Commun. math. Phys. 48, 137—154 (1976)

Communications in
Mathematical
Physics

On the Discrete Spectrum of the Schrödinger Operators of Multiparticle Systems

I. M. Sigal

Department of Physics and Astronomy, Tel-Aviv University, Ramat-Aviv, Tel-Aviv, Israel

Abstract. It is shown that the discrete spectrum of the n-particle Schrödinger operators in the center-of-mass frame is finite for short-range potentials.

Section 1

1. The general structure of the spectrum of Schrödinger operators of multiparticle systems, namely the location of the essential spectrum, was determined in [1, 2] (see also [3]). These results were subsequently improved by many authors (see [4–7] and References in these articles). Further and exhaustive information on the essential spectrum of the Schrödinger operator of a multiparticle system has been obtained in scattering theory [8]. Thus, the fundamental problem in this field is now to investigate the point spectrum and, in particular, the discrete spectrum[1].

2. It was shown in [1, 2] that the discrete spectrum of a Schrödinger operator is at most countable a set whose unique point of accumulation (if any) is the infimum of the continuous spectrum (for determination of this point, see below). The Schrödinger operators of atom-type systems have infinite discrete spectra [1]. Thus, the basic problem in a qualitative description of the discrete spectrum of the Schrödinger operator is to find classes of potentials for which the discrete spectrum is finite or infinite[2]. A solution to this problem is also important for scattering theory (see [9–11]), stability and spectra theory of quantum systems.

3. Familiar examples in physics indicate that one such class is apparently that of the so-called short-range potentials, i.e., potentials which decrease sufficiently rapidly at infinity.

There is already a considerable literature devoted to the proof that the discrete spectrum of the Schrödinger operator for short-range potentials is finite. The most complete results in this area may be found in [12–14]. These papers establish

---

[1] The set of isolated eigenvalues of finite multiplicity is called here the discrete spectrum. The point spectrum is the set of all eigenvalues of finite multiplicity.

[2] A useful discussion of these questions can be found in [16].

that the discrete spectrum is finite for systems with short-range interaction, on the assumption that the subsystems defined by the infimum of the limit spectrum are stable. The results of [12, 14] take the symmetry of the system into account.

4. This paper contains a result concerning the finiteness of the discrete spectrum of the Schrödinger operator of a multiparticle system with rapidly decreasing potentials; this result seems to be close to the best possible. The class of potentials for which the proof is given is described by inequality (1.2) and Condition A. We have made no attempt to carry out the proof for the broadest possible assumptions on the potentials. Condition (1.2) may be somewhat weakened as to the restrictions on the smoothness of the potentials, without serious modifications in our proof. The restriction on the rate of decrease in this condition cannot be relaxed: the potentials which fall at infinity as $|x|^{-2-\varepsilon}\forall\varepsilon>0$ satisfy the condition and if the potentials in atom-type system decrease at infinity as $|x|^{-2}$ with certain coefficients (obeying Condition A), the corresponding Schrödinger operator has an infinite discrete spectrum (see [16]).

It was proved in [15] that if Condition A fails to hold the Schrödinger operator of a three-particle system may have an infinite discrete spectrum, even if the potentials are infinitely differentiable and have compact support.

5. A few words about the method proposed in this paper to study the discrete spectrum of Schrödinger operators. As usual, $\sigma_d(A)$, $\sigma_c(A)$ and $D(A)$ will denote the discrete spectrum, continuous spectrum and domain of definition of an operator A.

Instead of a single Schrödinger operator H we shall consider a family of operators $H(g)$, $g\in\mathbb{R}^1$, such that $H(0)=H$. The family $H(g)$ will be associated with a family of operators $L(z,g)=L_H(z,g)$, $z\in\mathbb{C}\setminus\sigma_c(H)$, with the following properties:

(i) $L(z,g)$ are compact operators in some Banach space B;

(ii) $(H(g)-zE)\varphi=0$, $\varphi\in D(H)\leftrightarrow(E+L(z,g))\varphi=0$, $\varphi\in B$.

It turns out that the total multiplicity of the discrete spectrum of H is intimately related to the behavior of the operator-valued function $L(\mu,g)$ (where μ is the infimum of the continuous spectrum of H) in the neighborhood of $g=0$.

6. The Schrödinger operator of an n-particle system in the coordinate representation, in atomic units ($m_{e1}=1, e=1, h=1$), has the form

$$\begin{aligned} H_n f &= H_0^{(n)} f + V^{(n)} f \\ &= \sum_{i=1}^{n} (-1/2m_i)\Delta_i f(x) + \tfrac{1}{2}\sum_{i\neq j} V_{ij}(x_i-x_j)f(x), \end{aligned} \tag{1.1}$$

where $m_i>0$, Δ_i is the Laplacian with respect to x_i, and the functions $V_{ij}(x)$ (in this paper) satisfy the conditions

$$\begin{aligned} &\int |v_{ij}(k)|^{m_0}(1+|k|)^{\theta_0 m_0}dk<C, \\ &m_0=3+\varepsilon_0,\ \varepsilon_0>0,\ \theta_0>\tfrac{3}{2}(1-2/m_0), \end{aligned} \tag{1.2}$$

where $v_{ij}(k)=\int V_{ij}(x)e^{ikx}dx$. A sufficient condition for (1.2) to be valid for $V_{ij}(x)$ in the x-representation is that

$$\int |V_{ij}(x+h)-V_{ij}(x)|^{m_0'}dx<C|h|^{\theta_0 m_0'},\ 1/m_0'+1/m_0=1\,.$$

When condition (1.2) is satisfied, the operator H_n is defined on $S(\mathbb{R}^{3n})$ and the subordination inequality holds [17]:

$$\|V^{(n)}f\| \leqq \varrho^{\alpha}\|H_0^{(n)}f\| + \varrho^{-3m/2}\|f\|, \alpha>0, \tag{1.3}$$

where $\varrho>0$ is arbitrary. It follows from (1.3) (see [17]) that H_n has a unique selfadjoint extension.

7. We now define certain subspaces of the configuration space $\mathbb{R}^{3n}$ (see [7]): the subspace $R^{(0)}$ of relative movement of the system

$$R^{(0)} = \{x : x=(x_1, \ldots, x_n) \in \mathbb{R}^{3n}, m_1x_1 + \ldots + m_nx_n = 0\},$$

and the subspace $R^{(c)}$ of center-of-mass movement of the system:

$$R^{(c)} = \{x : x=(x_1, \ldots, x_n) \in \mathbb{R}^{3n}, x_1 = \ldots = x_n\}.$$

Defining a scalar product in $\mathbb{R}^{3n}$ by

$$(x, \tilde{x})_1 = \sum_{i=1}^{n} m_i(x_i, \tilde{x}_i), \tag{1.4}$$

where $(x_i, \tilde{x}_i)$ is the usual scalar product in $\mathbb{R}^3$, one readily shows ([7]) that the spaces $R^{(0)}$ and $R^{(c)}$ are mutually orthogonal in the sense of this scalar product, and moreover

$$R^{(0)} \oplus R^{(c)} = \mathbb{R}^{3n}. \tag{1.5}$$

It follows from (1.5) that

$$L_2(\mathbb{R}^{3n}) = L_2(R^{(0)}) \otimes L_2(R^{(c)})$$

(see, e.g., [20]). This decomposition of $L_2(\mathbb{R}^{3n})$ induces a representation of the operator H_n:

$$H_n = H \otimes E^{(c)} + E^{(0)} \otimes T^{(c)},$$

where $E^{(0)}$ and $E^{(c)}$ are the identity operators in $L_2(R^{(0)})$ and $L_2(R^{(c)})$, respectively, the operator H is defined in $L_2(R^{(0)})$ by

$$Hf = -\tfrac{1}{2}\Delta f + \tfrac{1}{2}\sum_{i \neq j} V_{ij}(x_i - x_j)f(x) = H_0 f + Vf,$$

where Δ is the Laplacian in $L_2(R^{(0)})$ [in the sense of the scalar product (1.4)], the operator $T^{(c)}$ is defined in $L_2(R^{(c)})$ by

$$T^{(c)} = -\tfrac{1}{2}\Delta^{(c)},$$

where $\Delta^{(c)}$ is the Laplacian in $L_2(R^{(c)})$ in the sense of the scalar product (1.4). The subordination inequality (1.3) for H_n implies (see [7]) a subordination inequality for the operator H:

$$\|Vf\| \leqq \varrho^{\alpha}\|H_0 f\| + \varrho^{-3n/2}\|f\|, \alpha>0, \tag{1.6}$$

where ϱ is an arbitrary positive number. By virtue of this inequality, H has a unique selfadjoint extension [17], for which we retain the same notation H.

