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Correlation Inequalities and Equilibrium States

M. Fannes* and A. Verbeure
Instituut voor Theoretische Fysica, Universiteit Leuven, B-3030 Heverlee, Belgium

Abstract. For an infinite dynamical system, idealized as a von Neumann algebra acted upon by a time translation implemented by a Hamiltonian H, we characterize equilibrium states (KMS) by stationarity, a Bogoliubov-type inequality and continuous spectrum of H, except at zero.

§ 1. Introduction

The equilibrium states of a finite volume system in statistical mechanics is usually given by the Gibbs-ensembles.

To describe bona fide physical phenomena it is well known that one has to take the so-called thermodynamic limit i.e. the volume tending to infinity, of any of the Gibbs ensembles. These "limit Gibbs' states" have an interesting property, they satisfy the so-called KMS-condition [1, 2].

In [3] Roepstorff derived a stronger version of the Bogoliubov inequality [4] for Gibbs states (for KMS-states see [5]).

Let $\langle . \rangle_{\beta H}$ denote the thermal average with respect to the Hamiltonian H and the inverse temperature $\beta = 1/kT$. For any pair of observables x, y the scalar product $(.,.)_\sim$ is defined by:

$$(x, y)_\sim = \frac{1}{\beta} \int_0^\beta d\lambda \langle \exp(\lambda H) x^* \exp(-\lambda H) y \rangle_{\beta H}$$

(see also [6]). In [3] the following inequality is derived

$$(x, x)_\sim \leqq [\langle xx^* \rangle_{\beta H} - \langle x^* x \rangle_{\beta H}] / \ln \langle xx^* \rangle_{\beta H} / \langle x^* x \rangle_{\beta H} . \tag{1}$$

Of course we have not to insist on the importance of the Bogoliubov inequality and its stronger version in statistical mechanics (see e.g. [7]).

* Aangesteld Navorser NFWO, Belgium

In this note we want to add an other argument in favor of the importance of the inequality (1). We prove that a basic concept like that of an equilibrium state is determined by the following three properties:

(i) Stationarity,
(ii) inequality (1),
(iii) spectral condition.

For the benefit of the reader we sketch here the argument for states on $\mathscr{B}(\mathbb{C}^n)$. We prove that (i) and (ii) imply the KMS-condition.

Let H be the Hamiltonian, as operator on $\mathbb{C}^n$ with spectral resolution

$$H=\sum_i \varepsilon_i E_i$$

$\varepsilon_i \neq \varepsilon_j$, (E_i) spectral family of H.

We suppose that ω is a state on $\mathscr{B}(\mathbb{C}^n)$ satisfying conditions (i) and (ii). From (i)

$$\omega(x)=\operatorname{Tr}\varrho x \qquad x\in\mathscr{B}(\mathbb{C}^n),$$

where ϱ is a density matrix of the form

$$\varrho=\sum_i R_i,$$

where

$$0\leqq R_i\leqq E_i \quad \text{for all } i.$$

Let $\mathscr{C}$ be the set partial isometries V of rank one such that

$$V^*V\leqq E_i$$
$$VV^*\leqq E_j$$

then from (ii) with $x=V$ one gets

$$\frac{\exp(\varepsilon_i-\varepsilon_j)-1}{\varepsilon_i-\varepsilon_j}\leqq\frac{\omega(VV^*)/\omega(V^*V)-1}{\ln\omega(VV^*)/\omega(V^*V)}.$$

From the strict monotonicity of the function $f(x)=\dfrac{x-1}{\ln x}$ one gets

$$\exp(\varepsilon_i-\varepsilon_j)\leqq\omega(VV^*)/\omega(V^*V)$$

substituting V by V^* yields

$$\exp(\varepsilon_j-\varepsilon_i)\leqq\omega(V^*V)/\omega(VV^*).$$

Hence

$$\omega(V^*V)/\omega(VV^*)=\exp(\varepsilon_j-\varepsilon_i).$$

Hence

$$\frac{\operatorname{Tr}R_iV^*V}{\operatorname{Tr}R_jVV^*}=\frac{\exp-\varepsilon_i}{\exp-\varepsilon_j}.$$

As this is true for any V in $\mathscr{C}$: there exists a constant α such that:

$$\operatorname{Tr} R_i V^* V = \alpha \exp(-\varepsilon_i)$$

and so

$$R_i = \alpha \exp(+\varepsilon_i) E_i .$$

From normalization: $\varrho = \exp(-H)/\operatorname{Tr}\exp(-H)$.

