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Stationary Solutions of the Bogoliubov Hierarchy Equations in Classical Statistical Mechanics. 2

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Abstract. In the preceding paper under the same title we have formulated a theorem which describes the set of states (i.e., probability measures on phase space of an infinite system of particles in R^ν) corresponding to stationary solutions of the BBGKY hierarchy. We have proved the following statement: if G is a Gibbs measure (Gibbs random point field) corresponding to a stationary solution of the BBGKY hierarchy, then its generating function satisfies a differential equation which is "conjugated" to the BBGKY hierarchy. The present paper deals with the investigation of the "conjugated" equation for the generating function in particular cases.

0. Preliminaries

This paper is the second part of a work of the authors published under the same title. The first part of the work is [1]. All references to the paper [1] are marked by the index I: Theorem 1, I, condition (G_1, I), formula (2.2, I), etc. In paper [1] we have formulated the main theorem which describes all stationary solutions of the Bogoliubov hierarchy equations (B.h.e.) in a class of probability measures on phase space. The proof of this theorem is naturally divided into two parts. In [1] we have proved a statement (Theorem 1, I) which is, in a sense, the first part of the main theorem. Namely, we have showed that under some restrictions [see conditions (I_1, I)–(I_4, I) and (G_1, I)–(G_6, I)], the generating function of a Gibbs random field corresponding to a stationary solution of the B.h.e. satisfies a differential equation[1] [see (2.8, I)] which may be considered as a conjugate equation to the B.h.e. The present paper deals with the proof of the second part of the main theorem.

The second part of the main theorem is formulated in [1] as an assertion (Theorem 2, I) according to which any function satisfying the Equation (2.8, I) [and conditions (G_1, I)–(G_6, I)] has the form (2.7, I), i.e. generates an equilibrium state (in

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[1] Or, if one prefers, a system of differential equations

the sense of Dobrushin-Lanford-Ruelle) corresponding to the given interaction potential $U(r)$ appearing in the B.h.e.[2] Here we prove the assertion of Theorem 2, I in a particular case where the generating function f corresponds to a binary interaction and also deduce some auxiliary facts used in the general situation.

The case of binary interactions, besides being interesting itself and requires an individual proof, is also important since it is an initial step of inductive schemes on which the proof of Theorem 2, I is based in the general case. In a separate paper we will give these induction schemes and finish the proof of Theorem 2, I.

For the reader's convenience we follow an exposition independent on that of [1]. This is motivated also by the fact that the assertion of Theorem 2, I is valid under more general assumptions than that of the main theorem (see Footnote[13] in [1]).

1. Formulation of Results

Remember some notations of [1]. We denote by M_1 the space $R^{2\nu}$, $\nu=1, 2, \ldots$, with the fixed representation $R^{2\nu}=R^\nu \times R^\nu$ (the space R^ν is always considered with a fixed basis). The points of M_1 are denoted x, y, etc., and also — when one uses the representation $R^{2\nu}=R^\nu \times R^\nu$ – as (q, v), (q', v'), etc., where $q, q' \in R^\nu$ and $v, v' \in R^\nu$, etc. The same notations are used for one-point subsets of M_1. The set M_1 is interpreted as the phase space of a one-particle system in R^ν. Denote by M_n, $n \geqq 2$ the collection of all (unordered) subsets of M_1 having n elements. The set M_n is interpreted as the phase space of a n-particle system in R^ν. The union $M^0 = \bigcup_{n=1}^{\infty} M_n$ is interpreted as the phase space of a system of an arbitrary (finite) number of particles in R^ν[3]. The points of M^0 are denoted $\bar{x}$, $\bar{y}$, etc.; this includes points of M_1 unless we need to distinguish them. Denote by $n(\bar{x})$ the number of elements in $\bar{x}$, $\bar{x} \in M_n$. Finally, let $\hat{M}$ denote the subset of $M_1 \times M^0$ defined by $\hat{M} = \{(y, \bar{x}) : y \in \bar{x}\}$.

Denote

$$(M_1^n)_{\neq} = \{(x_1, \ldots, x_n) : x_i \in M_1, 1 \leqq i \leqq n, x_i \neq x_j \text{ if } 1 \leqq i < j \leqq n\}, \quad n \geqq 1.$$

There is the natural symmetrisation map $s_n : (M_1^n)_{\neq} \to M_n$. The conjugate map $(s_n^* f)(\cdot) = f(s_n(\cdot))$ transforms a (real-valued) function $f : M_n \to R^1$ into a symmetric (real-valued) function on $(M_1^n)_{\neq}$. We say that a function f is of class C^k at a point $\bar{x} \in M_n$ if $s_n^* f$ is of class C^k at every point of $s_n^{-1} \bar{x} \subset (M_1^n)_{\neq}$.

Let $f \in C^1$ at a point $\bar{x} \in M^0$. The gradient map

$$((q, v), \bar{x}) \in \hat{M} \mapsto (\partial_q f(\bar{x}), \partial_v f(\bar{x})) \in R^\nu \times R^\nu$$

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[2] It should be noticed that the proof of Theorem 2, I (and hence of the Main theorem) becomes much more simple if in addition to above conditions one supposes that the generating function does not depend on velocities for configurations containing two or more particles. In this case one can replace rather restrictive condition (G_3) by a more natural one. A close result in a similar situation is recently obtained in [2]

[3] The set M_0 consisting of one element ϕ (vacuum) corresponding to the absence of particles is not considered

is defined as follows. Let $x=(q, v)\in\bar{x}$, and $\bar{x}_1=\bar{x}\setminus x$. Then, for fixed $\bar{x}_1$, the function $\tilde{f}(\tilde{q}, \tilde{v})=f(\bar{x}_1\cup(\tilde{q}, \tilde{v}))$ is of class C^1 at the point (q, v). We set

$$\partial_q f(\bar{x})=(\partial_{\tilde{q}}\tilde{f})\Big|_{\substack{\tilde{q}=q,\\ \tilde{v}=v}},\quad \partial_v f(\bar{x})=(\partial_{\tilde{v}}\tilde{f})\Big|_{\substack{\tilde{q}=q,\\ \tilde{v}=v}}.$$

The Poisson bracket of two functions $f, f'\in C^1$, is given by

$$\{f, f'\}(\bar{x})=\sum_{(q,v)\in\bar{x}}[\langle\partial_q f(\bar{x}), \partial_v f'(\bar{x})\rangle-\langle\partial_v f(\bar{x}), \partial_q f'(\bar{x})\rangle]$$

($\langle,\rangle$ denotes the inner product in R^ν).

Fix numbers $d_0\geqq 0$ and $d_1>d_0$ and a real-valued function $U(r)$, $d_0<r<\infty$, such that

(I_1') $U\in C^2(d_0, +\infty)$,

(I_2') $U\not\equiv 0$ on (d_0, d_1),

(I_3') $U(r)\equiv 0$ for $r\geqq d_1$.

