $k=0,1,2,\ldots$ ad inf., the first zero ($k=0$) being simple, while the others being double. The same consideration for the strips $-l\pi<\mathrm{Im}\,\theta<-(l-1)\pi$; $l=1,2,\ldots$ and the use of the condition (6) fixes the structure of singularities of $S_1(\theta)$ which is shown in Figure 1.

The analytical expression corresponding to Figure 1 is

$$S_1(\theta)=-\frac{i}{\pi}\mathrm{Sh}\left(\frac{8\pi}{\gamma}\theta\right)R(\theta)R(i\pi-\theta), \tag{9}$$

where

$$R(\theta)=\Gamma\left(1+i\frac{8\theta}{\gamma}\right)\prod_{l=1}^{\infty}\frac{\Gamma\left(2l\frac{8\pi}{\gamma}+i\frac{8\theta}{\gamma}\right)\Gamma\left(1+2l\frac{8\pi}{\gamma}+i\frac{8\theta}{\gamma}\right)}{\Gamma\left((2l+1)\frac{8\pi}{\gamma}+i\frac{8\theta}{\gamma}\right)\Gamma\left(1+(2l-1)\frac{8\pi}{\gamma}+i\frac{8\theta}{\gamma}\right)}. \tag{10}$$

The requirements (5), (6) will be satisfied if

$$S_2(\theta)=\frac{1}{\pi}\mathrm{Sin}\left(\frac{8\pi^2}{\gamma}\right)R(\theta)R(i\pi-\theta). \tag{11}$$

This expression automatically satisfies the crossing-symmetry condition (4). The singularity structure of (11) is shown in Figure 2.

For $\gamma\to 0$ functions (9) and (11) are converted into the known semiclassical expressions for soliton $S$-matrix [3, 4, 8, 12].

Expressions (9)–(11) are the main result of this paper. The author believes that these formulas give an exact two-particle $S$-matrix of the quantum sine-Gordon solitons. They pass through two independent and nontrivial tests. The first is based on Coleman's result [6, 7], that the sine-Gordon is equivalent to the massive Thirring model (MTM), the solitons being the fundamental fermions of MTM. For $\gamma \to 8\pi$ the formulas (9), (11) may be extended in powers of $\frac{2g}{\pi} = \frac{8\pi}{\gamma} - 1$ and compared with the perturbation theory results for MTM. Such a comparison was carried out to the second order in $g$ and the coïncidence was observed.

The lowest of the bound states (2) ($k=1$) is a "fundamental particle" ($FP$) of (1) [5]. So, the other test is to calculate the matrix element of $S$-matrix for the $FP+FP$ scattering and compare it with the sine-Gordon perturbation theory (expansion in powers of $\beta^2$). To do this, one should construct the $S$-matrix element of $A+A+\bar{A}+\bar{A}$ scattering, using the factorization of multiparticle sine-Gordon $S$-matrix [2, 9, 11], and the explicit expressions (9), (10). Then the residues should be taken at the poles corresponding to $FP$. The result is:

$$S(\theta') = \frac{\mathrm{Sh}\,\theta' + i\,\mathrm{Sin}\frac{\gamma}{8}}{\mathrm{Sh}\,\theta' - i\,\mathrm{Sin}\frac{\gamma}{8}}, \tag{12}$$

where $2m_1\,\mathrm{Ch}\,\theta' = s - 2m_1^2$. This expression was presented in [9, 10, 13], and was verified to the third order in $\beta^2$.

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### Note Added in Proof

Recently the exact $S$-matrices of $O(N)$-chiral field and of the fundamental fermions of Gross-Neveu model have been found by means of the analogous consideration. (A. B. Zamolodchikov, A. B. Zamolodchikov, in preparation)