Remark that the original Bogoliubov inequality

$$(x, x)_\sim \leqq 1/2\{\omega(xx^*) + \omega(x^*x)\} \tag{2}$$

is not sufficient for determining the KMS-property.

This can be checked on M_2. Take

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \varrho = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} \quad 0 \leqq \alpha \leqq 1 .$$

For α between

$$(e - 3/2)/(e-1) \leqq \alpha \leqq e/2(e-1)$$

the inequality is always satisfied, and the state need not to be KMS.

Therefore using the results of [3] and [5] we proved the equivalence of on the one hand the KMS-condition and on the other hand conditions (i) and (ii) i.e. stationarity and the inequality, which is an upper bound for the Duhamel two-point function.

We thank Professor E. Lieb for pointing out to us Ref. [8], where a different upper bound for the Duhamel two point function can be found. However it is unclear if this upper bound implies also the KMS-condition.

§ 2. The Main Theorem

Let $\mathfrak{M}$ be a von Neumann-algebra on a Hilbert space $\mathscr{H}$ and $(\alpha_t)_{t\in R}$ such that $\alpha_t(x) = \exp(itH) \times \exp(-itH)$ where H is a self-adjoint operator on $\mathscr{H}$. Let Ω be a cyclic vector of $\mathscr{H}$ for $\mathfrak{M}$ and let ω be the corresponding vector state i.e. $\omega(x) = (\Omega, x\Omega)$; $x \in \mathfrak{M}$. Furthermore suppose that $\mathfrak{M}\Omega$ belongs to the domain $\mathscr{D}(\exp(-tH/2))$ of $\exp(-tH/2)$ for all $t \in [0, 1]$. Then the following scalar product $(.,.)_\sim$:

$$(x, y)_\sim = \int_0^1 dt (\exp(-tH/2)x\Omega, \exp(-tH/2)y\Omega)$$

is well defined on $\mathfrak{M}$.

Lemma 2.1. *Suppose Ω cyclic and for all $x \in \mathfrak{M}$:*

$$(x, x)_\sim \leqq \frac{\omega(x^*x) - \omega(xx^*)}{\ln \omega(x^*x) - \ln \omega(xx^*)} . \tag{+}$$

Then Ω is separating.

Proof. Suppose $x^*\Omega=0$ then $\omega(xx^*)=0$. Suppose $x\Omega \neq 0$ then $\omega(x^*x)\neq 0$ and the right hand side of (+) vanishes; hence $(x,x)_\sim=0$.

As the integrand $\left(\exp\left(\frac{-tH}{2}\right)x\Omega, \exp\left(\frac{-tH}{2}\right)x\Omega\right)$ is continuous and positive

$$\exp\left(\frac{-tH}{2}\right)x\Omega=0$$

for all $t\in[0,1]$. For $t=0$ this yields $x\Omega=0$. Hence $x^*\Omega=0$ implies $x\Omega=0$. Therefore for all $y\in\mathfrak{M}$

$$yx^*\Omega=0,$$

or $(xy^*)^*\Omega=0$ implies $xy^*\Omega=0$.

As Ω is cyclic for $\mathfrak{M}$, this implies that $x=0$. Q.E.D.

Lemma 2.2. *Let* (i) Ω *cyclic and separating for* $\mathfrak{M}$.

(ii) $\mathfrak{M}\Omega\subset\mathscr{D}\left(\exp\left(\frac{-H}{2}\right)\right)$.

(iii) $\mathfrak{K}$ *linear, self-adjoint subspace of* $\mathfrak{M}$ *such that* $\mathfrak{K}\Omega$ *is dense in* $\mathscr{H}$.

(iv) *There exists a constant* $C\geqq 1$ *such that for all* $x\in\mathfrak{K}$

$$C^{-1}(\exp(-H/2)\alpha^*\Omega,\exp(-H/2)x^*\Omega)\leqq(x\Omega,x\Omega)$$
$$\leqq C(\exp(-H/2)x^*\Omega,\exp(-H/2)x^*\Omega) \qquad (*)$$

then $(*)$ *extends to all* $y\in\mathfrak{M}$.

Proof. Define the operator T on $\exp(-H/2)\mathfrak{K}\Omega$ *by*

$$T(\exp(-H/2)x^*\Omega)=x\Omega \quad x\in\mathfrak{K}.$$

By $(*)$ T is bounded by $\sqrt{C}$.