8. We now proceed to describe the compound systems, that is to say, the systems derived from the original system by neglecting the interaction between certain of its subsystems.

A partition of the set $\{1, \ldots, n\}$ is defined as a collection of disjoint nonempty subsets c_i such that $\cup c_i = \{1, \ldots, n\}$. We shall sometimes use the term "system" for the whole set $\{1, \ldots, n\}$ and the terms "subsystems" (clusters) for the subsets c_i. Partitions will be denoted by lower case roman letters: $a, b, c, \ldots$. The number of subsets in a partition a is denoted by $k(a)$, and the set of all partitions by $\mathscr{A}$; in addition, we set $\mathscr{A}_s = \{a \in \mathscr{A} : k(a) = s\}$. We partially order the partitions as follows. If b is obtained from a by breaking up certain subsystems in a, we write $b \subset a$ ("b is contained in a"). The smallest partition containing a and b is denoted by $a \cup b : a \cup b = \sup(a, b)$, and the largest partition contained in a and b by $a \cap b : a \cap b = \inf(a, b)$.

9. Let $H_a^{(n)}$ denote the operator obtained from H_n by deleting the operators $V_{ij}^{(n)}$ for which i and j belong to different subsets in the partition a. Set

$$\mu = \min_{a, k(a) > 1} \inf H_a^{(n)} .$$

The fundamental theorem on the spectrum of Schrödinger operators is:

Theorem ([1, 3])[3]. (i) *The essential spectrum of the operator H coincides with the half-line $[\mu, +\infty)$.*

(ii) *The discrete spectrum of H (if it exists) lies to the left of the point μ and its only point of accumulation (if any) is μ.*

10. All the analytical manipulations in this paper will be carried out in the so-called momentum space, in which they are somewhat more easily described. This representation is obtained from the more familiar coordinate representation by the Fourier transformation F of the basic space $L_2(\mathbb{R}^{3n})$ and the corresponding transformation FH_nF^{-1} of the operator H_n. We shall therefore translate some of our definitions into the language of the momentum representation and add some new definitions. The old notation will be retained for the operators H_n, H, H_0, V, $H_a^{(n)}$ in the new representation; this will involve no confusion, since all our deliberations will be in the momentum language.

The Schrödinger operator H_n in the momentum representation is

$$H_n = H_0^{(n)} + V^{(n)}, \; V^{(n)} = \tfrac{1}{2} \sum_{i \neq j} V_{ij}^{(n)} ,$$

$$(H_0^{(n)} f)(p) = \sum_{i=1}^{n} (p_i^2 / 2m_i) f(p) ,$$

$$(V_{ij}^{(n)} f)(p) = \int v_{ij}(p_i - q_i) \delta(p_i + p_j - q_i - q_j) \prod_{k \neq i, j} \delta(p_k - q_k) f(q) d^n q .$$

11. We define a scalar product in the momentum space $\mathbb{R}^{3n}$ (see [11]) which is dual (in the sense of the Fourier transform) to the scalar product (1.4) in the configuration (coordinate) space:

$$(p, \tilde{p})' = \sum_{i=1}^{n} m_i^{-1} (p_i, \tilde{p}_i) . \tag{1.7}$$

[3] See also [2, 7].

To each partition a there corresponds a pair of spaces R^a and R_a (the space of relative movement of particles in subsystems of a and the space of CM movement of these subsystems), defined by

$$R^a=\left\{p:p=(p_1,\ldots,p_n)\in\mathbb{R}^{3n},\ \sum_{k\in c_i}p_k=0,\ c_i\in a\right\},$$

$$R=R^a,\ k(a)=1,$$

and

$$R_a=\{p:p=(p_1,\ldots,p_n)\in R,\ m_j^{-1}p_i=m_j^{-1}p_j$$

if there exists $c_k\in a$ such that $i,j\in c_k\}\cap R$.

With these definitions, we have [in the sense of the scalar product (1.7)]

$$R^a\perp R_a,\ R^a\oplus R_a=R. \tag{1.8}$$

If $k(a)=1$, the spaces R^a and R_a are the Fourier-duals of $R^{(0)}$ and $R^{(c)}$, respectively. It follows from (1.8) that

$$L_2(R)=L_2(R^a)\otimes L_2(R_a). \tag{1.9}$$

In accordance with (1.5), the operator $H_a^{(n)}$ admits the decomposition

$$H_a^{(n)}=H^a\otimes E_a^{(n)}+E^a\otimes T_a^{(n)},$$

where E^a and $E_a^{(n)}$ are the identity operators in $L_2(R^a)$ and $L_2(R_a)$, respectively; H^a is the internal movement operator of the subsystems $c_i\in a$, each relative to its center of mass, defined in the space $L_2(R^a)$; $T_a^{(n)}$ is the relative movement operator of the centers of mass of subsystems $c_i\in a$, defined in $L_2(R_a)$ by

$$(T_a^{(n)}f)(p_a)=\sum_{i=1}^{k(a)}\left(2\sum_{j\in c_i}m_j\right)^{-1}\left(\sum_{j\in c_i}p_j\right)^2 f(p_a).$$

The operators H^a satisfy subordination inequalities similar to (1.6) and so possess unique selfadjoint extensions. For $a\in\mathscr{A}_1$, we set $H=H^a$, $E=E^a$. The operator H has the same structure as $H^{(n)}$:

$$H=H_0+\tfrac{1}{2}\sum_{i\neq j}V_{ij}.$$

Let $\tilde{\mathscr{A}}=\{a\in\mathscr{A},\ \sigma_d(H^a)\ni\mu\}\cup\mathscr{A}_n$. Let $\psi^{a,k}$ be the eigenfunctions of the operator H^a[4], $a\in\tilde{\mathscr{A}}$, $k(a)<n$, belonging to the eigenvalue μ.

We shall use the notation

$$\tau(p)=\tfrac{1}{2}(p,p)',\ p\in R.$$

12. We can now proceed to the main topic of this paper – the discrete spectrum of the operator H. We associate with H a one-parameter family of Berezin's

[4] Henceforth the index k of the eigenfunctions will be omitted.

operators $L(z)=L_H(z)$ ([18], see also [19]), defined by induction on partitions a in the space $D(H_0)$:

$$L_u(z)=R_0(z)V_\alpha,\ u=u_\alpha=\left\{(\alpha)\prod_{k\notin\alpha}(k)\right\}\in\mathscr{A}_{n-1},$$

$$E+L_a(z)=\prod_{s=k(a)+1}^{n-1}\prod_{b\in\mathscr{A}_s,b\subset a}(E+L_b(z))^{-1}\left(E+\sum_{b\in\mathscr{A}_{n-1},b\subset a}L_b(z)\right), \quad 1\leq k(a)<n-1, \tag{1.10}$$

$$L(z)=L_a(z)(a\in\mathscr{A}_1).$$

The symbol $\prod_{b\in\mathscr{A}_s}$ denotes a product of the appropriate operator factors in some arbitrary but fixed order. Note that if z lies outside the continuous spectrum of H^a, then $L_a(z)$ is a bounded operator in $D(H_0)$ and $E+L_a(z)$, $z\notin\sigma(H^a)$, has a bounded inverse. We set

$$F(z)=\overset{\curvearrowright}{\prod_{s=1}^{n-1}}\prod_{a\in\mathscr{A}_s}(E+L_a(z))^{-1}R_0(z). \tag{1.11}$$

It is evident from (1.10) and (1.11) that the operators $L(z)$ and $H-zE$ are related by

$$E+L(z)=F(z)(H-zE). \tag{1.12}$$

13. If z lies in the continuous spectrum of H, the operator $L(z)$ is no longer bounded on $D(H_0)$. We shall farther be interested in the limit of the family of operators $L(\lambda)$ as $\lambda\to\mu-0$. This limit will be studied in specially chosen scales of Banach spaces, to whose construction we now proceed.

