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# Field Theory with an External Potential

W. J. M. A. Hochstenbach
Institut für Theoretische Physik, Universität Göttingen, D-3400 Göttingen,
Federal Republic of Germany

**Abstract.** A quantum theory for charged spin zero particles interacting with an external potential is constructed for a certain class of time-independent potentials.
For potentials of a different class, a group of Bogoliubov transformations generated by the solutions of a classical differential equation with an external potential is defined in the free one-particle space. We give necessary and sufficient conditions on the potential for this group to be unitarily implementable in the Fock space of the free field.

## 1. Introduction

Although perturbation theory constitutes the basis for the practical calculations in quantum electrodynamics, a direct justification for applying perturbation theory is prevented by the existence of a Euclidean symmetry group. Generally one tries to circumvent this implication of Haag's theorem by breaking the symmetry.

For charged spin zero particles in an external time-independent potential we prove in Section 3 a conjecture of Schroer, Seiler and Swieca [1]: the interaction Hamiltonian does not exist in the Fock space of the free field if the external potential contains a three-vector part. Thus, in a field theory with an external, time-independent vector potential $(0, \boldsymbol{A})$ the assertion of Haag's theorem is valid, although its assumptions are not fulfilled.

To be more precise, let $(A^\mu)$ $(\mu=0, 1, 2, 3)$ be an element of the function class $\mathcal{Q}$ specified in Section 3; the time evolution of the interacting field in the Fock space of the free particles is determined by a one-parameter group of Bogoliubov transformations in the free one-particle space; at time $t=0$ the interacting field coincides with the free field. Then the time evolution is described by a strongly continuous, one-parameter group of unitary operators in the Fock space of the free field, if and only if $\boldsymbol{A}=0$ on $\mathbb{R}^3$ and

$$Q_0(t)=\int \frac{d\boldsymbol{p}d\boldsymbol{p}'}{\omega(\boldsymbol{p})\omega(\boldsymbol{p}')}\left(\frac{\omega(\boldsymbol{p})-\omega(\boldsymbol{p}')}{\omega(\boldsymbol{p})+\omega(\boldsymbol{p}')}\right)^2 |\tilde{A}_0(\boldsymbol{p}-\boldsymbol{p}')|^2 \sin^2\frac{(\omega(\boldsymbol{p})+\omega(\boldsymbol{p}'))t}{2}$$

is finite for all times $t$ and continuous in $t$ at 0.

In the spin zero case, the vanishing of $A$ stems from the presence of derivatives in the interaction part of the Hamiltonian in the free one-particle space.

In Section 2 we show the existence of a non-trivial class of potentials, for which one can construct a Fock quantization with a unique vacuum for charged spin zero particles in an external potential. Furthermore, it is shown that every quantization with a unique vacuum, whose generating functional is in the Fock sector, is unitarily equivalent to the Fock quantization.

## 2. Quantization of the Klein-Gordon Equation with an External Potential

Our main concern will be proving the existence of a non-trivial class $\mathscr{P}$ of potentials, for which one can define a one-particle Hilbert space $\mathscr{H}_P$ and a one-parameter strongly continuous group with a strictly positive generator $\overline{|B|}$.

Let $\mathscr{H}_{E_0}$ be the direct sum of the Sobolev space $W^1(\mathbb{R}^3)$ and the Hilbert space $L^2(\mathbb{R}^3)$. The operator $B_0$ in $\mathscr{H}_{E_0}$, with domain $D(B_0)=C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$ and defined by $B_0=\begin{pmatrix} 0 & 1 \\ -\Delta+m^2 & 0 \end{pmatrix}$ on $D(B_0)$ $(m\neq 0)$, is essentially self-adjoint on $D(B_0)$.

Its closure $\bar{B}_0$ has domain $D(\bar{B}_0)=W^2(\mathbb{R}^3)\oplus W^1(\mathbb{R}^3)$, and is given by $\bar{B}_0=\begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}$, where $\omega^2$ is the closure of the operator $-\Delta+m^2$ on $C_0^\infty(\mathbb{R}^3)$ as a mapping from $W^1(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. We close the operator $B_0$ and denote this closure again by $B_0$. The self-adjoint operator $B_0$ has the property that $B_0^{-1}$ is bounded with $\|B_0^{-1}\|=m^{-1}$. In the space $\mathscr{H}_{E_0}$ we introduce the perturbation $B_{\text{int}}:=\begin{pmatrix} eA_0 & 0 \\ e^2\boldsymbol{A}^2+ie\nabla\cdot\boldsymbol{A}+2ie\boldsymbol{A}\cdot\nabla & eA_0 \end{pmatrix}$ as a bounded operator for potentials $(A^\mu)$ $(\mu=0,1,2,3)$ in $C_B^1(\mathbb{R}^3)$, the class of real bounded continuous functions with bounded and continuous first partial derivatives.