We interpret $U(r)$ as the potential of a pair interaction of particles. Clearly, conditions (I_1')–(I_3') are weaker than $(\mathrm{I}_1, \mathrm{I})$–$(\mathrm{I}_4, \mathrm{I})$. Denote[4]

$$D^0=\{\bar{x}\in M^0:\ \min_{q,q'\in\bar{x},q\neq q'}|q-q'|>d_0\},\tag{1.1}$$

and

$$H(\bar{x})=\frac{1}{2}\sum_{v\in\bar{x}}\langle v, v\rangle+\sum_{q,q'\in\bar{x},q\neq q'}U(|q-q'|),\quad \bar{x}\in D^0.\tag{1.2}$$

Let $n_0=2, 3, \ldots$; denote by $\mathscr{C}_{n_0}$ the class of the real-valued functions $f(\bar{x})$, $\bar{x}\in D^0$, satisfying the conditions

(G_1') $f\in C^2$ at every point $\bar{x}\in D^0$,

(G_2') for any fixed $v\in R^\nu$, $f^{(1)}(q, v)$, as a function of $q\in R^\nu$, is bounded below [$f^{(1)}(q, v)$ denotes here the restriction of f to M_1],

(G_3') $f(\bar{x})\equiv 0$ if $\bar{x}\in D^0\cap M_n$ and $n>n_0$,

(G_4') for any $n=2, \ldots, n_0$, every $\bar{x}\in D^0\cap M_{n-1}$ and $v\in R^\nu$,

$$\lim_{|q|\to\infty}|f(\bar{x}\cup(q, v))|=0.$$

Clearly, (G_1')–(G_4') for $d_0>0$ follow from $(\mathrm{G}_2\mathrm{a}, \mathrm{I})$, $(\mathrm{G}_2\mathrm{b}, \mathrm{I})$, $(\mathrm{G}_3, \mathrm{I})$, and $(\mathrm{G}_5, \mathrm{I})$. Let $\sum\mathscr{C}_{n_0}$ denote the set of functions h of the form

$$h(\bar{x})=h_f(\bar{x})=\sum_{\bar{y}\subseteqq\bar{x}}f(\bar{y}),\ \bar{x}\in D^0, f\in\mathscr{C}_{n_0}.\tag{1.3}$$

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4 As in [1], we use the notations $q\in\bar{x}$ and $v\in\bar{x}$ instead of $(q, v)\in\bar{x}$

The reader can easily write down conditions on h under which $h\in\sum'\mathscr{C}_{n_0}$. For the most part of the present paper we consider the case $n_0=2$. Our main result is the following

Theorem 2′. *Let U satisfy the conditions (I'_1)–(I'_3). Suppose $f\in\mathscr{C}_2$ and that at every point $\bar{x}\in D^0$ the corresponding function $h(\bar{x})$ satisfies the equation*

$$\{h(\bar{x}), H(\bar{x})\}=0. \tag{1.4}$$

Then f is given by

$$f(\bar{x})=\begin{cases}\frac{1}{2}c_1\langle v,v\rangle+\langle v,v_0\rangle+c_2, & \bar{x}=x=(q,v)\in M_1,\\ \\ c_1U(|q-q'|), & \bar{x}=\{(q,v),(q',v')\}\in D^0\cap M_2,\end{cases} \tag{1.5}$$

where c_1, $c_2\in R^1$ are constants and $v_0\in R^\nu$ is a fixed vector. The function $h(\bar{x})$, corresponding to (1.5), has the form

$$h(\bar{x})=c_1H(\bar{x})+\left\langle\sum_{v\in\bar{x}}v,v_0\right\rangle+c_2n(\bar{x}),\quad \bar{x}\in D^0. \tag{1.6}$$

Theorem 2, I [in the particular case when $n_0=2$ in the condition $(\mathrm{G}_3, \mathrm{I})$] follows from Theorem 2′. In fact, Equation (1.4) is equivalent to (2.8, I) [see (3.41, I)]. Hence if, in addition to (G'_1)–(G'_4), the function f obeys $(\mathrm{G}_2\mathrm{c}, \mathrm{I})$, then c_1 in (1.5) must be positive. This means that f has the form (2.7, I).

Equation (1.4) means that the function h is invariant w.r.t. the shift along the trajectories of the system of the Hamilton differential equations corresponding to the interaction potential U. More precisely, fix $\bar{x}\in D^0$ and consider the Cauchy problem for vector-valued functions $\mathbf{q}(t;(q,v))$ and $\mathbf{v}(t;(q,v))$ labelled by the pairs $(q,v)\in\bar{x}$:

$$\begin{cases}\dfrac{\partial\mathbf{q}}{\partial t}(t;(q,v))=\mathbf{v}(t;(q,v)), & \\ & (q,v)\in\bar{x},\\ \dfrac{\partial\mathbf{v}}{\partial t}(t;(q,v))=-\displaystyle\sum_{(q',v')\in\bar{x}:\,q'\neq q}[\partial_{\tilde{q}}U(|\tilde{q}|)|_{\tilde{q}=\mathbf{q}(t;(q,v))-\mathbf{q}(t;(q',v'))}], &\end{cases} \tag{1.7}$$

with the initial data

$$\mathbf{q}(0;(q,v))=q,\quad \mathbf{v}(0;(q,v))=v,\quad (q,v)\in\bar{x}. \tag{1.8}$$

For any $\bar{x}\in D^0$ one can find $t_0=t_0(\bar{x})>0$ such that the solution of the initial value problem (1.7) and (1.8) exists and is unique for $t\in(-t_0, t_0)$ and

$$\bar{x}_t=\{(\mathbf{q}(t;(q,v)),\mathbf{v}(t;(q,v))),(q,v)\in\bar{x}\}\in D^0,\quad |t|<t_0.$$

Equation (1.4) means that $h(\bar{x}_t)=h(\bar{x})$, $|t|<t_0$, i.e., h is a first integral of the system (1.7).

From this point of view, Theorem 2′ means that under conditions (I'_1)–(I'_3) all first integrals of (1.7) belonging to $\sum\mathscr{C}_2$ have the form of linear combinations of the classical ones: total energy, total momentum and the number of particles.

We conclude this section by the following remark. The study of the structure of first integrals of a Hamilton system has been initiated by results of Bruns, Painlevé, and Poincaré (see [3], Chapt. XIV). A valuable contribution was made by Siegel [4, 5] who considered a Hamilton system of a general type and studied its analytic first integrals (see also [6]). The main result of [5] is that for "amost all" Hamilton systems their analytic first integrals are series in the Hamiltonian H.

We consider the Hamilton systems of a special kind: our H is of the form (1.2). Further, we consider the first integrals which are, in general, non-analytic, but have special properties given by conditions (G_2')–(G_4'). This approach is natural from the point of view of Statistical Mechanics. It is interesting to note that in our situation there is a family of "exeptional" systems (i.e., of interaction potentials) having, in general, other first integrals than these given by Theorem 2'. These exceptions are a priori excluded by conditions (I_1')–(I_3').

The following sections deal with the proof of Theorem 2'.

2. Notations and Auxiliary Tools

Before going to the proof of Theorem 2' we introduce some notations and formulate auxiliary statements we use below. The proof of these statements is carried out in the Appendix. We use different type faces to denote scalar-, vector-, and matrix-valued functions of the variables $(q_1, \ldots, q_n; v_1, \ldots, v_n) \in R^{n\nu} \times R^{n\nu}$, $n = 1, 2, \ldots$. These functions are denoted as a, $\mathbf{a}$, and A respectively[5] (as exceptions, the vector functions q_i and v_i, $i = 1, 2, \ldots, n$, are denoted as scalar ones).