Let

$$(N_a(\theta)f)(p_a;\theta)=N_a(p_a;\theta)f(p_a),$$

where $N_a(p_a;\theta)$, $a\in\mathscr{A}\backslash\mathscr{A}_1$, are the estimating functions introduced in [10]. We recall one property of these functions:

$$\int[N_a(p_a;\theta)]^k dp_a<C,\ \theta k>3.$$

The Banach spaces in question are defined as follows:

$$B_{m,\theta}(R_a)=N_a(\theta)L_m(R_a),\ \|f\|_{B_{m,\theta}(R_a)}=\|N_a^{-1}(\theta)f\|_{L_m(R_a)},$$

and

$$\hat{B}_{m,\theta}=\sum_{a\in\tilde{\mathscr{A}}}\oplus B_{m,\theta}(R_a),\ \|\hat{f}\|_{\hat{B}_{m,\theta}}=\sum_{a\in\tilde{\mathscr{A}}}\|f_a\|_{B_{m,\theta}(R_a)},\ \hat{f}=\{f_a(p_a),a\in\tilde{\mathscr{A}}\}.$$

We define a linear mapping of $\hat{B}_{m,\theta}$ by

$$\pi\hat{f}=\sum_{a\in\tilde{\mathscr{A}}}\frac{\psi^a(p^a)f_a(p_a)}{\tau(p_a)-\mu\delta_{k(a),n}},\ \hat{f}=\{f_a(p_a),a\in\tilde{\mathscr{A}}\}.$$

Denote the image of $\hat{B}_{m,\theta}$ under the homomorphism π by $B_{m,\theta}$:

$$B_{m,\theta}=\pi\hat{B}_{m,\theta}.$$

$B_{m,\theta}$ becomes a Banach space if we define the norm by

$$\|f\|_{B_{m,\theta}} = \inf_{\pi\hat{f}=f} \|\hat{f}\|_{\hat{B}_{m,\theta}}.$$

14. In the sequel we shall need a certain relationship which holds in the above spaces. Let $f \in B_{m,\theta}$, $\psi \in B_{m,\theta}(R)$. Then

$$|(f,\varphi)| = |\int f(p)\bar{\varphi}(p)dp| \leqq C\|f\|_{B_{m,\theta}}\|\varphi\|_{B_{m,\theta(R)}}, \quad m>6,\ \theta>\tfrac{3}{2}(1-2/m). \tag{1.13}$$

Throughout this section we shall assume that the following condition is fulfilled by the potentials[5]. Let

$$\mathring{\mathscr{A}} = \{a \in \mathscr{A},\ k(a)>1;\ \text{there exists } b \subsetneqq a \text{ such that } \mu \in \sigma_d(H^b) \text{ and } b \in \mathscr{A}_{k(a)+1}\}.$$

Let $L^a(z)$ denote the restriction of the operator $L_a(z)$ to $D(H^a)$.

Condition A. The potentials $V_{ij}(x)$ have the property that the equation $\varphi + L^a(\mu)\varphi = 0$ has no nontrivial solutions of the form

$$\sum_{\substack{b \subset a \\ b \in \tilde{\mathscr{A}}}} \frac{\psi^b(p^b) f_b(p_b^a)}{\tau(p_b^a) - \mu\delta_{k(b),n}}, \quad f_b \in N_b^a(\theta) L_m(R_b \cap R^a),\ 6<m\leqq 2m_0,$$

$\theta > \frac{3}{2}(1-2/m)$, for any a in $\mathring{\mathscr{A}}$. (See the note added in proof.)

15. **Lemma 1.1.** *The operator $L(\lambda)$, $\lambda \leqq \mu$, is uniformly bounded, continuous in the operator topology with respect to λ, and compact in $B_{m,\theta}$, $m>6$, $\theta>\frac{3}{2}(1-2/m)$.*

Lemma 1.2. *The operator $F(\lambda)$, $\lambda \leqq \mu$, are bounded from $B_{m,\theta}(R)$ to $B_{m,\theta}$; for $\lambda=\mu$ the operator has an inverse which maps $B_{m,\theta}$ continuously into $B_{m,\theta}(R)$, $m>6$, $\theta>\frac{3}{2}(1-2/m)$.*

Lemma 1.3. *The operator $H-\mu E$ is continuous from $B_{m,\theta}$ to $B_{m,\theta}(R)$, $m>6$, $\theta>\frac{3}{2}(1-2/m)$.*

The proofs of these lemmas will be given in the next section.

16. Let $\mathscr{W}$ denote the class of operators in $L_2(R)$ satisfying the conditions: (i) the operators of $\mathscr{W}$ are nonpositive, (ii) the operators of $\mathscr{W}$ are continuous from $L_1(R)+L_2(R)$ to $S(R)$.

Consider the family of operators

$$H(g) = H + gW,\ W \in \mathscr{W},\ 0 \leqq g \leqq 1.$$

With the family $H(g)$ we associate a two-parameter family of operators $L(z,g)$, defined by[6]

$$L(z,g) = L(z) + gF(z)W. \tag{1.14}$$

It follows from (1.13), (1.14) that

$$E + L(z,g) = F(z)(H(g) - zE). \tag{1.15}$$

17. Using Equation (1.15) and Lemma 1.1, one obtains the following

---

[5] For the definition of the estimating functions $N_b^a(p_b^a;\theta)$, see [19]. We note here only that $\int (N_b^a(p_b^a;\theta))^k dp_b^a < C$, $k\theta>3$.

[6] The operators $L(z,g)$ are obtained from $H(g)$ according to the same principle as $L(z)$ from H.

Lemma 1.4. *If $\lambda(g)$ is a point of the discrete spectrum of $H(g)$ and $\psi(g)$ is a corresponding eigenfunction, then $\psi(g)\in B_{m,\theta}$ and $\psi(g)$ is an eigenfunction of the operator $L(\lambda(g), g)$ belonging to the eigenvalue -1.*

18. Assume that the operator $L(\mu)$ has an eigenvalue -1. We wish to evaluate the first derivatives with respect to g, at the point $g=0$, of the eigenvalues of $L(\mu, g)$ which equal -1 at $g=0$.

Let $\mathscr{L}$ be the eigensubspace of the operator $L(\mu, 0)=L(\mu)$ that corresponds to the point -1 of the spectrum, and ψ an arbitrary function in $\mathscr{L}$ for which there exists an eigenfunction $\psi(g)$ of $L(\mu, g)$ corresponding to a real eigenvalue such that $\psi(0)=\psi$. Let $\lambda_\psi(g)$ be the eigenvalue corresponding to $\psi(g)$, $\lambda_\psi(0)=-1$. The defining equation for $\psi(g)$ is

$$L(\mu, g)\psi(g)=\lambda_\psi(g)\psi(g)\,. \tag{1.16}$$

By Lemmas 1.1 and 1.2, the vector-valued functions $\psi(g)$ in $B_{m,\theta}$ and the function $\lambda_\psi(g)$ are analytic for $g\in[0, \varepsilon]$, where ε is some positive number. Let

$$\psi^{(1)}=\left.\frac{d\psi(g)}{dg}\right|_{g=0}\in B_{m,\theta},\ \lambda^{(1)}=\left.\frac{d\lambda_\psi(g)}{dg}\right|_{g=0}. \tag{1.17}$$

Differentiating the Equation (1.16) with respect to g and setting $g=0$, we obtain

$$F(\mu)W\psi+L(\mu)\psi^{(1)}-\lambda^{(1)}\psi+\psi=0\,. \tag{1.18}$$

In view of Equation (1.13) for $z=\mu$ and Lemmas 1.2, 1.3, we transform Equation (1.18) to the form

$$F(\mu)(W\psi-\lambda^{(1)}F^{-1}(\mu)\psi+(H-\mu E)\psi^{(1)})=0\,. \tag{1.19}$$

By Lemma 1.2 and the assumptions on W, we deduce from (1.19) the equation

$$W\psi+(H-\mu E)\psi^{(1)}-\lambda^{(1)}F^{-1}(\mu)\psi=0\,. \tag{1.20}$$

We now take the scalar product [in the sense of $L_2(R)$] of Equation (1.20) with ψ; this is legitimate in view of (1.13) and Lemmas 1.2, 1.3. The result is

$$\lambda^{(1)}=\frac{(W\psi, \psi)+((H-\mu E)\psi^{(1)}, \psi)}{(F^{-1}(\mu)\psi, \psi)}\,. \tag{1.21}$$