Now we prove the result by proving that there exists a closable extension $\tilde{T}$ of T. Define $\tilde{T}$ on $\exp(-H/2)\mathfrak{M}\Omega$ by

$$\tilde{T}(\exp(-H/2)x^*\Omega)=x\Omega, \quad x\in\mathfrak{M}.$$

$\tilde{T}$ is well defined [$\exp(-H/2)x^*\Omega=0$ implies $x^*\Omega=0$ and by (i) $x\Omega=0$] on a dense set $\exp(-H/2)\mathfrak{M}\Omega$, as $\exp(-H/2)$ is invertible.

We prove that the adjoint $\tilde{T}^*$ of $\tilde{T}$ is densily defined. First we prove that $\mathfrak{M}'\Omega\subseteq\mathscr{D}(\exp(H/2))$, where $\mathfrak{M}'$ is the commutant of $\mathfrak{M}$.

As $\exp(-H/2)$ is a self-adjoint invertible operator, $\exp(-H/2)\mathfrak{K}\Omega$ is dense in $\mathscr{H}$ and for all $y'\in\mathfrak{M}'$ and $x\in\mathfrak{K}$ using $(*)$:

$$|(y'\Omega,\exp(H/2)\exp(-H/2)x\Omega)|^2$$
$$=|(x^*\Omega,y'^*\Omega)|^2\leqq\|x^*\Omega\|^2\,\|y'^*\Omega\|^2$$
$$\leqq C\|\exp(-H/2)x\Omega\|^2\,\|y'^*\Omega\|^2.$$

Hence

$$y'\Omega\in\mathscr{D}(\exp(H/2)).$$

Define the operator $\tilde{T}^+$ on $\mathfrak{M}'\Omega$ [dense in $\mathscr{H}$, because of (i) by]:

$$\tilde{T}^+(y'\Omega)=\exp(H/2)y'^*\Omega\,, \qquad y'\in\mathfrak{M}'.$$

$\tilde{T}^+$ is well defined because Ω is cyclic for $\mathfrak{M}$.

Now $\tilde{T}^*$ is an extension of $\tilde{T}^+$, because for all $x\in\mathfrak{M}$ and $y'\in\mathfrak{M}'$:

$$\begin{aligned}&(\tilde{T}^+y'\Omega,\exp(-H/2)x\Omega)\\&=(\exp(H/2)y'^*\Omega\,,\ \exp(-H/2)x\Omega)\\&=(y'^*\Omega,x\Omega)=(x^*\Omega,y'\Omega)=(\tilde{T}\exp(-H/2)x\Omega,y'\Omega)\,.\end{aligned}$$

Therefore the second inequality $(*)$ extends to all $x\in\mathfrak{M}$. Analogously for the other inequality. Q.E.D.

Theorem 2.3. *Let ω and α_t be as above, if ω satisfies:*

(i) *ω is α_t-invariant (stationnary state).*

(ii) *for all $x\in\mathfrak{M}$:*

$$(x,x)_\sim \leqq [\omega(xx^*)-\omega(x^*x)]/\ln\omega(xx^*)/\omega(x^*x)\,.$$

(iii) *the spectrum of H is continuous except for the point zero.*

Then ω satisfies the KMS*-condition for the evolution α_t at $\beta=1$, i.e. $\forall x,y\in\mathfrak{M}$:* $(\exp(-H/2)y\Omega,\ \exp(-H/2)x\Omega)=(x^*\Omega,\ y^*\Omega)$.

Proof. Suppose $E\in\mathrm{sp}(H)\subseteqq R$ and $\delta>0$,

$$\Delta=[E-\delta,E+\delta]\,,\qquad \Delta^-=[-E-\delta,\ -E+\delta]$$

such that zero is not an endpoint of Δ.

Let $H=\int\lambda dF(\lambda)$ be the spectral resolution of H with spectral family $\{F(\lambda)/\lambda\in\mathbb{R}\}$; $F_\Delta=\int_\Delta dF(\lambda)$.