For potentials in $C_B^1(\mathbb{R}^3)$ with the additional property $|eA_0|\leqq m$ an inner product $(\cdot,\cdot)_E$ is defined on $C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$ by

$$(f,g)_E=\int\{((-i\nabla-e\boldsymbol{A})f_1)^*(-i\nabla-e\boldsymbol{A})g_1+(m^2-e^2A_0^2)f_1^*g_1 \\ +(eA_0f_1+f_2)^*(eA_0g_1+g_2)\}d\boldsymbol{x}$$

for all $f,g\in C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$.

The completion of $C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$ in the energy norm is denoted by $\mathscr{H}_E$; the Hilbert spaces $\mathscr{H}_{E_0}$ and $\mathscr{H}_E$ are related as sets by $\mathscr{H}_{E_0}\subset\mathscr{H}_E$. In the same way as one proves the Kato-Rellich theorem, one obtains the following result.

**Lemma 2.1.** *If $(A^\mu)\in C_B^1(\mathbb{R}^3)$ and $|eA_0|\leqq m$, then the operator $B:=B_0+B_{\text{int}}$ is essentially self-adjoint on $D(B_0)$ in the Hilbert space $\mathscr{H}_E$.*

If one assumes that the norms $\|\cdot\|_{E_0}$ and $\|\cdot\|_E$ on $C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$ are equivalent, one gets a stronger result: the operator $B$ is self-adjoint on $D(B_0)$ in $\mathscr{H}_E$; $C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$ is a core for $B$.

Let $\mathscr{P}_0$ be the class of potentials $(A^\mu)$ with $A^\mu\in C_B^1(\mathbb{R}^3)$ and $|eA_0|\leqq m$ such that there exists $c>0$ with $\frac{1}{c}\|f\|_{E_0}\leqq\|f\|_E\leqq c\|f\|_{E_0}$ for all $f\in C_0^\infty(\mathbb{R}^3)\oplus C_0^\infty(\mathbb{R}^3)$. The class $\mathscr{P}_0$ is non-trivial; in fact one can show the following lemma.

**Lemma 2.2.** *If* $A_0=0$ *and* $A^i \in C_B^1(\mathbb{R}^3)$ $(i=1,2,3)$, *then* $(A^\mu)\in\mathscr{P}_0$. *If* $|eA_0|\leqq m$, $A_0\in C_B^1(\mathbb{R}^3)$ *and* $A_0(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1-\varepsilon})$ *for* $|\boldsymbol{x}|\to\infty(\varepsilon>0)$ *and* $A^i=0$ $(i=1,2,3)$, *then* $(A^\mu)\in\mathscr{P}_0$.

To prove the second statement, one needs an estimate from [2]: if $A_0^2\in L^2(\mathbb{R}^3)$ and $A_0^2(\boldsymbol{x})=O(|\boldsymbol{x}|^{-2-\varepsilon})$ for $|\boldsymbol{x}|\to\infty(\varepsilon>0)$, then there exists $c>0$, such that for all $f\in C_0^\infty(\mathbb{R}^3)$ $\int |eA_0(\boldsymbol{x})f(\boldsymbol{x})|^2 d\boldsymbol{x}\leqq c\int|\nabla f(\boldsymbol{x})|^2 d\boldsymbol{x}$.

As a further restriction on the class of potentials, we only admit potentials in $\mathscr{P}$, the subset of $\mathscr{P}_0$ such that 0 is a point of the resolvent set $\varrho(B)$. The class $\mathscr{P}$ is non-trivial as the next lemma shows.

**Lemma 2.3.** *For every* $(A^\mu)$ *in Lemma 2.2 there exists* $r>0$, *such that* $(\alpha A^\mu)\in\mathscr{P}$ *for all* $\alpha\in\mathbb{R}$ *with* $|\alpha|<r$.