Consider the second term in the numerator of Equation (1.21). Using (1.13) and the fact that $S(R)$ is everywhere dense in $B_{m,\theta}$ one readily shows that the operator $H-\mu E$ may be transferred to the second factor in the scalar product:

$$((H-\mu E)\psi^{(1)}, \psi)=(\psi^{(1)}, (H-\mu E)\psi)\,. \tag{1.22}$$

The term on the right of this equality vanishes. Indeed, it follows from (1.13) and Lemmas 1.1–1.3 that

$$(\varphi, (H-\mu E)\psi)=(\varphi, F^{-1}(\mu)(E+L(\mu))\psi)=0 \tag{1.23}$$

for all $\varphi\in B_{m,\theta}$. (Recall that ψ is an eigenfunction of the operator $L(\mu)$ in B belonging to the eigenvalue -1.) Using (1.17), (1.22), and (1.23), we obtain from (1.21)

$$\left.\frac{d\lambda_\psi(g)}{dg}\right|_{g=0}=\frac{(W\psi, \psi)}{(F^{-1}(\mu)\psi, \psi)}\,. \tag{1.24}$$

19. Having prepared all the necessary material for the proof of our main theorem, we now proceed to state and prove it.

Theorem 2. *Let the potentials V_{ij} in the operator H satisfy inequality* (1.2) *and Condition A. Then the discrete spectrum of H is finite.*

Proof. Suppose, on the contrary, that H has an infinite discrete spectrum. By Theorem 1, its only point of accumulation is μ. Define $H(g)$, $L(z, g)$, $\mathscr{L}$ and $\lambda_\psi(g)$, $\psi\in\mathscr{L}$, as before.

Since W is a compact operator in $L_2(R)$, it follows that the operator $H(g)$ has the same limit spectrum as H, for any finite g. Since W is a nonpositive operator, it thus follows that for any positive g the operator $H(g)$ also has an infinite discrete spectrum, whose point of accumulation is precisely the infimum μ of the continuous spectrum of H. Let $\lambda_n(g)$, $n=1, 2, \ldots$, denote the eigenvalues of $H(g)$, $0\leqq g\leqq 1$, belonging to the discrete spectrum. It was shown previously that

$$\lambda_n(g)\to\mu \quad \text{as} \quad n\to\infty, g\in[0, 1]\,. \tag{1.25}$$

It follows from Lemma 1.4 that

$$L(\lambda_n(g), g)\ni -1, g\in[0, 1]\,. \tag{1.26}$$

By (1.25), (1.26), Lemma 1.1 and the theorem stating that the set of singular points of a compact operator is closed (see Appendix), it follows that

$$\sigma(L(\mu, g))\ni -1, 0\leqq g\leqq 1\,, \tag{1.27}$$

for any operator W in $\mathscr{W}$.

We now prove a contradiction to (1.27): there exist operators W in $\mathscr{W}$ for each of which there is a positive ε such that $\sigma(L(\mu, g))\not\ni -1$ for $0<g<\varepsilon$. To this end, we assume that, in addition to (i), (ii), the operator W satisfies the condition: W is strictly negative on $\mathscr{L}$, i.e., there exists $C>0$ such that

$$(W\psi, \psi)< -C, \psi\in\mathscr{L}\,. \tag{1.28}$$

But by Lemma 1.2

$$|(F^{-1}(\mu)\psi, \psi)|<\infty, \psi\in B_{m,\theta}\,,$$

and so it follows from (1.24) and (1.28) that for any function ψ

$$\left.\frac{d\lambda_\psi(g)}{dg}\right|_{g=0}\neq 0\,.$$

This is the desired contradition, and the proof is complete.

Section 2

In this section we shall prove Lemmas 1.1 and 1.2 of Section 1. The integral equations for multiparticle systems outside the continuous spectrum of the Schrödinger operator have been studied by many authors ([2, 3, 11, 21, 22]). There are also results on the behavior of the Faddeev-Yakubovskii equation at the end-point of the continuous spectrum and the Berezin equation on the continuous spectrum. Our investigation differs from all these in the conditions imposed on the potentials and the estimates that the various integral operators in the equation are desired to satisfy.

It follows readily from Equation (1.10) that the operator $L(\lambda)$ admits a representation as a linear combination of monomials of the type $\Pi[R_{a_l}(\lambda)V_{\alpha_l}]$; an analogous representation holds for $F(\lambda)$. Thus, in order to establish estimates for $L(\lambda)$ we must first study the operators $R_a(\lambda)$. The operators $R_a(\lambda)$ will be studied with the help of equations involving the operators $L_a(\lambda)$, $F_a(\lambda)$, and so on. According to this approach, the natural way of investigating the operators $L(\lambda)$, $F(\lambda)$ is to proceed by induction on partitions.

Furthermore, it is clear that the induction hypothesis may be phrased either in terms of $L_a(\lambda)$ or in terms of $R_a(z)$. Both methods of proof are equivalent; we shall use the second.

In order to avoid cumbersome estimates with complicated estimating functions (see [11]) we shall use stronger assumptions on the potentials than adopted previously. The assumptions are stronger only as regards smoothness in the coordinate representation of the potential functions. The new restrictions on the potentials are as follows:

$$\int |v_{ij}(k)|^{m_0}(1+|k|)^{\theta_0}dk < C,\ m_0<3,\ \theta_0>\tfrac{3}{2}(n-1)(1-1/m_0),\ (2.1m_0)\,.$$

To apply the induction method, we shall need some additional definitions and propositions concerning compound systems. Let

$$R^a_b = R^a \cap R_b,\ b \subset a\,. \tag{2.2}$$

We cite a few relations for these spaces, analogous to those presented in Section 1. For R^a_b, $b\subset a$, we have decompositions

$$R^a_b = R^d_b \oplus R^a_d,\ b \subseteqq d \subseteqq a\,,$$

whence it follows that

$$L_2(R^a_b) = L_2(R^d_b)\otimes L_2(R^a_d),\ b\subseteqq d\subseteqq a\,. \tag{2.3}$$

Let V^a_α, $u_\alpha \subset a$, be the restriction of the operator V_α to $L_2(R^a)$. We recall that

$$u_{ij} = \{(1)\ldots(i-1)(i+1)\ldots(j-1)(j+1)\ldots(n)\}\,.$$

We now define

$$(T^a_b f)(p^a_b) = \tau(p^a_b) f(p^a_b),\ T^a = T^a_b\ (b\in\mathscr{A}_n)\,,$$

and

$$H^a_b = T^a + \tfrac{1}{2}\sum_{\substack{i\neq j\\ u_{ij}\subset b}} V^a_{ij},\ b\subset a\,.$$

These operators admit a decomposition following from (2.3):

$$H^a_b = H^d_b \otimes E^a_d + E^d \otimes T^a_d,\ b\subseteqq d\subseteqq a\,, \tag{2.4}$$

where E_d^a is the identity operator in $L_2(R_d^a)$. Let $R_{b,a}(z)$ be the resolvent of the operator H_b^a. It follows from (2.4) that

$$R_{b,a}(z) = R_{b,d}(z - \tau(p_d^a)) \otimes E_d^a;$$

this relation should be understood as follows:

$$R_{b,a}(z)f = R_{b,d}(z - \tau(p_d^a))f_{p_b^a}, \qquad f \in L_2(R^a), \tag{2.4'}$$

where $f_{p_b^a}$ denotes $f \in L_2(R^a)$, treated as a vector-valued function from R_b^a to $L_2(R^b)$. For $b \in \mathscr{A}_n$, we set

$$H_b^a = H_0^a,\ R_{b,a}(z) = R_{0,a}(z),\ b \in \mathscr{A}_n.$$

Define

$$\mu^a = \min_{b \subset a} \inf H_b^a.$$

Let $L_{b,a}(z)$ be the restriction of the operator $L_b(z)$ to $L_2(R^a)$, which is defined by induction along the lines of (1.10):

$$\left.\begin{aligned} &L_{u,a}(\lambda) = R_{0,a}(\lambda)V_\alpha^a,\ u = u_\alpha \in \mathscr{A}_{n-1}, \\ &E^a + L_{b,a}(\lambda) = \overset{n-1}{\overrightarrow{\prod_{s=k(b)+1}}} \prod_{\substack{c \in \mathscr{A}_s \\ c \subset b}} (E^a + L_{c,a}(\lambda))^{-1} \Bigg(E^a + \sum_{\substack{u \in \mathscr{A}_{n-1} \\ u \subset b}} L_{u,a}(\lambda)\Bigg). \end{aligned}\right\} \tag{2.5}$$