A function of one of the mentioned types is called a C^k-function at a point of $R^{n\nu} \times R^{n\nu}$ if all its components are (scalar) functions of class C^k at this point. The gradient of a scalar C^1-function a at a point of $R^{n\nu} \times R^{n\nu}$ w.r.t. the variable q_i (resp., v_i) is denoted as above by $\partial_{q_i} a(\cdot)$ [resp., $\partial_{v_i} a(\cdot)$]. The derivative $\partial_{q_i}\mathbf{a}(\cdot)$ [resp., $\partial_{v_i}\mathbf{a}(\cdot)$] of a vector C^1-function $\mathbf{a} = (a^1, \ldots, a^\nu)$ at a point of $R^{n\nu} \times R^{n\nu}$ w.r.t. the variable $q_i = (q_i^1, \ldots, q_i^\nu)$ [resp., $v_i = (v_i^1, \ldots, v_i^\nu)$] is the matrix whose (k, l)-th element equals $\partial/\partial_{q_i^k} a^j$ (resp., $\partial/\partial_{v_i^k} a^j$). By definition, for a C^2-function a, $\partial^2_{q_i, q_j} a(\cdot) = \partial_{q_j}(\partial_{q_i} a(\cdot))$, $\partial^2_{q_i, v_j} a(\cdot) = \partial_{v_j}(\partial_{q_i} a(\cdot))$, etc.

In what follows A^* denotes the adjoint matrix; Aa and $A_1 A_2$ denote as usually the product of a matrix and a vector and of two matrices, respectively.

In Section 3 we employ repeatedly the following assertions whose proof is contained in Appendix.

Proposition 2.1. *Let $U(r)$, $d_0 < r < \infty$, be of class C^2. Then*

i) $\partial_q U(|q|) = \dfrac{U'(|q|)}{|q|} q, \qquad q \in R^\nu, \qquad |q| > d_0,$

ii) $\det \partial^2_{qq} U(|q|) = U''(|q|) \left[\dfrac{U'(|q|)}{|q|}\right]^{\nu-1}, \qquad q \in R^\nu, \qquad |q| > d_0.$

Proposition 2.2. *Any real solution of the matrix equation*

$$(\partial^2_{q_2, q_2} U(|q_1 - q_2|) A(q_1) = A(q_2)(\partial^2_{q_1, q_1} U(|q_1 - q_2|)), \quad q_1, \ q_2 \in R^\nu, \ |q_1 - q_2| > d_0$$

has the form $A(q) = aE$, $q \in R^\nu$, where $a \in R^1$ is a constant and E is the identity matrix.

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[5] We deal with ν-vectors and $\nu \times \nu$-matrices; for $\nu = 1$ all the functions are of course, scalars and this notation system is not necessary

3. Proof of Theorem 2′

Let a function $f \in \mathscr{C}_2$; we identify f with the pair of functions $(f^{(1)}, f^{(2)})$, where

$$f^{(1)} = f^{(1)}(q, v) = f(q, v), \qquad q, v \in R^\nu, \tag{3.1a}$$

$$f^{(2)} = f^{(2)}(q_1, q_2; v_1, v_2), \quad q_i, v_i \in R^\nu, \quad i = 1, 2, \quad |q_1 - q_2| > d_0, \tag{3.1b}$$

and $f^{(2)}$ has the following symmetry property:

$$f^{(2)}(q_1, q_2; v_1, v_2) = f^{(2)}(q_2, q_1; v_2, v_1). \tag{3.1c}$$

Condition (G_4) for $f^{(2)}$ may be written in the form

$$\lim_{|q| \to \infty} f^{(2)}(q, q_1; v, v_1) = 0, \qquad q_1, v_1, v \in R^\nu. \tag{3.1d}$$

In terms of the functions $f^{(1)}$ and $f^{(2)}$

$$h(\bar{x}) = \sum_{(q,v) \in \bar{x}} f^{(1)}(q, v) + \tfrac{1}{2} \sum_{\substack{(q,v) \in \bar{x}, (q',v') \in \bar{x}, \\ q \neq q'}} f^{(2)}(q, q'; v, v'), \qquad \bar{x} \in D^0.$$

Equation (1.4) is equivalent to the system of three equations:

$$\langle \partial_q f^{(1)}(q, v), v \rangle = 0, \qquad q, v \in R^\nu, \tag{3.2a}$$

$$\begin{aligned} &\langle \partial_{q_1} f^{(2)}(q_1, q_2; v_1, v_2), v_1 \rangle + \langle \partial_{q_2} f^{(2)}(q_1, q_2; v_1, v_2), v_2 \rangle \\ &- \langle \partial_{v_1} f^{(2)}(q_1, q_2; v_1, v_2) + \partial_{v_1} f^{(1)}(q_1, v_1), \partial_{q_1} U(|q_1 - q_2|) \rangle \\ &- \langle \partial_{v_2} f^{(2)}(q_1, q_2; v_1, v_2) + \partial_{v_2} f^{(1)}(q_2, v_2), \partial_{q_2} U(|q_1 - q_2|) \rangle = 0, \\ &q_i, v_i \in R^\nu, \quad i = 1, 2, \quad |q_1 - q_2| > d_0, \end{aligned} \tag{3.2b}$$

and

$$\begin{aligned} &\langle \partial_{v_1} f^{(2)}(q_1, q_2; v_1, v_2), \partial_{q_1} U(|q_1 - q_3|) \rangle \\ &+ \langle \partial_{v_2} f^{(2)}(q_1, q_2; v_1, v_2), \partial_{q_2} U(|q_2 - q_3|) \rangle \\ &+ \langle \partial_{v_1} f^{(2)}(q_1, q_3; v_1, v_3), \partial_{q_1} U(|q_1 - q_2|) \rangle \\ &+ \langle \partial_{v_3} f^{(2)}(q_1, q_3; v_1, v_3), \partial_{q_3} U(|q_2 - q_3|) \rangle \\ &+ \langle \partial_{v_2} f^{(2)}(q_2, q_3; v_2, v_3), \partial_{q_2} U(|q_1 - q_2|) \rangle \\ &+ \langle \partial_{v_3} f^{(2)}(q_2, q_3; v_2, v_3), \partial_{q_3} U(|q_1 - q_3|) \rangle = 0, \\ &q_i, v_i \in R^\nu, \quad i = 1, 2, 3, \quad |q_j - q_k| > d_0, \quad 1 \leqq j < k \leqq 3. \end{aligned} \tag{3.2c}$$

Our aim is to show that, if C^2-functions $f^{(1)}$ and $f^{(2)}$ obey (G_2') and (G_4') respectively and satisfy (3.2a)–(3.2c), then

$$f^{(1)} = f^{(1)}(v) = \tfrac{1}{2} c_1 \langle v, v \rangle + \langle v, v_0 \rangle + c_2, \qquad v \in R^\nu, \tag{3.3a}$$

and

$$f^{(2)} = f^{(2)}(q_1, q_2) = c_1 U(|q_1 - q_2|), \quad q_1, q_2 \in R^\nu, \quad |q_1 - q_2| > d_0, \tag{3.3b}$$

where $c_1, c_2 \in R^1$ and $v_0 \in R^\nu$ are fixed.

It is not hard to verify that this assertion follows from the two lemmas below.