*Proof.* Range$(B)=$ Range$(BB_0^{-1})=$ Range$(\mathbb{1}+B_{\text{int}}B_0^{-1})=\mathscr{H}_E$ if $\|B_{\text{int}}B_0^{-1}\|<1$. There exists $c>0$ such that for all $\alpha$ in $\mathbb{R}$ with $|\alpha|\leqq 1$, $\|B_{\text{int}}((\alpha A^\mu))B_0^{-1}\|\leqq|\alpha|mc$. By choosing $\alpha$ sufficiently small Range$(B((\alpha A^\mu)))=\mathscr{H}_E$.

Let $V(\cdot)$ be the one-parameter strongly continuous unitary group, generated by the self-adjoint operator $B$ in $\mathscr{H}_E$; $P_+$ and $P_-$ are the spectral projections of $B$, connected with the sets $(0,\infty)$ and $(-\infty,0)$. Because of $(A^\mu)\in\mathscr{P}$ the bilinear functional $(\cdot,\cdot)_Q$ on $\mathscr{H}_E$, defined by $(f,g)_Q:=(f,B^{-1}g)_E$ for all $f,g\in\mathscr{H}_E$, is bounded;

$$(f,g)_Q=(f_1,g_2)+(f_2,g_1) \quad \text{and} \quad (V(t)f,V(t)g)_Q=(f,g)_Q$$

for all $f,g\in\mathscr{H}_E$ and for all $t\in\mathbb{R}$.

For potentials in $\mathscr{P}$ one can introduce a new inner product in $\mathscr{H}_E$; for all $f,g\in\mathscr{H}_E$, $(f,g)_I:=(f,|B|^{-1}g)_E$. $\mathscr{H}_I$ denotes the completion of $\mathscr{H}_E$ with respect to the norm $\|\cdot\|_I$. The operators $V(t)$, $P_+$, $P_-$ and the bilinear functional $(\cdot,\cdot)_Q$ are $\mathscr{H}_I$-bounded on $\mathscr{H}_E$; so one can extend them to bounded linear operators (resp. bilinear functional) on $\mathscr{H}_I$. We denote these extensions by the same symbols. The operator $B$ is essentially self-adjoint in $\mathscr{H}_I$; its closure $\bar{B}$ in $\mathscr{H}_I$ is the generator of the one-parameter strongly continuous unitary group $V(\cdot)$. The starting point for quantizing a classical equation $i\partial_t f(t)=\bar{B}f(t)$ in $\mathscr{H}_I$ is a triple $(L,\sigma,V(\cdot))$, where $(L,\sigma)$ is a symplectic space and $V(\cdot)$ a one-parameter group of symplectic transformations. We choose as the triple $L=\mathscr{H}_I^r$ (the underlying real linear space of the Hilbert space $\mathscr{H}_I$), $\sigma(f,g)=\operatorname{Im}(f,g)_Q$ for all $f,g\in\mathscr{H}_I$ and $V(\cdot)$ the one-parameter strongly continuous group of unitary operators in $\mathscr{H}_I$.

From [3] we adopt the notion of quantization. A quantization over $(L,\sigma,V(\cdot))$ is a quadruple $(\mathscr{K},W,\Omega,U(\cdot))$ such that 1) $(\mathscr{K},W,\Omega)$ is a cyclic Weyl system over $(L,\sigma)$; 2) $t\mapsto U(t)$ is a one-parameter strongly continuous group of unitary operators on $\mathscr{K}$; $U(t)=e^{-iHt}$ for all $t\in\mathbb{R}$ with $H\geqq 0$; 3) the symplectic transformations are unitarily implemented by the $U(t)$'s and $U(t)\Omega=\Omega$ for all $t\in\mathbb{R}$; $W(V(t)f)=U(t)W(f)U(t)^{-1}$ for all $t\in\mathbb{R}$ and $f\in L$. Two quantizations $(\mathscr{K},W,\Omega,U(\cdot))$ and $(\mathscr{K}_0,W_0,\Omega_0,U_0(\cdot))$ over $(L,\sigma,V(\cdot))$ are unitarily equivalent, if there exists a unitary intertwiner $U$ for the cyclic Weyl systems $(\mathscr{K},W,\Omega)$ and $(\mathscr{K}_0,W_0,\Omega_0)$.