Here, as in (1.10), the symbol Π denotes the product of the operator factors indicated, in some arbitrary but fixed order. We set

$$L^a(\lambda) = L_{a,a}(\lambda).$$

It follows from (2.5) that

$$E^a + L^a(\lambda) = F^a(\lambda)(H^a - \lambda E^a), \tag{2.6}$$

where

$$F^a(\lambda) = \overset{n-1}{\overrightarrow{\prod_{s=k(a)+1}}} \prod_{\substack{b \in \mathscr{A}_s \\ b \subset a}} (E^a + L_{b,a}(\lambda))^{-1} R_{0,a}(\lambda). \tag{2.7}$$

We now introduce the Banach spaces in which the operators $L^a(\lambda)$ will be considered:

$$B_{m,\theta,t}(R_b^a) = (t + \tau(p_b^a))^{\theta/2} L_m(R_b^a),\ \theta > 3(k(b) - k(a))(m-2)/2m,$$

and

$$\hat{B}_{m,\theta,t}^a = \sum_{\substack{b \in \tilde{\mathscr{A}} \\ b \subset a}} \oplus B_{m,\theta,t}(R_b^a),$$

where the norms are defined in the usual way. We define a linear mapping on $B_{m,\theta,t}^a$ by

$$\pi^a \hat{f} = \sum_{\substack{b \in \tilde{\mathscr{A}} \\ b \subset a}} \frac{\psi^b(p^b) f_b(p_b^a)}{\tau(p_b^a) - \mu\delta_{k(b),n}},\ \hat{f} = \{f_b\} \in \hat{B}_{m,\theta,t}^a,$$

where $\psi^b \equiv 1$ for $b \in \mathscr{A}_n$. Denote

$$B_{m,\theta,t}^a = \pi^a \hat{B}_{m,\theta,t}^a.$$

If $B^a_{m,\theta,t}$ is provided with the norm

$$\|f\|_{B^a_{m,\theta,t}} = \inf_{\pi^a f = f} \|\hat{f}\|_{\hat{B}^a_{m,\theta,t}},$$

it becomes a Banach space. In this new situation, we use a strengthened version of condition A:

Condition A_m. The potentials V_{ij} have the property that the equation $\varphi + L^a(\mu)\varphi = 0$ has no nontrivial solutions in $\mathring{B}^a_{m,\theta,t}$ for any $a \in \mathscr{A}$.

Let P^a denote the projection onto the eigen-subspace of the operator H^a corresponding to the point μ of the discrete spectrum, where $a \in \tilde{\mathscr{A}}(\sigma_d(H^a) \ni \mu)$; if these last conditions do not hold, we put $P^a = 0$. It follows from standard theorems of functional analysis that the operator $R^a(\lambda)$, $a \in \mathscr{A}$, admits a representation

$$R^a(\lambda) = (\mu - \lambda)^{-1} P^a + R^a_1(\lambda), \tag{2.8}$$

where $R^a_1(\lambda)$ is a continuous operator from $L_2(R^a)$ to $D(H^a)$, analytic in λ for $\lambda \leqq \mu$.

Proposition 2.1. *For $\lambda \leqq \mu$, the operator $R^a_1(\lambda)$, $a \in \mathscr{A} \setminus \mathscr{A}_n$, satisfies the relations:*

$$\|R^a_1(\lambda)\|_{B_{m,\theta,t}(R^a) \to B^a_{m,\theta,t}} < C, \quad \lambda \leqq \mu^a, \tag{2.9a}$$

$$\|R^a_1(\lambda') - R^a_1(\lambda)\|_{B_{m,\theta,t}(R^a) \to B^a_{m,\theta,t}} \to 0 \quad \text{as} \quad \lambda' \to \lambda,\ \lambda', \lambda \leqq \mu^a. \tag{2.10a}$$

Proof. We shall prove the relations by induction. Assuming them to be true for all $b, b \subset a$, we shall prove them for the partition a. This will complete the proof for partitions in $\mathscr{A}_{n-1}$, since then it follows from $b \subset a$ that $b \in \mathscr{A}_n$ and $R^b(\lambda) = \lambda^{-1}$.

To study the operator $R^a_1(\lambda)$, we shall use an equation that follows from (2.5), (2.8):

$$R^a_1(\lambda) + L^a(\lambda) R^a_1(\lambda) = F^a_1(\lambda), \tag{2.11}$$

where

$$\begin{aligned} F^a_1(\lambda) &= F^a(\lambda) - (E^a + L^a(\lambda)) P^a(\lambda) \\ &= F^a(\lambda) - F^a(\lambda)(H^a - \lambda E^a) P^a(\lambda) \\ &= F^a(\lambda)(E^a - P^a). \end{aligned} \tag{2.12}$$

Lemma 2.1. *For $\lambda \leqq \mu$, the operator $L^a(\lambda)$, $a \in \mathscr{A}$, $k(a) < n$, satisfies the relations*

$$\|L^a(\lambda)\|_{B^a_{m,\theta,t} \to B^a_{m,\theta,t}} < C, \tag{2.13}$$

$$\|L^a(\lambda') - L^a(\lambda)\|_{B^a_{m,\theta,t} \to B^a_{m,\theta,t}} \to 0 \quad \text{as} \quad \lambda' \to \lambda. \tag{2.14}$$

Proof. It follows readily from (2.5)[7] with $b = a$ that the operator $L^a(\lambda)$ may be expressed as a linear combination of monomials

$$\overset{\curvearrowleft}{\prod_{i=1}^{k}} [R_{b_i,a}(\lambda) V^a_{\alpha_i}], \tag{2.15}$$

[7] In [11] formula (2.5) is transformed to a form from which these conditions are easily deduced.

where b_i, α_i, $i=1, \ldots, k$, satisfy the conditions:

$$\left.\begin{aligned}
&b_i \subset a,\ u_{\alpha_i} \subseteqq a,\ \bigcup_1^k u_{\alpha_t} = a\,,\\
&\bigcup_{j+1}^{i} u_{\alpha_t} \subseteqq b_j \to b_i \subseteqq \bigcup_{j+1}^{i} u_{\alpha_t},\ i>j,\ i,j=1,\ldots,k\,,\\
&b_i \subset \bigcup_1^i u_{\alpha_t},\ \bigcup_{j+1}^{k} u_{\alpha_t} \nsubseteq b_j\,.
\end{aligned}\right\} \tag{2.16}$$

We begin our study of the operator $L^a(\lambda)$ by establishing estimates for the operators $R_{b,a}(\lambda)$, $b \subset a$.

Lemma 2.2. *Let the operator* $B(\lambda)$ *satisfy the estimates* (2.9*b*) *and* (2.10*b*). *Then the operator*

$$B'(\lambda)f = B(\lambda - \tau(p_b^a))f(p_b^a)$$

will satisfy estimates (2.9*a*), (2.10*a*).

Proof. We have

$$\begin{aligned}
\|B'(\lambda)f\|_{B^a_{m,\theta,t}} &= \|\, \|B(\lambda-\tau(p_b^a))f_{p_b^a}\|_{B^b_{m,\theta,t+\tau(p_b^a)}}\|_{L_m(R_b^a)}\\
&\leqq C\|\, \|f_{p_b^a}\|_{B_{m,\theta,t+\tau(p_b^a)}(R^b)}\|_{L_m(R_b^a)}\\
&= C\|f\|_{B_{m,\theta,t}(R^a)}\,.
\end{aligned}$$

The proof of the second estimate is similar. This completes the proof of Lemma 2.2.

It follows from Lemma 2.2, the induction hypothesis and Equations (2.4′) that the operators $R_{b,a}(\lambda) - (\mu + \tau(p_b^a) - \lambda)^{-1}(P^b \otimes E_b^a)$, $b \subset a$, satisfy estimates (2.9a), (2.10a).