Lemma 3.1. *Let $f^{(1)} \in C^1$, $f^{(2)} \in C^2$, and $f^{(2)}$ obey* (G$'_4$). *Suppose the pair of functions* $(f^{(1)}, f^{(2)})$ *satisfies* (3.2a)–(3.2c). *Then*

$$f^{(2)}(q_1, q_2; v_1, v_2) \equiv 0 \quad \text{if} \quad |q_1 - q_2| > d_0 . \tag{3.4}$$

Lemma 3.2. *Let a pair* $(f^{(1)}, f^{(2)})$ *of C^2-functions satisfy* (3.2a)–(3.2c) *and $f^{(1)}$ obey* (G$'_2$), *$f^{(2)}$ obey* (G$'_4$). *Then*

i) $f^{(2)} = f^{(2)}(q_1, q_2)$, $\quad q_1, q_2 \in R^\nu$, $\quad |q_1 - q_2| > d_0$,

ii) $\partial_{q_i} f^{(2)}(q_1, q_2) = c_1 \partial_{q_i} U(|q_1 - q_2|)$, $\quad |q_1 - q_2| > d_0$, $\qquad i = 1, 2$,

iii) *$f^{(1)}$ is of the form* (3.3a).

To avoid a repetition in the proof of Theorem 2.1 for arbitrary n_0 we prove here a general assertion, whose particular case is Lemma 3.1. Let $f^{(n)}$ be a function defined on the set

$$\mathcal{O}^{(n)} = \{(q_1, \dots, q_n; v_1, \dots, v_n) \in R^{n\nu} \times R^{n\nu} : \min_{1 \leqq i < j \leqq n} |q_i - q_j| > d_0\}, \qquad n = 2, 3, \dots .$$

We say that $f^{(n)}$ obeys (G$'_4$) if for any $(q_2, \dots, q_n; v_2, \dots, v_n) \in \mathcal{O}^{(n-1)}$ and any $v \in R^\nu$

$$\lim_{|q| \to \infty} f^{(n)}(q, q_2, \dots, q_n; v, v_2, \dots, v_n) = 0 . \tag{3.5}$$

Denote

$$\boldsymbol{x}^{(n)} = (q_1, \dots, q_n; v_1, \dots, v_n), \quad \boldsymbol{x}^{(n),i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n; v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n),$$
$$1 \leqq i \leqq n .$$

Lemma 3.1′. *Let $f^{(n-1)} \in C^1$, $f^{(n)} \in C^2$ and $f^{(n)}$ obey* (G$'_4$). *Suppose the pair* $(f^{(n-1)}, f^{(n)})$ *satisfies the equations*

$$\begin{aligned}
&\sum_{i=1}^{n} \langle \partial_{q_i} f^{(n)}(\boldsymbol{x}^{(n)}), v_i \rangle - \sum_{i=1}^{n} \sum_{1 \leqq j \leqq n, j \neq i} \langle \partial_{v_i} f^{(n)}(\boldsymbol{x}^{(n)}), \partial_{q_i} U(|q_i - q_j|) \rangle \\
&\qquad - \sum_{i=1}^{n} \sum_{1 \leqq j \leqq n, j \neq i} \langle \partial_{v_j} f^{(n-1)}(\boldsymbol{x}^{(n),i}), \partial_{q_j} U(|q_i - q_j|) \rangle = 0 , \\
&\boldsymbol{x}^{(n)} = (q_1, \dots, q_n; v_1, \dots, v_n) \in \mathcal{O}^{(n)} ,
\end{aligned} \tag{3.6a}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n+1} \sum_{1 \leqq j \leqq n+1, j \neq i} \langle \partial_{v_j} f^{(n)}(\boldsymbol{x}^{(n+1),i}), \partial_{q_j} U(|q_i - q_j|) \rangle = 0 , \\
&\boldsymbol{x}^{(n+1)} = (q_1, \dots, q_{n+1}; v_1, \dots, v_{n+1}) \in \mathcal{O}^{(n+1)} .
\end{aligned} \tag{3.6b}$$

Then

$$f^{(n)}(\boldsymbol{x}^{(n)}) = 0 \quad \text{if} \quad \min_{1 \leqq i < j \leqq n} |q_i - q_j| > d_1 . \tag{3.7}$$

Proof of Lemma 3.1. From now on d_1 is supposed to be chosen so that $d_1 = \inf[d' : U(r) \equiv 0 \text{ for } r \geqq d']$. Let $\boldsymbol{x}^{(n)} = (q_1, \dots, q_n; v_1, \dots, v_n)$, where $\min_{1 \leqq i < j \leqq n} |q_i - q_j| > d_1$. We say that q_i is an external point for $(q_1, \dots, q_n)$ if q_i is a vertex of the smallest convex polyhedron containing $q_1, \dots, q_n$. It is easy to see that any external point q_i has the following properties: one can find

a) an open set $Q_i = Q_i(q_1, \ldots, q_n) \subset R^\nu$ such that for any $q \in Q_i$, $U'(|q-q_i|) \neq 0$ and $\min\limits_{1 \leqq j \leqq n, j \neq i} |q-q_j| > d_1$;

b) an open connected unbounded set $Q_i' = Q_i'(q_1, \ldots, q_n) \subset R^\nu$, $Q_i' \ni q_i$, such that any $q' \in Q_i'$ is an external point for $(q_1, \ldots, q_{i-1}, q', q_{i+1}, \ldots, q_n)$ and $\min\limits_{1 \leqq j \leqq n, j \neq i} |q'-q_j| > d_1$.

For definiteness suppose that q_1 is an external point for $q_1, \ldots, q_n$. Let $v \in R^\nu$ and $q \in Q_1$ be arbitrary vectors. Denote $x_{q,v}^{(n),1} = (q, q_2, \ldots, q_n; v, v_2, \ldots, v_n)$. Then Equations (3.6a) and (3.6b) take the form

$$\sum_{i=1}^{n} \langle \partial_{q_i} f^{(n)}(x^{(n)}), v_i \rangle = 0, \tag{3.8a}$$

$$\langle \partial_{v_1} f^{(n)}(x^{(n)}, \partial_{q_1} U(|q_1 - q|) \rangle + \langle \partial_v f^{(n)}(x_{q,v}^{(n),1}), \partial_q U(|q_1 - q|) \rangle = 0. \tag{3.8b}$$

Both terms in LHS of (3.8b) are of class C^1. Apply to (3.8b) the operator ∂_{v_1}[6]. This gives the vector equality

$$\partial_{v_1,v_1}^2 f^{(n)}(x^{(n)}) \partial_{q_1} U(|q_1 - q|) = 0,$$

and, according to (2.1a) and the condition $U'(|q_1 - q|) \neq 0$,

$$\partial_{v_1,v_1}^2 f^{(n)}(x^{(n)}) (q_1 - q) = 0.$$

The last equality holds for the open set Q_1 of vectors q. Thus we have the matrix equality

$$\partial_{v_1,v_1}^2 f^{(n)}(x^{(n)}) = 0. \tag{3.9}$$

The general solution of Equation (3.9) is

$$f^{(n)}(x^{(n)}) = a_1(x^{(n)}) + \langle \mathbf{a}_1(x^{(n)}), v_1 \rangle, \tag{3.10}$$

$\min\limits_{1 \leqq i < j \leqq n} |q_i - q_j| > d_1$, where q_1 is an external point for $(q_1, \ldots, q_n)$ and a_1 and $\mathbf{a}_1$ do not depend on v_1. Substituting (3.10) into (3.8a) we obtain:

$$\langle \partial_{q_1} a(x^{(n)}), v_1 \rangle + \langle \partial_{q_1} \mathbf{a}_1(x^{(n)}) v_1, v_1 \rangle$$
$$+ \sum_{i=2}^{n} \langle \partial_{q_i} a_1(x^{(n)}) + \partial_{q_i} \mathbf{a}_1(x^{(n)}) v_1, v_i \rangle = 0.$$