**Lemma 2.4.** *If* $(A^\mu)\in\mathscr{P}$, *then there exists a quantization* $(\mathscr{K}_0,W_0,\Omega_0,U_0(\cdot))$ *over* $(L,\sigma,V(\cdot))$ *with* $\Omega_0\in\mathscr{K}_0$ *being the only* $(\neq 0)$ *vector in* $\mathscr{K}_0$, *which is invariant under* $U_0(t)$ *for all* $t\in\mathbb{R}$.

*Proof.* We extend the symplectic space $(L, \sigma)$ to a Hilbert space $\mathscr{H}_P$ such that the underlying real linear space is $L$, the imaginary part of the inner product is the symplectic form $\sigma$ and $V(\cdot)$ is a one-parameter strongly continuous group of unitary operators in $\mathscr{H}_P$ with positive generator. Define $\mathscr{H}_I^+ = P_+\mathscr{H}_I$ and $\mathscr{H}_I^- = P_-\mathscr{H}_I$, then $\mathscr{H}_P = \mathscr{H}_I^+ \oplus \overline{\mathscr{H}_I^-}$, where $\overline{\mathscr{H}_I^-}$ is the Hilbert space conjugate to $\mathscr{H}_I^-$. The multiplication with complex numbers is defined by $if := jf$, where $j$ is an operator in $\mathscr{H}_I$; $j := i(P_+ - P_-)$. For all $f, g \in \mathscr{H}_I$ $(f, g)_P = \mathrm{Re}(f, g)_I + i\sigma(f, g)$. Because $jV(t) = V(t)j$, the operators $V(t)$ are unitary in $\mathscr{H}_P$. From $i\partial_t V(t)f = BV(t)f$ for all $f \in D(B)$ in $\mathscr{H}_E$ follows $j\partial_t V(t)f = |B|\, V(t)f$ for all $f \in D(B)$; the operator $|B|$ is essentially self-adjoint in $\mathscr{H}_P$; so the $\mathscr{H}_P$-unitary group $V(\cdot)$ has $\overline{|B|}$ as its generator. $\overline{|B|}$ is positive, because $\sigma(\overline{|B|}) \subset \sigma(|B|) \subset (0, \infty)$ taking into account $(A^\mu) \in \mathscr{P}$. ($\sigma(\overline{|B|})$ is the spectrum of the operator $\overline{|B|}$ in $\mathscr{H}_P$ (or $\mathscr{H}_I$); $\sigma(|B|)$ is the spectrum of the operator $|B|$ in $\mathscr{H}_E$.) So $0 \in \mathscr{H}_P$ is the only vector in $\mathscr{H}_P$, invariant under $V(t)$ for all $t \in \mathbb{R}$. The lemma is proven, if one considers the Fock quantization over $(\mathscr{H}_P, V(\cdot))$.

**Lemma 2.5.** *If $(A^\mu) \in \mathscr{P}$ and $(\mathscr{K}, W, \Omega, U(\cdot))$ is a quantization over $(L, \sigma, V(\cdot))$ with a unique vacuum and if the Weyl systems $(\mathscr{K}, W)$ and $(\mathscr{K}_0, W_0)$ (from Lemma 2.4) are unitarily equivalent, then the quantizations are unitarily equivalent.*

*Proof.* The generating functional $E$ of the quantization $(\mathscr{K}, W, \Omega)$ has the form $E(f) = \exp(-\frac{1}{4}\|f\|_P^2 + iF(f))$ for all $f \in L = \mathscr{H}_P^r = \mathscr{H}_I^r$ [3]; $F$ is a $\mathbb{R}$-linear function on $L$ and invariant under $V(t)$ for all $t \in \mathbb{R}$. If $(\mathscr{K}, W)$ and $(\mathscr{K}_0, W_0)$ are unitarily equivalent, then $F$ is a continuous functional on $L$. From $0 \notin \sigma(\overline{|B|})$ one deduces $F = 0$ on $L$. Application of the G.N.S. theorem proves the lemma.

## 3. The Interaction Hamiltonian in the Fock Space of the Free Field

For the special case $A^\mu = 0$ $(\mu = 0, 1, 2, 3)$ one can construct $\mathscr{H}_I = W^{\frac{1}{2}}(\mathbb{R}^3) \oplus W^{-\frac{1}{2}}(\mathbb{R}^3)$, the free one-particle space $\mathscr{H}_P$ and the Fock system $(\mathscr{K}, W, \Omega)$ over $\mathscr{H}_P$.