To derive estimates for the operator $(\tau(p_b^a) + \mu - \lambda)^{-1}(P^b \otimes E_b^a)$ we must first estimate the eigenfunctions of the discrete spectrum of H^b, $b \subset a$. We observe from (2.6) that the eigenfunctions of H^a, $a \in \tilde{\mathscr{A}}$, belonging to a point λ of the discrete spectrum satisfy the equation

$$\psi + L^a(\lambda)\psi = 0\,. \tag{2.17}$$

Lemma 2.3. *If* $\lambda < \mu^a$, *the operator* $L^a(\lambda)$ *maps* $R_{0,a}(i)B_{m,\theta,t}(R^a)$, $m \geqq 2$, $\theta > \frac{3}{2}(n-k(a))(1-2/m)$ $(\theta=0,\ m=2)$, *continuously into* $R_{0,a}(i)B_{m',\theta',t}(R^a)$, $m'>m$, $\theta' > \frac{3}{2}(n-k(a))(1-2/m')$.

Proof. The assertion of the lemma follows from the representation of $L^a(\lambda)$ in terms of monomials (2.15) satisfying the condition $\bigcup_1^k u_{\alpha_t} = a$ and the following easily verified relations:

$$V_\alpha^a R_{0,a}(i) : B_{m',\theta',t}(R^b) \otimes B_{m,\theta,t}(R_b^a) \to B_{m',\theta',t}(R^{b \cup u_\alpha}) \otimes B_{m,\theta,t}(R^a_{b \cup u_\alpha})\,, \tag{2.18}$$

$$R_{b,a}(\lambda) : B_{m',\theta',t}(R^c) \otimes B_{m,\theta,t}(R_c^a) \to R_{0,a}(i)(B_{m',\theta',t}(R^c) \otimes B_{m,\theta,t}(R_c))\,, \tag{2.19}$$

where $c \supseteqq b$, $\lambda < \mu^b$, and m', θ', m, θ are the same as in the statement of the lemma. This completes the proof.

Lemma 2.3 and the Equation (2.17) for the eigenfunctions of the discrete spectrum imply

Lemma 2.4. *The eigenfunctions of the discrete spectrum of the operator H^a (if it exists) belong to the space $R_{0,a}(i)B_{m_0,\theta_0,1}(R^a)$.*

It follows from Lemma 2.4 that the operator $(\tau(p_b^a)+\mu-\lambda)^{-1}(P^b\otimes E_b^a)$ satisfies estimates (2.9a), (2.10a).

For the application of Equations (2.15)–(2.16) to establish estimates for $L^a(\lambda)$, it remains only to study the operators V_α^a and $R_{0,a}(\lambda)V_\alpha^a$. We first observe that

$$\|V_\alpha^b\psi^b\|_{B_{m,\theta,t}(R^b)}<C,\ u_\alpha\subset b,\ b\in\mathscr{A}\,. \tag{2.20}$$

Now let $R_{T_b^a}(z)$ denote the resolvent of the operator T_b^a.

Lemma 2.5. *If $s\geqq 0$ and $u_\alpha\not\subset b$, the operator $V_\alpha^a R_{T_b^a}(-s)$ satisfies the following estimates in $L_2(R^b)\times L_2(R_b^a)$*[8]*:*

$$\|V_\alpha^a R_{T_b^a}(-s)\varphi f\|_{B_{m,\theta,t}(R^a)}\leqq C\|\varphi\|_{B_{m,\theta,1}(R^b)}\|f\|_{B_{m,\theta,t}(R_b^a)}\,,$$

$$\|V_\alpha^a(R_{T_b^a}(-s')-R_{T_b^a}(-s))\varphi f\|_{B_{m,\theta,t}(R^a)}\leqq C|s'-s|^\delta\|\varphi\|_{B_{m,\theta,1}(R^b)}\|f\|_{B_{m,\theta}(R_b^a)}\,,$$

where $m>6$, $\delta<(3/2)(1/3-1/m)$.

Proof. To simplify the notation, we shall assume that $a\in\mathscr{A}_1$, $b\in\mathscr{A}_2$. This involves no loss of generality. Indeed, let $(\bar b,\bar a)$, $\bar b\subset\bar a$, be any pair. Consider the pair $(\bar b,a')$, where $a'=\bar b\cup u_\alpha$. The estimates for this pair follow directly from estimates for the special case under consideration [for the whole system $a\in\mathscr{A}_1$ one can take the union of two subsystems of the partition $\bar b$ containing the indices i and j, $(ij)=\alpha$, and the partition $b\in\mathscr{A}_2$ may be chosen as the combination of these two subsystems]. Estimates for the pair $(\bar b,\bar a)$ may be derived from estimates for $(\bar b,a')$, using exactly the same arguments as in the proof of Lemma 2.2.

Let $b=\{c_1,c_2\}$, $i\in c_1$, $j\in c_2$, $M_c=\sum_{r\in c}m_r$. We go over from the sequence p^b to independent variables $\bar p^b=(p_k,\,p_k\in p^b,\,k\neq i,j)$. Since $a\in\mathscr{A}_1$, we shall now omit the index a in the notation for operators and variables. The function $V_\alpha R_{T_b}(-s)\varphi f$ admits an integral representation:

$$(V_\alpha R_{T_b}(-s)\varphi f)(p,s)=\varphi(\bar p^b)\int\frac{v_\alpha(p_{c_1}-k)f(k)dk}{k^2/2M_{c_1}+(p_{c_1}+p_{c_2}-k)^2/2M_{c_2}+s}\,.$$

Using the inequalities

$$k^2/2M_{c_1}+(p_{c_1}+p_{c_2}-k)^2/2M_{c_2}>Cq^2,\ C>0\,,$$

and

$$|(s'+a)^{-1}-(s+a)^{-1}|<C|s'-s|^\delta a^{-1-\delta},\ s',s\geqq 0,\ a>0\,,$$

one readily finds the estimates

$$|V_\alpha R_{T_b}(-s)f|<C\int\frac{|v_\alpha(p_{c_1}-k)f(k)|dk}{|k|^2}$$

and

$$|V_\alpha(R_{T_b}(-s')-R_{T_b}(-s))f|<C|s'-s|^\delta\int\frac{|v_\alpha(p_{c_1}-k)f(k)|dk}{|k|^{2(1+\delta)}}\,.$$

We must thus estimate the integral

$$g_\delta(p)=\int\frac{|v(p-k)f(k)|dk}{|k|^{2(1+\delta)}}\,.$$

[8] $L_2(X)\times L_2(Y)=\{f=\varphi\cdot g,\ \varphi\in L_2(X),\ g\in L_2(Y)\}$.

To do this, we use Hölder's inequality and the inequality

$$(1+|p-k|)^{-\alpha}(s+|k|)^{-\alpha}<C(s+|p|)^{-\alpha}[(1+|p-k|)^{-\alpha}+(1+|k|)^{-\alpha}].$$

We obtain

$$\begin{aligned} g_\delta(p) &\leqq \left[\int \frac{(1+|p-k|^2)^{-\frac{\theta m_1'}{2}}(t+k^2)^{-\frac{\theta m_1'}{2}}}{|k|^{2(1+\delta)m_1'}}\,dk\right]^{1/m_1'}[\int|v(p-k)|^{m/2}dk]^{2/m} \\ &\quad\times[\int|(1+|p-k|^2)^{\theta/2}v^{1/2}(p-k)(t+k^2)^{\theta/2}f(k)|^m dk]^{1/m} \\ &\leqq C(t+p^2)^{-\theta/2}[\int|(1+|p-k|)^{\theta/2}v^{1/2}(p-k)(t+k^2)^{\theta/2}f(k)|^m dk]^{1/m}, \end{aligned}$$

where $1/m_1'+2/m=1, 2(1+\delta)m_1'<3$. Hence:

$$\|(t+p^2)^{\theta/2}g_\delta\|_{L_m(\mathbb{R}^3)}\leqq C\|(1+p^2)^{\theta/2}v\|_{L_{1/2m}(\mathbb{R}^3)}\|(t+p^2)^{\theta/2}f\|_{L_m(\mathbb{R}^3)},$$

where $\delta<(3/2)(1/3-2/m)$, $m>6$. This completes the proof of Lemma 2.5.