Taking the terms of the first and second order in v_1 we have

$$\partial_{q_1} a_1(x^{(n)}) + \sum_{i=2}^{n} (\partial_{q_i} \mathbf{a}_1(x^{(n)}))^* v_i = 0 \tag{3.11a}$$

and

$$\langle \partial_{q_1} \mathbf{a}_1(x^{(n)}) v_1, v_1 \rangle = 0,$$

i.e.

$$\partial_{q_1} \mathbf{a}_1(x^{(n)}) = -(\partial_{q_1} \mathbf{a}_1(x^{(n)}))^*. \tag{3.11b}$$

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[6] By application of an operator ∂ to a given equality we mean the operation of taking the corresponding derivative of functions in both its sides

Proposition 3.3. *Let* $\mathbf{a}(q)$ *be a vector* C^2*-function on an open connected subset* $Q \subset R^\nu$, *and* $\partial_q \mathbf{a}(q)$, $q \in Q$, *is an antisymmetric matrix. Then the components of* $\mathbf{a}$ *are polynomial functions of the components of* q.

For the proof of Proposition 3.3 see Appendix. According to Proposition 3.3, the components of $\mathbf{a}_1(\cdot)$ are polynomials in $q \in Q'_1$ ($q_2, \ldots, q_n$; v_1, $v_2, \ldots, v_n$ are supposed to be fixed). Due to (3.11a), the same is true for $\partial_{q_1} a_1(\cdot)$ and hence $a_1(\cdot)$ is also a polynomial in q_1. Now return to (3.10) and set $v_1 = 0$. We obtain

$$f^{(n)}(x^{(n)})|_{v_1=0} = a_1(x^{(n)}), \qquad q_1 \in Q'_1 .$$

Using (3.5) and the property b) above, we get

$$\lim_{|q_1| \to \infty, q_1 \in Q'_1} a_1(x^{(n)}) = 0 .$$

Since a_1 is a polynomial in q_1, it must vanish. Now (3.5) and (3.10) imply

$$\lim_{|q_1| \to \infty, q_1 \in Q'_1} \langle \mathbf{a}_1(x^{(n)}), v_1 \rangle = 0 \quad \text{for any} \quad v_1 \in R^\nu .$$

Thus the vector function $\mathbf{a}_1$ whose components are polynomials in q_1 also vanishes. This completes the proof of Lemma 3.1.

Proof of Lemma 3.2i). Let $q_1, q_2 \in R^\nu$, $|q_1 - q_2| > d_0$. Using conditions (I'_1)–(I'_3) and our choice of d_1 it is not hard to check that there exists $r > \max[d_0, d_{\frac{1}{2}}]$ such that $U'(r)U''(r) \neq 0$ (and hence, an interval of such r's). Therefore, there exists a non-empty open set $\mathcal{O} \subset R^\nu$ such that $|q_1 - q| > d_1$ and $U'(|q - q_2|)\, U''(|q - q_2|) \neq 0$ whenever $q \in \mathcal{O}$. By Lemma 3.1, if $q_3 \in \mathcal{O}$, Equation (3.2c) takes the form

$$\begin{aligned} &\langle \partial_{v_2} f^{(2)}(q_1, q_2; v_1, v_2), \partial_{q_2} U(|q_2 - q_3|) \rangle \\ &+ \langle \partial_{v_2} f^{(2)}(q_2, q_3; v_2, v_3), \partial_{q_2} U(|q_1 - q_2|) \rangle = 0 . \end{aligned} \tag{3.12}$$

Applying to (3.12) the operator ∂_{v_1} gives

$$\partial^2_{v_2, v_1} f^{(2)}(q_1, q_2; v_1, v_2)\, \partial_{q_2} U(|q_2 - q_3|) = 0 ,$$

i.e., according to Proposition 2.1i),

$$\partial^2_{v_2, v_1} f^{(2)}(q_1, q_2; v_1, v_2)(q_2 - q_3) = 0 .$$

This equality holds for the open set $\mathcal{O}$ of vectors q_3. Hence,

$$\partial^2_{v_2, v_1} f^{(2)}(q_1, q_2; v_1, v_2) = 0 . \tag{3.13}$$

Equation (3.13) has the following general symmetric solution:

$$f^{(2)}(q_1, q_2; v_1, v_2) = a_2(q_1, q_2; v_1) + a_2(q_2, q_1; v_2), \qquad |q_1 - q_2| > d_0 , \tag{3.14}$$

where a_2 is a C^2-function.

Substitute (3.14) into (3.2b) and apply to the equation obtained the operator ∂_{v_1}:

$$\begin{aligned} &\partial_{q_1} a_2(q_1, q_2; v_1) + (\partial^2_{q_1, v_1} a_2(q_1, q_2; v_1)) v_1 + \partial_{q_1} a_2(q_2, q_1; v_2) \\ &+ (\partial^2_{q_2, v_1} a_2(q_1, q_2; v_1)) v_2 - (\partial^2_{v_1, v_1} a_2(q_1, q_2; v_1) + \partial_{v_1} f^{(1)}(q_1, v_1)) \partial_{q_1} U(|q_1 - q_2|) = 0. \end{aligned}$$

We see that the vector $\partial_{q_1} a_2(q_2, q_1; v_2)$ depends on v_2 linearly:

$$\partial_{q_1} a_2(q_2, q_1; v_2) = \mathbf{a}_2(q_2, q_1; v_1) + A_1(q_2, q_1; v_1) v_2. \tag{3.15}$$

Taking $v_2 = 0$ in (3.15) we have

$$\partial_{q_1} a_2(q_2, q_1; 0) = \mathbf{a}_2(q_2, q_1; v_1)$$

i.e., $\mathbf{a}_2(q_2, q_1; v_1)$ does not depend on v_1. We write $\mathbf{a}_2(q_2, q_1)$ instead of $\mathbf{a}_2(q_2, q_1; v_1)$, and, similarly, $A_1(q_2, q_1)$ instead of $A_1(q_2, q_1; v_1)$. Thus

$$\partial_{q_1} a_2(q_2, q_1; v_2) = \mathbf{a}_2(q_2, q_1) + A_1(q_2, q_1) v_2. \tag{3.15'}$$

Now return to (3.12). Apply the operator ∂_{q_3} and take into account (3.14) and (3.15′):

$$\partial^2_{q_2, q_3} U(|q_2 - q_3|) \partial_{v_2} a_2(q_2, q_1; v_2) + A_1(q_2, q_3) \partial_{q_1} U(|q_1 - q_2|) = 0.$$

We obtain that for any two vectors $v', v'' \in R^\nu$

$$\partial^2_{q_2, q_3} U(|q_2 - q_3|) [\partial_{v_2} a_2(q_2, q_1; v_2)|_{v_2 = v'} - \partial_{v_2} a_2(q_2, q_1; v_2)|_{v_2 = v''}] = 0.$$

Due to Proposition 2.1ii), the matrix $\partial^2_{q_2, q_3} U(|q_2 - q_3|)$, $q_3 \in \mathcal{O}$, is invertible and hence $\partial_{v_2} a_2(q_2, q_1; v_2)|_{v_2 = v'} - \partial_{v_2} a_2(q_2, q_1; v_2)|_{v_2 = v''} = 0$, i.e. $a_2(q_2, q_1; v_2)$ depends on v_2 linearly:

$$a_2(q_2, q_1; v_2) = \langle \mathbf{a}_3(q_2, q_1), v_2 \rangle + a_3(q_2, q_1), \quad |q_1 - q_2| > d_0. \tag{3.16}$$