Let $\mathscr{Q}$ be the set of real potentials $(A^\mu)$ on $\mathbb{R}^3$ with Fourier transforms $\tilde{A}^\mu$, such that the functions $\boldsymbol{p} \mapsto (1 + |\boldsymbol{p}|)\tilde{A}^\mu(\boldsymbol{p})$ are integrable over $\mathbb{R}^3$ $(\mu = 0, 1, 2, 3)$.

**Lemma 3.1.** *If $(A^\mu) \in \mathscr{Q}$, then*

1) *$B_{\mathrm{int}}$ with $D(B_{\mathrm{int}}) = \mathscr{H}_E$ can be extended to a bounded operator $\bar{B}_{\mathrm{int}}$ on $\mathscr{H}_I$.*

2) *The equation $i\partial_t f(t) = (\bar{B}_0 + \bar{B}_{\mathrm{int}})f(t)$ has for all $f \in D(\bar{B}_0)$ a unique differentiable solution $f(\cdot)$ in $\mathscr{H}_I$ with $f(0) = f$. Define for all $f = f(0) \in D(\bar{B}_0)$, $V(t)f = f(t)$; $V(\cdot)$ is a one-parameter strongly continuous group of bounded operators in $\mathscr{H}_I$.*

3) *$(V(t)f, V(t)g)_Q = (f, g)_Q$ for all $t \in \mathbb{R}$ and for all $f, g \in \mathscr{H}_I$.*

*Proof.* 1) is proven by the same method as in the appendix of [1]. Define $\tilde{V}(t) = \sum_{n=0}^{\infty} R_n(t)$ with $R_0(t) = \mathbb{1}$ and $R_n(t) = \int_0^t B(s)R_{n-1}(s)ds$ $(n \geqq 1)$ and

$$B(t) = -ie^{i\bar{B}_0 t}\bar{B}_{\mathrm{int}}e^{-i\bar{B}_0 t}.$$

The series is uniformly convergent in norm on any compact interval in $\mathbb{R}$; the function $t \mapsto \tilde{V}(t)$ is norm continuous. Because $\tilde{V}(t)D(\bar{B}_0) \subset D(\bar{B}_0)$, $V(t) = \exp(-i\bar{B}_0 t)\,\tilde{V}(t)$ represents the solution of equation 2). The domain $D(B_0)$ is also left invariant by the operators $V(t)$; on $D(B_0)$ one shows 3) by direct computation; it then holds on $\mathscr{H}_I$ by continuity. We remark that the operators $V(t)$ are not linear in $\mathscr{H}_P$, because they do not commute with multiplication by $i$.

**Lemma 3.2.** *If* $(A^\mu)\in\mathscr{Q}$ *and* $V(\cdot)$ *is the solution of the differential equation* $i\partial_t f(t)=(\bar{B}_0+\bar{B}_{\text{int}})f(t)$ *in* $\mathscr{H}_I$, *then*

1) *for all* $t\in\mathbb{R}$ $P_+V(t)P_+$ *is a bounded, one-to-one operator from* $\mathscr{H}_I^+$ *onto* $\mathscr{H}_I^+$.

2) *The mapping* $t\mapsto(P_+V(t)P_+)^{-1}\phi$ *is continuous for all* $\phi\in\mathscr{H}_I^+$; *the mapping* $t\mapsto(P_+V(t)P_+)^*$ *is strongly continuous.*

3) *For all* $f\in\mathscr{H}_I^+$ $(P_+V(t)P_-)^*(P_+V(t)P_+)f-(P_-V(t)P_-)^*(P_-V(t)P_+)f=0$.

*Proof.* The operator $P_+V(t)P_+$ is invertible and $\text{Range}(P_+V(t)P_+)$ is dense in $\mathscr{H}_I^+$ ([4]). For the inverse of the operator $P_+V(t)P_+$ one finds $(P_+V(t)P_+)^{-1}=(P_+V(t)^{-1}P_+V(t)P_+)^{-1}P_+V(t)^{-1}P_+$ with $\|(P_+V(t)^{-1}P_+V(t)P_+)^{-1}\|\leqq 1$. Because $(P_+V(t)P_+)^{-1}$ is a bounded operator on $\text{Range}(P_+V(t)P_+)$, the range is the whole of $\mathscr{H}_I^+$. The continuity of the mapping $t\mapsto(P_+V(t)P_+)^{-1}\phi$ is now trivial. The mapping $t\mapsto\tilde{V}(t)^*$ is norm continuous, so $t\mapsto P_+e^{i\bar{B}_0 t}\tilde{V}(t)^*P_+$ is strongly continuous. The result 3) can be found in [4] or [5].