We now consider the operator $R_{0,a}(\lambda)V_\alpha^a$. Let $\varphi^b\in L_2(R^b)$, $f_b\in L_2(R_b^a)$, $b\subset a$, $b\in\tilde{\mathscr{A}}$. We have

$$\begin{aligned} &R_{0,a}(\lambda)V_\alpha^a\sum_{\substack{b\subset a\\ b\in\tilde{\mathscr{A}}}}R_{T_b^a}(\mu\delta_{k(b),n})\varphi^b f_b \\ &=-R_{0,a}(\lambda)\sum_{\substack{b\supseteq u_\alpha\\ b\in\tilde{\mathscr{A}}}}\tilde{\varphi}^b f_b+\sum_{b\supseteq u_\alpha}R_{T_b^a}(0)\tilde{\varphi}^b f_b+\sum_{b\supseteq u_\alpha}V_\alpha R_{T_b^a}(\mu\delta_{k(b),n})\varphi^b f_b, \end{aligned} \tag{2.21}$$

where

$$\tilde{\varphi}^b(p^b)=(\tau(p^b)-\lambda)^{-1}(V_\alpha^b\varphi^b)(p^b),\ b\supset u_\alpha,\ b\in\tilde{\mathscr{A}}, \tag{2.22}$$

$$\|\tilde{\varphi}^b\|_{R_{0,b}(i)B_{m,\theta,1}(R^a)}\leqq C\|\varphi^b\|_{R_{0,b}(i)B_{m,\theta,1}(R^a)}. \tag{2.23}$$

Applying the estimates for $R_{b,a}(\lambda)$, $b\subset a$, Lemma 2.5 and Equations (2.20)–(2.23) to formulas (2.15) and (2.16), we finally obtain the estimates required in Lemma 2.1. This completes the proof of Lemma 2.1.

The operator $F^a(\lambda)$ also admits a representation of type (2.15), (2.16). Applying the same results as before, we obtain

Lemma 2.6. *The operator $F^a(\lambda)$, $a\in\mathscr{A}\setminus\mathscr{A}_n$, satisfies the estimates* (2.9a), (2.10a) *for $\lambda\leqq\mu$.*

Lemma 2.7. *For $\lambda\leqq\mu$, the operator $L^a(\lambda)$, $a\in\mathscr{A}\setminus\mathscr{A}_n$, is compact in $B_{m,\theta,t}^a$, $m>3$, $\theta>\frac{3}{2}(n-k(a))(1-2/m)$.*

Proof. We first observe that it will suffice to prove the lemma for $\lambda<\mu$. It will then follow from (2.14) for the case $\lambda=\mu$.

Employing the same reasoning as in the proof of Lemma 2.5, one proves that

$$V_\alpha^a R_{T_c^a}(-s): B_{m',\theta',t}(R^b)\otimes B_{m,\theta,t}(R_b^a)\to B_{m',\theta',t}(R^{b\cup u_\alpha})\otimes B_{m,\theta,t}(R_{b\cup u_\alpha}^a),\ s>0, \tag{2.24}$$

for $u_\alpha\nsubseteq c$, $\theta>3/2$, where $m'>m$, $\theta'>\frac{3}{2}(n-k(a))(1-2/m')$. It follows from the representation of $L^a(\lambda)$ in terms of monomials (2.15) with the conditions

$$\bigcup_1^k u_{\alpha_t}=a,\ b_i\subset\bigcup_1^i u_{\alpha_t},$$

and from Equations (2.18)–(2.20), (2.24), that the operator $L^a(\lambda)$, $\lambda<\mu^a$, maps $B^a_{m,\theta,t}$, $m>3$, $\theta>\frac{3}{2}(n-k(a))(1-2/m)$, continuously into $B^a_{m',\theta',t}$, $m_0\geqq m'\geqq m$, $\theta_0\geqq\theta'>\frac{3}{2}(n-k(a))(1-2/m')$, and moreover m, θ may be chosen in such a way that $m<m'$, $\theta<\theta'-3(n-k(a))(1/m-1/m')$. Under these assumptions, the space $B^a_{m',\theta',t}$ is compactly embedded in $B^a_{m,\theta,t}$. Thus $L^a(\lambda)$, $\lambda<\mu^a$, is a compact operator in $B^a_{m,\theta,t}$ for such values of m, θ. This proves Lemma 2.7.

Consider the equation

$$f+L^a(\lambda)f=F^a_1(\lambda)\varphi,\ \varphi\in B_{m,\theta,t}(R^a)\,, \tag{2.25}$$

in the space $B^a_{m,\theta,t}$. Lemmas 2.5 and 2.6 show that the Fredholm alternative applies to this equation. The following proposition for the corresponding homogeneous equation

$$f+L^a(\lambda)f=0\,. \tag{2.26}$$

is valid:

Lemma 2.8. *Let the function f_λ satisfy Equation* (2.26) *for some λ, $\lambda<\mu^a$ or $\lambda=\mu^a$, $a:\{b:b\subset a,\ b\in\mathscr{A}_{k(a)+1}\cap\tilde{\mathscr{A}}\}=\emptyset$. Then f_λ is an eigenfunction of the operator H^a with eigenvalue λ.*

Proof. In view of the equation

$$H^a-\mu E^a=\overset{n-1}{\overset{\curvearrowright}{\prod_{s=k(a)+1}}}\ \prod_{\substack{b\in\mathscr{A}_s\\ b\subset a}}{}'\ (E+L_{b_\lambda a}(\mu))^{-1}(E+L^a(\mu))\,,$$

where Π' denotes the product in the order opposite to Π, it will suffice to prove that $f_\lambda\in D(H^a)$. For a such that $\mu^a=\mu^b$ for all $b\subset a$, $k(b)=k(a)+2$, this follows from the embedding $B^a_{m,\theta,t}\subset D(H^a)$. For arbitrary a and $\lambda<\mu^a$, it follows from Lemma 2.7 and the Equation (2.26) for f_λ. This completes the proof of Lemma 2.8.

Lemma 2.8 shows that Equation (2.26) with $\lambda\leqq\mu$ has solutions in $B_{m,\theta,t}$ only for $\lambda=\mu$. For a such that $\mu^a=\mu$, this is impossible by virtue of Condition A. In all other cases, the solution is an eigenfunction of the discrete spectrum of H^a with eigenvalue μ.

It follows from Lemmas 2.1, 2.6, 2.7, and the relation

$$(E^a+L^a(\lambda'))^{-1}-(E^a+L^a(\lambda))^{-1}=(E^a+L^a(\lambda'))^{-1}(L^a(\lambda')-L^a(\lambda))(E^a+L^a(\lambda))^{-1}$$

that if condition A_m holds then Equation (2.25) has a unique solution for $\lambda\leqq\nu^a$, where $\nu^a=\mu-\varepsilon$ (ε being some small number) if $\mu\in\sigma_d(H^a)$, and $\nu^a=\mu$ if $\mu\notin\sigma_d(H^a)$. This solution defines an operator satisfying estimates (2.9a), (2.10a). It is readily seen that if $\lambda\notin\sigma(H^a)$ this operator is continuous from $L_2(R^a)$ to $D(H^a)$. Hence, by the uniqueness of the solution of Equation (2.11) for $\lambda\notin\sigma(H^a)$ in the class $\mathscr{L}(L_2(R^a), D(H^a))$, it follows that the solution is precisely $R^a_1(\lambda)$.

In order to establish estimates for $R^a_1(\lambda)$ when $\mu-\varepsilon\leqq\lambda\leqq\mu$ for a such that $\mu\in\sigma_d(H^a)$, we must use the fact that $\mu^a>\mu$ and $R^a_1(\lambda)$, $\lambda<\mu^a$, is continuous from $L_2(R^a)$ to $D(H^a)$, and apply Lemma 2.3 to the equation obtained by iteration of Equation (2.11). This completes the proof of Proposition 2.1.

If $a\in\mathscr{A}_1$, Lemmas 2.1 and 2.7 yield the desired assertion for the operator $L(\lambda)$, subject to suitable assumptions on the potentials. We formulate this as a separate proposition:

Proposition 2.2. *Let* $B_{m,\theta}=B^a_{m,\theta,1}$, $a\in\mathscr{A}_1$. *Suppose that the potentials* v_{ij} *satisfy inequality* ($2.1m_0$) *and condition* A_m, $m>6$. *Then the operator* $L(\lambda)$, $\lambda\leqq\mu$, *is bounded and continuous in the uniform operator topology with respect to* λ, *and is compact in* $B_{m,\theta}$, $\theta_0>\theta>\frac{3}{2}(n-1)(1-2/m)$.