Denote $a_4(q_1, q_2) = a_3(q_1, q_2) + a_3(q_2, q_1)$; a_4 is a symmetric function: $a_4(q_1, q_2) = a_4(q_2, q_1)$. Due to (3.14) and (3.16), we have

$$f^{(2)}(q_1, q_2; v_1, v_2) = \langle \mathbf{a}_3(q_1, q_2), v_1 \rangle + \langle \mathbf{a}_3(q_2, q_1), v_2 \rangle + a_4(q_1, q_2), \quad |q_1 - q_2| > d_0. \tag{3.17}$$

Our aim is to show that $\mathbf{a}_3 = 0$. Substitute (3.17) into Equation (3.12). We get

$$\langle \mathbf{a}_3(q_2, q_1), \partial_{q_2} U(|q_2 - q_3|) \rangle + \langle \mathbf{a}_3(q_2, q_3), \partial_{q_2} U(|q_1 - q_2|) \rangle = 0$$

or, due to Proposition 2.1i),

$$\frac{U'(|q_2 - q_3|)}{|q_2 - q_3|} \langle \mathbf{a}_3(q_2, q_1), q_2 - q_3 \rangle + \frac{U'(|q_1 - q_2|)}{|q_1 - q_2|} \langle \mathbf{a}_3(q_2, q_3), q_2 - q_1 \rangle = 0. \tag{3.18}$$

First suppose $U'(|q_1 - q_2|) = 0$. Then for $q_3 \in \mathcal{O}$

$$\langle \mathbf{a}_3(q_2, q_1), q_2 - q_3 \rangle = 0.$$

Since $\mathcal{O}$ is open, this implies that $\mathbf{a}_3(q_2, q_1) = 0$. By symmetry of $f^{(2)}$, $\mathbf{a}_3(q_1, q_2) = 0$. Hence,

$$f^{(2)}(q_1, q_2; v_1, v_2) = a_4(q_1, q_2) \tag{3.19}$$

for $q_1, q_2 \in R^\nu$ such that

$$|q_1 - q_2| > d_0, \quad U'(|q_1 - q_2|) = 0. \tag{3.20}$$

Next consider the case $U'(|q_1-q_2|)\neq 0$. The set $\{r\geq d_0: U'(r)\neq 0\}$ is the union of intervals β_k, $k=1,2,\ldots$. Let $|q_1-q_2|\in\beta_k$, $q_3\in\mathcal{O}$. We can rewrite Equation (3.18) in the form

$$\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\langle \mathbf{a}_3(q_2,q_1), q_2-q_3\rangle + \frac{|q_2-q_3|}{U'(|q_2-q_3|)}\langle \mathbf{a}_3(q_2,q_3), q_2-q_1\rangle = 0. \quad (3.21)$$

Notice that Equation (3.21) remains true for small changes of the variables q_1 and q_3. Applying to (3.21) successively the operators ∂_{q_3} and ∂_{q_1}, we have

$$\partial_{q_1}\left(\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)\right) + \left[\partial_{q_3}\left(\frac{|q_2-q_3|}{U'(|q_2-q_3|)}\mathbf{a}_3(q_2,q_3)\right)\right]^* = 0. \quad (3.22)$$

This means that, for fixed q_2, the matrix $\partial_{q_1}\left(\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)\right)$ does not change for a small change of q_1.

From now on it is convenient to consider separately the cases $\nu>1$ and $\nu=1$. In this Section we give the proof for the case $\nu>1$; the modifications we have to make for $\nu=1$ are presented in Section 4. The vector function $\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)$ is of class C^2 at any point (q_1,q_2) such that $|q_1-q_2|\in\beta_k$. For $\nu>1$ the set

$$T_k(q_2)=\{q_1\in R^\nu: |q_1-q_2|\in\beta_k\}$$

is connected. Hence, for fixed q_2, the matrix $\partial_{q_1}\left(\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)\right)$ is constant whenever $|q_1-q_2|\in\beta_k$, i.e.

$$\partial_{q_1}\left(\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)\right) = A_2(q_2;k), \qquad |q_1-q_2|\in\beta_k. \quad (3.23)$$

Equation (3.22) takes now the form

$$A_2(q_2;k)+A_2(q_2,k')^*=0 \quad (3.24)$$

whenever there exist q_1, $q_3\in R^\nu$ such that

$$|q_1-q_2|\in\beta_k,\ |q_2-q_3|\in\beta_{k'},\ |q_1-q_3|>d_1,\ k,k'=1,2,\ldots. \quad (3.25)$$

Clearly, such q_1, q_3 exist for a pair (k,k') if $t_k+t_{k'}>d_1$, where $\beta_k=(s_k,t_k)$, $\beta_{k'}=(s_{k'},t_{k'})$. Using conditions (I'_1)–(I'_3) and our choice of d_1, it is not hard to check that there exists k_0 such that $t_{k_0}>d_1-d_0$ and hence, $t_{k_0}+t_{k'}>d_1$ for any k'. This means that $A_2(q_2;k)$ does not depend on k:

$$A_2(q_2;k)=A_2(q_2),\ k=1,2,\ldots;\quad A_2(q_2)+A_2(q_2)^*=0. \quad (3.26)$$

The general solution of Equation (3.23) may be written as

$$\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)=A_2(q_2)q_1+\mathbf{a}_4(q_2;k), \qquad |q_1-q_2|\in\beta_k. \quad (3.27)$$

Now consider Equation (3.2c) where

$$|q_1-q_2|\in\beta_k,\quad |q_2-q_3|\in\beta_{k'},\quad |q_1-q_3|>d_1.$$

On account of (3.17) and (3.27), we obtain

$$\langle A_2(q_2)q_1+\mathbf{a}_4(q_2;k), q_2-q_3\rangle+\langle A_2(q_2)q_3+\mathbf{a}_4(q_2;k'), q_2-q_1\rangle=0. \qquad (3.28)$$

Notice that Equation (3.28) with fixed k and k' remains true for small changes of the variables q_1 and q_3. Thus considering the terms in LHS of (3.28) which contain q_3, but not q_1, we have

$$\mathbf{a}_4(q_2;k)=A_2(q_2)^*q_2, \qquad k=1,2,\ldots,$$

and, returning to (3.27) and using (3.26) and again Proposition 2.1i),

$$\mathbf{a}_3(q_2,q_1)=A_2(q_2)\partial_{q_1}U(|q_1-q_2|). \qquad (3.29)$$

On account of (3.17) and (3.29),

$$f^{(2)}(q_1,q_2;v_1,v_2)=\langle A_2(q_2)v_2-A_2(q_1)v_1, \partial_{q_1}U(|q_1-q_2|)\rangle+a_4(q_1,q_2), \qquad (3.30)$$

whenever

$$|q_1-q_2|>d_0, \qquad U'(|q_1-q_2|)\neq 0.$$

Notice that the first term in RHS of (3.30) vanishes whenever $U'(|q_1-q_2|)=0$ and so, due to (3.19) and (3.20), formula (3.30) may be considered as a general representation of $f^{(2)}$ for $|q_1-q_2|>d_0$. Substitute (3.30) into (3.2b) and apply successively the operators ∂_{v_1} and ∂_{v_2}. We obtain the equality

$$A_2(q_2)\partial^2_{q_1,q_1}U(|q_1-q_2|)-\partial^2_{q_2,q_2}U(|q_1-q_2|)A_2(q_1)=0, \qquad |q_1-q_2|>d_0. \qquad (3.31)$$