In the space $\mathscr{H}_P$ the one-parameter group $V(\cdot)$ of $\mathscr{H}_P^r$-bounded operators is symplectic (Lemma 3.1); so the operators $W_t(f)$, defined by $W_t(f)=W(V(t)^{-1}f)$ constitute for every time $t$ a Weyl system in the Fock space $\mathscr{K}$ over $\mathscr{H}_P$.

**Lemma 3.3.** *Let* $(A^\mu)$ *be an element of* $\mathscr{Q}$.

*The one-parameter sympletic group* $V(\cdot)$ *in* $\mathscr{H}_P$ *is unitarily implementable in the Fock space* $\mathscr{K}$ *of the free field, if and only if the mapping* $t\mapsto P_-V(t)P_+$ *is Hilbert-Schmidt continuous in* $\mathscr{H}_I$.

*Proof.* Let $V(t)=U(V(t))\,(V(t)^TV(t))^{1/2}$ be the polar decomposition of $V(t)$ in $\mathscr{H}_P^r$; $U(V(t))$ is an orthogonal operator and $V(t)^T$ is the adjoint of $V(t)$ in $\mathscr{H}_P^r$. $V(t)^T=V(t)^*=-jV(t)^{-1}j$; $V(t)^*$ is the adjoint of $V(t)$ in $\mathscr{H}_I$.

According to a result of Shale [6] it is necessary and sufficient for the one-parameter symplectic group in $\mathscr{H}_P$ to be unitarily implementable in the Fock space $\mathscr{K}$, that $t\mapsto U(V(t))$ is strongly continuous in $\mathscr{H}_P^r$ and $t\mapsto(V(t)^TV(t))^{1/2}-\mathbb{1}$ is Hilbert-Schmidt (H.S.) continuous in $\mathscr{H}_P^r$. In our case the mapping $t\mapsto V(t)$ is strongly continuous, so the strong continuity of $t\mapsto U(V(t))$ is automatically satisfied. A further simplification is obtained by observing that the H.S. continuity of the mapping $t\mapsto(V(t)^TV(t))^{1/2}-\mathbb{1}$ in $\mathscr{H}_P^r=\mathscr{H}_I^r$ is equivalent to the H.S. continuity of the mapping $t\mapsto V(t)^TV(t)-\mathbb{1}$ in $\mathscr{H}_I$. The strong continuity of the function $t\mapsto V(t)$ entails the equivalence of the last statement with H.S. continuity of the mapping $t\mapsto P_-V(t)P_+-P_+V(t)P_-$ on $\mathscr{H}_I$.

With Lemma 3.2 3) one proves that the function $t\mapsto P_+V(t)P_-$ is H.S. continuous in $\mathscr{H}_I$, if the function $t\mapsto P_-V(t)P_+$ is H.S. continuous in $\mathscr{H}_I$.

The next lemma shows that it is sufficient to consider the Born term of the time-evolution $V(t)$.

**Lemma 3.4.** *If* $(A^\mu)\in\mathscr{Q}$, *then the mapping* $t\mapsto P_-V(t)P_+$ *is H.S. continuous in* $\mathscr{H}_I$, *if and only if the mapping* $t\mapsto P_-R_1(t)P_+$ *is H.S. continuous in* $\mathscr{H}_I$.

*Proof.* Adapting the proof of Lemma 3 in [7] one can show that the H.S. continuity of the Born term $P_- R_1(t)P_+$ is sufficient. To prove that this condition is also necessary, one observes that the function $t \mapsto P_- R_1(t)P_+$ is the uniquely determined solution of the integral equation

$$X(t) = C(t)A(t)^{-1} + \int_0^t X(s)D(s)A(t)^{-1}ds$$

with $X(t) \in \mathscr{B}(\mathscr{H}_I^+, \mathscr{H}_I^-)$ for all $t \in \mathbb{R}$, $t \mapsto X(t)$ strongly continuous and $X(0) = 0$. The coefficients of the equation are defined by $A(t) = P_+ \tilde{V}(t) P_+$, $C(t) = P_- \tilde{V}(t) P_+ - \int_0^t P_- B(s) P_- \tilde{V}(s) P_+ ds$ and $D(t) = P_+ B(t) \tilde{V}(t) P_+$.