Take any element $y\in\mathfrak{M}$ such that

$$y\Omega\in F_\Delta\mathscr{H}\,,$$

then

$$\begin{aligned}&\int_0^1 d\lambda(\exp(-tH/2)y\Omega,\exp(-tH/2)y\Omega)\\&\geqq\int_0^1 d\lambda\exp(-t(E+\delta))\,(y\Omega,y\Omega)\\&=[(\exp(-(E+\delta))-1)/-(E+\delta)]\omega(y^*y)\,.\end{aligned}$$

By (ii):

$$(\exp(-(E+\delta))-1)/-(E+\delta)\leqq\frac{\omega(yy^*)/\omega(y^*y)-1}{\ln\omega(yy^*)/\omega(y^*y)}\,.$$

By the monotonicity of the function

$$\lambda\to\frac{\lambda-1}{\ln\lambda}\,,\qquad \lambda\in\mathbb{R}^+$$

this yields

$$\exp(-(E+\delta)) \leqq \omega(yy^*)/\omega(y^*y).$$

Also:

$$(\exp(-H/2)y\Omega, \exp(-H/2)y\Omega) \leqq \exp(\delta - E)\omega(y^*y).$$

Hence

$$\exp(-2\delta)(\exp(-H/2)y\Omega, \exp(-H/2)y\Omega) \leqq \omega(y^x). \tag{1}$$

Analogously, remarking that $y^*\Omega \in F_\Delta - \mathscr{H}$ yields

$$\omega(yy^*) \leqq (\exp(-H/2)y\Omega, \exp(-H/2)y\Omega \exp(2\delta). \tag{2}$$

From (1) and (2):

$$\exp(-2\delta)(\exp(-H/2)y\Omega, \exp(-H/2)y\Omega) \leqq \omega(yy^*)$$
$$\leqq (\exp(-H/2)y\Omega, \exp(-H/2)y\Omega)\exp(2\delta). \tag{3}$$

Let $\{\Delta_k^n/k \in Z\}$ be a partition of the real line such that

$$\Delta_k^n = \left[\frac{2k-1}{2n}, \frac{2k+1}{2n}\right)$$

for $n \in N$.

Let

$$\mathfrak{K}_k^n = \{y \in \mathfrak{M}/y\Omega \in F_{\Delta_k^n}\mathscr{H}\}$$

and $\mathfrak{K}^n$ be the linear span of the $\mathfrak{K}_k^n$ for all $k \in Z$. We prove that $\mathfrak{K}^n\Omega$ is dense in $\mathscr{H}$.

Take any $\psi \in F_{\Delta_k^n}\mathscr{H}$, for any $\varepsilon > 0$, there exists an element $x \in \mathfrak{M}$ such that

$$\|\psi - x\Omega\| < \varepsilon.$$

Take

$$x(f) = \int f(t)\alpha_t(x)dt$$

with $f \in L^1(R)$ and support of the Fourier transform $\hat{f}$ of f in Δ_k^n, then $x(f)\Omega \in F_{\Delta_k^n}\mathscr{H}$.

Because of condition (iii), it is furthermore possible to choose f such that

$$\|x(f)\Omega - F_{\Delta_k^n}x\Omega\| < \varepsilon.$$

Then

$$\|\psi - x_{(f)}\Omega\| \leqq \|\psi - F_{\Delta_k^n}x\Omega\| + \|F_{\Delta_k^n}\alpha\Omega - x(f)\Omega\| < 2\varepsilon$$

proving that $\mathfrak{K}^n\Omega$ is dense in $\mathscr{H}$.

The inequalities (3) are easily extended to $\mathfrak{K}^n$. Take

$$x = \sum_{k=1}^{N} x_k, x_k \in \mathfrak{K}_k^n$$

then

$$\exp(-1/n)(\exp(-H/2)x\Omega, \exp(-H/2)x\Omega)$$
$$=\exp(-1/n)\sum_n(\exp(-H/2)x_k\Omega, \exp(-H/2)x_k\Omega)$$
$$\leqq \sum_k (x_k^*\Omega, x_k^*\Omega) = (x^*\Omega, x^*\Omega)$$

and analogously for the second inequality.

Now we are in a position to use Lemma 2.2 yielding (3) for all $x \in \mathfrak{M}$. As this is true for all n we get for all $x \in \mathfrak{M}$:

$$(\exp(-H/2)x\Omega, \exp(-H/2)x\Omega) = (x^*\Omega, x^*\Omega).$$

By polarization, for all x and $y \in \mathfrak{M}$, we get

$$(\exp(-H/2)y\Omega, \exp(-H/2)x\Omega) = (x^*\Omega, y^*\Omega)$$

which is a particular form of the KMS-equation. Q.E.D.

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