According to Proposition 2.2, $A_2(q_2)=a_6E$ where a_6 is a constant. Due to the antisymmetry of A_2 [see (3.26)], $a_6=0$. Hence, $A_2(q_1)=A_2(q_2)\equiv 0$. Q.E.D.

ii) The proof given here holds for the both cases: $v>1$ and $v=1$. On account of i), Equation (3.2b) takes the form

$$\begin{aligned}&\langle\partial_{q_1}f^{(2)}(q_1,q_2), v_1\rangle+\langle\partial_{q_2}f^{(2)}(q_1,q_2), v_2\rangle\\ &-\langle\partial_{v_1}f^{(1)}(q_1,v_1), \partial_{q_1}U(|q_1-q_2|)\rangle\\ &-\langle\partial_{v_2}f^{(1)}(q_2,v_2), \partial_{q_2}U(|q_1-q_2|)\rangle=0, \quad |q_1-q_2|>d_0.\end{aligned} \qquad (3.32)$$

Applying to (3.32) the operator ∂_{v_1} gives

$$\partial_{q_1}f^{(2)}(q_1,q_2)-\partial^2_{v_1,v_1}f^{(1)}(q_1,v_1)\ \partial_{q_1}U(|q_1-q_2|)=0.$$

Hence, for any v, $v'\in R^v$,

$$[\partial^2_{v_1,v_1}f^{(1)}(q_1,v_1)|_{v_1=v'}-\partial^2_{v_1,v_1}f^{(1)}(q_1,v_1)|_{v_1=v''}]\,\partial_{q_1}U(|q_1-q_2|)=0. \qquad (3.33)$$

Fix $q_1\in R^v$ and consider Equation (3.33) on the open set $\{q_2\in R^v: |q_1-q_2|>d_0, U'(|q_1-q_2|)\neq 0\}$. Using Proposition 2.1i), we obtain

$$\partial^2_{v_1,v_1}f^{(1)}(q_1,v_1)|_{v_1=v'}=\partial^2_{v_1,v_1}f^{(1)}(q_1,v_1)|_{v_1=v''},$$

i.e.,

$$\partial^2_{v_1,v_1}f^{(1)}(q_1,v_1)=A_3(q_1), \qquad (3.34)$$

where A_3 is a symmetric matrix.

The general solution of (3.34) is

$$f^{(1)}(q_1, v_1) = \langle A_3(q_1)v_1, v_1\rangle + \langle \mathbf{a}_5(q_1), v_1\rangle + a_7(q_1). \tag{3.35}$$

Substitute (3.35) into (3.32) and aquate with zero the first order terms in v_1 and v_2 respectively. We obtain the equations:

$$\partial_{q_1} f^{(2)}(q_1, q_2) = 2A_3(q_1)\,\partial_{q_1} U(|q_1 - q_2|), \quad |q_1 - q_2| > d_0,$$

$$\partial_{q_2} f^{(2)}(q_1, q_2) = 2A_3(q_2)\,\partial_{q_2} U(|q_1 - q_2|), \quad |q_1 - q_2| > d_0.$$

Apply to (3.36a) and (3.36b) the operators ∂_{q_2} and ∂_{q_1} respectively. Using the relation $\partial^2_{q_1,q_2} f^{(2)}(q_1, q_2) = (\partial^2_{q_2,q_1} f^{(2)}(q_1, q_2))^*$ we arrive at the equation

$$\partial^2_{q_1 q_2} U(|q_1 - q_2|)\,A_3(q_1) = A_3(q_2)\,\partial_{q_1 q_2} U(|q_1 - q_2|), \quad |q_1 - q_2| > d_0.$$

According to Proposition 2.2,

$$A_3(q_1) = A_3(q_2) = a_8 E,$$

where a_8 is a constant. Now Equations (3.36a) and (3.36b) give ii) with $c_1 = 2a_8$.

iii) As in ii), we consider the both cases: $\nu > 1$ and $\nu = 1$ simultaneously. On account of (3.35) and (3.37), we have to prove that $\mathbf{a}_5(q_1)$ and $a_7(q_1)$ are constant. Equation (3.2b) takes now the form

$$\langle \mathbf{a}_5(q_1) - \mathbf{a}_5(q_2), \partial_{q_1} U(|q_1 - q_2|)\rangle = 0, \quad |q_1 - q_2| > d_0.$$

If $U'(|q_1 - q_2|) \neq 0$, then

$$\langle \mathbf{a}_5(q_1) - \mathbf{a}_5(q_2), q_1 - q_2\rangle = 0$$

and applying successively the operators ∂_{q_2} and ∂_{q_1} gives

$$\partial_{q_1}\mathbf{a}_5(q_1) + (\partial_{q_2}\mathbf{a}_5(q_2))^* = 0.$$

This means that $\partial_{q_1}\mathbf{a}_5(q_1)$ is a constant antisymmetric matrix, say, A_4, i.e.,

$$\mathbf{a}_5(q_1) = A_4 q_1 + \mathbf{a}_6,$$

where $\mathbf{a}_6$ is a constant vector.

Now use Equation (3.1c). Due to (3.35), (3.37), and (3.38), we have

$$\langle \partial_q a_7(q), v\rangle = 0, \quad q, v \in R^\nu.$$

Hence, $\partial_q a_7(q) = 0$ and $a_7(q) = c_2 = \text{const}$. For $f^{(1)}$ we obtain

$$f^{(1)}(q, v) = \tfrac{1}{2}c_1\langle v, v\rangle + \langle A_4 q + \mathbf{a}_6, v\rangle + c_2.$$

Suppose $v \in R^\nu$ to be fixed. Condition (G_2') implies that $A_4 = 0$. Setting $\mathbf{a}_6 = v_0$, we obtain iii). Lemma 3.2 is proved.

4. One-dimensional Case

For $\nu = 1$ we have to modify a part of the proof of Lemma 3.2i)[7]. The set $T_k(q_2)$ is now non-connected: it consists of two connected components $T_k^{\pm}(q_2) = \{q_1 \in R^1 : q_1$

[7] As we noticed above, for $\nu = 1$ all the functions considered are scalars. We use the same notation system as in the case $\nu > 1$ for more complete analogy with corresponding arguments in the preceding section

$-q_2 \gtrless 0, \pm(q_1 - q_2) \in \beta_k\}$. Equations (3.23) and (3.24) take the form

$$\partial_{q_1}\left(\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1)\right) = A_2^{\pm}(q_2;k), \quad \pm(q_1-q_2)\in\beta_k, \tag{4.1}$$

and

$$A_2^{\pm}(q_2;k) + A_2^{\mp}(q_2,k') = 0, \tag{4.2}$$

if there exist $q_1, q_3 \in R^1$ such that

$$\pm(q_1-q_2)\in\beta_k, \quad \pm(q_2-q_3)\in\beta_k, \quad \pm(q_1-q_3) > d_1 .$$