The function $t \mapsto (A(t)^{-1})^*$ is strongly continuous and from our assumption the function $t \mapsto C(t)$ is H.S. continuous; so the function $t \mapsto \varrho_0(t) = C(t)A(t)^{-1}$ is H.S. continuous. Also the function $t \mapsto D(t)^*$ is strongly continuous; so the functions $t \mapsto \varrho_n(t) = \int_0^t \varrho_{n-1}(s)D(s)A(t)^{-1}ds$ are H.S. continuous. On the interval $[-a, a]$ $(a > 0)$ the series $\sum_{n=0}^{\infty} \varrho_n(t)$ is uniformly convergent in $\mathscr{I}_2(\mathscr{H}_I^+, \mathscr{H}_I^-)$ (the Banach space of H.S. operators from $\mathscr{H}_I^+$ to $\mathscr{H}_I^-$ with H.S. norm). This proves that the function $t \mapsto \sum_{n=0}^{\infty} \varrho_n(t)$ is H.S. continuous on $[-a, a]$. The series satisfies the integral equation; by uniqueness of the solution $P_- R_1(t)P_+ = \sum_{n=0}^{\infty} \varrho_n(t)$ for all $t \in \mathbb{R}$.

**Theorem 3.5.** *Let $(A^\mu)$ be an element of $\mathscr{Q}$.*

*The one-parameter symplectic group $V(\cdot)$ in $(\mathscr{H}_P^r, \sigma)$ is unitarily implementable in the Fock space $\mathscr{K}$ of the free field, if and only if $\boldsymbol{A} = 0$ on $\mathbb{R}^3$ and for all $t \in \mathbb{R}$ $Q_0(t)$ is finite and the function $t \mapsto Q_0(t)$ is continuous at $t = 0$.*

*Proof.* By applying Fubini's theorem twice one obtains for all $t \in \mathbb{R}$

$$\|P_- R_1(t) P_+\|_2 = \int |K_t(\boldsymbol{p}, \boldsymbol{p}')|^2 \frac{d\boldsymbol{p}\, d\boldsymbol{p}'}{2\omega(\boldsymbol{p}) 2\omega(\boldsymbol{p}')}$$

with

$$K_t(\boldsymbol{p}, \boldsymbol{p}') = i(-1 + \exp(-i(\omega(\boldsymbol{p}) + \omega(\boldsymbol{p}'))t)) 2\omega(\boldsymbol{p}') B_{\mathrm{int}}^{-+}(\boldsymbol{p}, \boldsymbol{p}') / (\omega(\boldsymbol{p}) + \omega(\boldsymbol{p}')),$$
$$B_{\mathrm{int}}^{-+}(\boldsymbol{p}, \boldsymbol{p}') = (-\omega(\boldsymbol{p}')^{-1}\omega(\boldsymbol{p}) + 1)\tilde{A}_0(\boldsymbol{p} - \boldsymbol{p}') + \omega(\boldsymbol{p})^{-1} \tilde{\boldsymbol{A}} * \tilde{\boldsymbol{A}}(\boldsymbol{p} - \boldsymbol{p}')$$
$$- \omega(\boldsymbol{p}')^{-1}(\boldsymbol{p} + \boldsymbol{p}') \cdot \tilde{\boldsymbol{A}}(\boldsymbol{p} - \boldsymbol{p}').$$

If there exists a point $\boldsymbol{p} \in \mathbb{R}^3$ with $\tilde{\boldsymbol{A}}(\boldsymbol{p}) \neq 0$, then one obtains a contradiction with $K_t \in L^2\left(\mathbb{R}^3 \times \mathbb{R}^3, \frac{d\boldsymbol{p}}{\omega(\mathbf{p})} \frac{d\boldsymbol{p}'}{\omega(\mathbf{p}')}\right)$ Setting $\boldsymbol{A} = 0$ on $\mathbb{R}^3$ the H.S. continuity of the function $t \mapsto P_- R_1(t) P_+$ implies the continuity of the mapping $t \mapsto Q_0(t)$ at $t = 0$.

*Remark.* For spin one-half particles this theorem was generalized independently in [8] and [9].

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