Repeating the arguments used in the case $v>1$ we obtain instead of (3.26) and (3.27),

$$A_2^{\pm}(q_2;k) = A_2^{\pm}(q_2), \quad k=1,2,\dots, \quad A_2^{+}(q_2)+A_2^{-}(q_2)=0 \tag{4.3}$$

and

$$\frac{|q_1-q_2|}{U'(|q_1-q_2|)}\mathbf{a}_3(q_2,q_1) = A_2^{\pm}(q_2)\,q_1 + \mathbf{a}_4^{\pm}(q_2;k), \quad \pm(q_1-q_2)\in\beta_k . \tag{4.4}$$

As above, one can easely make sure that $\mathbf{a}_4^{\pm}(q_2;k) = A_2^{\mp}(q_2)\,q_2$ and therefore

$$\mathbf{a}_3(q_2,q_1) = A_2^{\pm}(q_2)\,\partial_{q_1}U(|q_1-q_2|), \quad \pm(q_1-q_2) > d_0 .$$

Instead of (3.30) we now have

$$\begin{aligned} &f^{(2)}(q_1,q_2;v_1,v_2) \\ &= (A_2^{\pm}(q_2)v_2 - A_2^{\mp}(q_1)v_1)\,\partial_{q_1}U(|q_1-q_2|) + a_4(q_1,q_2), \end{aligned}$$

where

$$\pm(q_1-q_2) > d_0, \; U'(|q_1-q_2|) \neq 0 .$$

On account of (3.19) and (3.20), this representation holds for any $q_1, q_2 \in R^1$ such that $|q_1-q_2| > d_0$. As above, we obtain the equation

$$A_2^{\pm}(q_2)\,U''(|q_1-q_2|) - U''(|q_1-q_2|)\,A_2^{\mp}(q_1) = 0, \quad |q_1-q_2| > d_0 \tag{4.5}$$

which is analogous to (3.31). Now we may cancel $U''(|q_1-q_2|)$ out of (4.5) and get

$$A_2^{\pm}(q_2) = A_2^{\mp}(q_1) = A_2 .$$

Due to (4.3), $A_2 = 0$. This gives i) for $v=1$.

Appendix

Proof of Proposition 2.1. Statement i) is trivial and we start with the proof of ii). We use the following formula

$$\partial^2_{q,q}U(|q|) = \frac{U'(|q|)}{|q|}E + \left(U''(|q|) - \frac{U'(|q|)}{|q|}\right)(e_q \otimes e_q), \quad |q| > d_0 ,$$

where $e_q = |q|^{-1} q$ and $e_q \otimes e_q$ denotes the matrix whose (ij)-th element is $e_q^i e_q^j$. It is easy to check that

$$(\partial^2_{q,q} U(|q|))\, q = U''(|q|) q \tag{A.1}$$

and

$$(\partial^2_{q,q} U(|q|))\, \mathbf{a} = \frac{U'(|q|)}{|q|} \mathbf{a} \tag{A.2}$$

for any $\mathbf{a} \in R^\nu$ which is orthogonal to q. Thus, $\partial^2_{q,q} U(|q|)$ has two eigenvalues, $U''(|q|)$ and $|q|^{-1} U'(|q|)$, of multiplicity 1 and $(\nu - 1)$ respectively. This gives (2.1b).

Proof of Proposition 2.2. The change of variables $q_1 \leftrightarrow q_2$ in (2.2) and the fact that $\partial^2 U(|q_1 - q_2|)$ is a symmetric matrix lead to the equality

$$\partial^2_{q_1,q_1} U(|q_1 - q_2|)\, A(q_2) = A(q_1)\, \partial^2_{q_2,q_2} U(|q_1 - q_2|)$$

which together with (2.2) gives

$$\begin{aligned} &A(q_1)(\partial^2_{q_1,q_1} U(|q_1 - q_2|))^2 \\ &= (\partial^2_{q_2,q_2} U(|q_1 - q_2|))^2 A(q_1), \quad |q_1 - q_2| > d_0 . \end{aligned} \tag{A.3}$$

By the above arguments,

$$\begin{aligned} &\partial^2_{q_i,q_i} U(|q_1 - q_2|) \\ &= U''(|q_1 - q_2|)^2 P_{q_1 - q_2} + \left(\frac{U'(|q_1 - q_2|)}{|q_1 - q_2|} \right)^2 (E - P_{q_1 - q_2}), \quad i = 1, 2, \end{aligned} \tag{A.4}$$

where $P_{q_1 - q_2}$ is the projection on the one-dimensional subspace spanned by $q_1 - q_2$. Using (A.3) and (A.4) we have

$$\begin{aligned} &U''(|q_1 - q_2|)^2 A(q_1)(q_1 - q_2) \\ &= \partial^2_{q_i,q_i} U(|q_1 - q_2|) A(q_1)(q_1 - q_2), \quad |q_1 - q_2| > d_0 , \end{aligned}$$

i.e., $A(q_1)(q_1 - q_2)$ is an eigenvector of $\partial^2_{q_i,q_i} U(|q_1 - q_2|)$ with the eigenvalue $U''(|q_1 - q_2|)^2$. Now we use the fact that for any fixed q_1 there exists an open set of vectors q_2 for which $U''(|q_1 - q_2|)^2 \neq |q_1 - q_2|^{-2} U'(|q_1 - q_2|)^2$. The proof is simple and is given below. Due to (A.1), (A.2), for any fixed q_1 there exists an open set of vectors q_2 and hence, of vectors $(q_1 - q_2)]$ such that

$$A(q_1)(q_1 - q_2) = a(q_1, q_2)(q_1 - q_2)$$

i.e., $A(q_1)$ has an open set of eigenvectors. Hence, $A(q_1)$ is a scalar matrix and a depends only on $q_1 : A(q_1) = a(q_1) E$. Substituting this formula into (A.3), it is not hard to see that $a(q_1) = \text{const}$.

Let us prove the mentioned fact. Let $\beta = (s, t)$ be an arbitrary connected component of the set $\{r > d_0 : U'(r) \neq 0\}$. Consider the sets $\Delta_\pm = \{r \in \beta : U''(r) = \pm r^{-1} U'(r)\}$. Clearly, $\Delta_+ \cap \Delta_- = \phi$. We have to prove that $\Delta_+ \cup \Delta_- \neq \beta$. The sets $\Delta_\pm$ are closed in β. Since β is connected, either Δ_+ or Δ_- is empty. Consider, e.g., the case $\Delta_- = \phi$ and suppose $\beta = \Delta_+$. The general solution of the equation $U''(r) = r^{-1} U'(r)$ is $U(r) = a_1 r^2 + a_2$. By definition of β, $U'(t) = 0$, but $a_1 \neq 0$. Thus we have the contradiction. The case $\Delta_+ = \phi$ may be treated in a similar way.

Proof of Proposition 3.3. Due to the antisymmetry,

$$\frac{\partial \mathbf{a}^i(q)}{\partial q^j} + \frac{\partial \mathbf{a}^j(q)}{\partial q^i} = 0, \quad q \in Q, \tag{A.5}$$

and, in particular, $\frac{\partial \mathbf{a}^i(q)}{\partial q^i} = 0$. Taking $\partial/\partial q^i$ of (A.5) we obtain

$$\frac{\partial^2 \mathbf{a}^i(q)}{\partial q^i \partial q^j} + \frac{\partial^2 \mathbf{a}^j(q)}{(\partial q^i)^2} = \frac{\partial^2 \mathbf{a}^j(q)}{(\partial q^i)^2} = 0.$$

This means that the components $\mathbf{a}^j(q)$ are locally polynomials in q. Since Q is supposed to be connected, they are polynomials.

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