We can now prove Lemma 1.3. Let $H_a=H^d_a$, $T_a=T^d_a$, $d\in\mathscr{A}_1$, $I_a=\sum_{u_{ij}\nsubseteq a} V_{ij}$, $\hat{f}=\{f_a\}\in\hat{B}_{m,\theta}$. We have

$$\begin{aligned}(H-\mu E)\pi\hat{f} &= \sum_{a\in\tilde{\mathscr{A}}}(H_a-\mu E+I_a)R_{T_a}(\mu\delta_{k(a),n})\psi^a f_a\\ &=\sum_{a\in\tilde{\mathscr{A}}}\psi^a f_a+\sum_{a\in\tilde{\mathscr{A}}}\sum_{u_\alpha\nsubseteq a}V_\alpha R_{T_a}(\mu\delta_{k(a),n})\psi^a f_a\,. \end{aligned}\tag{2.27}$$

Applying Lemma 2.5 to the second term on the right of (2.27), we obtain the estimate

$$\|(H-\mu E)\pi\hat{f}\|_{B_{m,\theta}(R)}<C\|\hat{f}\|_{\hat{B}_{m,\theta}},\ m>6,\ \theta>\tfrac{3}{2}(n-1)(1-2/m)\,,$$

and this implies the assertion of Lemma 1.3 when ($2.1m_0$) and condition A_m are satisfied.

To establish estimates for $F(\lambda)$, $\lambda\leqq\mu$, we put $a\in\mathscr{A}_1$ in Lemma 2.6. To estimate $F^{-1}(\mu)$, we first observe that formula (2.11) implies readily that $F^{-1}(\mu)-R^{-1}(\mu)$ admits a representation as a linear combination of monomials

$$R_0^{-1}(\mu)L_{f_1}(\mu)\ldots L_{f_t}(\mu)\,,\tag{2.28}$$

where $f_1,\ldots,f_t$ satisfy the conditions:

$$\begin{aligned}&f_1\in\mathscr{A}_{n-1},\ k(f_s)\geqq k(f_{s+1}),\ k(f_t)\geqq k(a)+1\,,\\ &\bigcup_1^s f_i\subsetneqq f_{s+1},\ s=1,\ldots,t\,.\end{aligned}\tag{2.29}$$

The fact that monomials of type (2.28) are bounded operators from $B_{m,\theta}$ to $B_{m,\theta}(R)$ is proved in the same way as the boundedness of $L(\lambda)$, $\lambda\leqq\mu$, in $B_{m,\theta}$, except that in addition one uses conditions (2.29).

Acknowledgement. I wish to thank Professor G.M. Zhislin for many valuable discussions concerning this article.

Appendix

We prove here a lemma stated in [8] for a certain specific operator.

Recall that a singular point of a family of compact operators $A(\lambda)$ in a Banach space B is defined as a parameter value λ such that the operator $A(\lambda)$ has the eigenvalue -1 (or any other number).

Lemma. *Let* $A(\lambda)$ *be a family of compact operators in a Banach space B, defined in some closed region of the complex* λ*-plane and continuous there with respect to* λ *in the uniform operator topology. Then the set of singular points of the family* $A(\lambda)$ *(in the region of interest) is closed.*

Proof. Let $\{\lambda_n\}$ be a sequence of singular points of the family $A(\lambda)$ converging to a point λ_0 at which $A(\lambda)$ is defined. We claim that λ_0 is also a singular point of the family $A(\lambda)$. Let φ_n be an eigenfunction of the operator $A(\lambda_n)$ belonging to the eigenvalue $-1, n=1, 2, \ldots$ Since the operator $A(\lambda_0)$ is compact, the sequence $\{\psi_n = A(\lambda_0)\varphi_n\}$ contains a subsequence $\{\psi_{n'}\}$ which is convergent in B. Let $\psi = \lim_{n'\to\infty} \psi_{n'}$. ψ is an eigenfunction of the operator $A(\lambda_0)$, belonging to the eigenvalue -1. Indeed, this follows from the estimates

$$\begin{aligned}\|A(\lambda_0)\psi+\psi\| &\leqq \|A(\lambda_0)(\psi_{n'}-\psi)\| + \|\psi_{n'}-\psi\| + \|A(\lambda_0)\psi_{n'}+\psi_{n'}\| \\ &\leqq \|A(\lambda_0)\| \; \|\psi_{n'}-\psi\| + \|\psi_{n'}-\psi\| + \|A(\lambda_0)\| \; \|\psi_{n'}+\varphi_{n'}\| ,\\ \|\psi_{n'}+\varphi_{n'}\| &= \|A(\lambda_0)\varphi_{n'}+\varphi_{n'}\| \\ &\leqq \|(A(\lambda_0)-A(\lambda_{n'}))\varphi_{n'}\| + \|A(\lambda_{n'})\varphi_{n'}+\varphi_{n'}\| \\ &\leqq \|A(\lambda_0)-A(\lambda_{n'})\| \; \|\varphi_{n'}\| .\end{aligned}$$

This completes the proof.

References

1. Zhislin, G. M.: Tr. Mosk. Mat. Obsc. **9**, 81 (1960)
2. Hunziker, W.: Helv. Phys. Acta **39**, 451 (1966)
3. Winter, C., van: Mat. Fys. Skr. Dan. Vid. Selsk **1**, Nos. 8, 10 (1964/1965)
4. Combes, J. M.: Commun. math. Phys. **12**, 253 (1969)
5. Zhislin, G. M.: Izv. Akad. Nauk, SSSR, math. **33**, 590 (1969)
6. Jorgens, K., Weidmann, J.: Spectral Properties of Hamiltonian Operators, Lecture Notes in Math. . ol. 313. Berlin-Heidelberg-New York: Springer 1973
7. Sigalov, A. G., Sigal, I. M.: Teor. Mat. Fiz. **5**, 73 (1970), English transl. Theor. Math. Phys. (1971)
8. See, for example Hunziker, W.: Lectures Theoret. Phys. XA, (Barut, ed.). New York: Gordon and Breach 1969;
   Combes, J. M.: Scatt. Theory in Math. Phys., (LaVita, ed.). Dordrecht-Boston: D. Reidel Publ. Comp. 1974
   The recent rigorous results in multiparticle scattering can be found in [11].
9. Faddeev, L. D.: Tr. Steklov Mat. Ins. **69** (1963), English transl. Mathematical Aspects of the Three-body Problem in the Quantum Theory of Scattering, Israel Program of Scientific Translations, Jerusalem (1965)
10. Hepp, K.: Helv. Phys. Acta **42**, 425 (1965)
11. Sigal, I. M.: Preprint of Tel-Aviv University (1975)
12. Zhislin, G. M.: Teor. Mat. Fiz. **7**, 332 (1971); English Transl. Theor. Math. Phys. **7**, 521 (1971); Dokl. Akad. Nauk. SSSR, **207**, 25 (1972); Soviet Math. Dokl. **13**, 1445 (1972)
13. Jafaev, D. R.: Funct. Anal. Appl. **6**, 346 (1972)
14. Antonets, M. A., Zhislin, G. M., Shereshevskii, I. A.: Teoret. Mat. Fiz. **16**, 235 (1973). English Transl. Theor. Math. Phys. **16**, 800 (1974)
15. Jafaev, D. R.: Mat. Sb. (N.S.) **94** (136), 567, 655 (1974)
16. Simon, B.: Helv. Phys. Acta **43**, 607 (1970)
17. Kato, T.: Trans. Amer. Math. Soc. **70**, 195 (1951)
18. Berezin, F. A.: Dokl. Akad. Nauk. SSSR **163**, 795 (1965); Soviet Math. Dokl. **6**, 997 (1965)
19. Sigal, I. M.: Dokl. Akad. Nauk. SSSR **204**, 795 (1972); Soviet Math. Dokl. **13**, 756 (1972)
20. Berezanskii, Ju. M.: Expansions in eigenfunctions of sul adjoint operators. Providence: Rhode Island: Am. Math. Soc. 1965
21. Weinberg, S.: Phys. Rev. **133**, B232 (1964)
22. Jakubovskii, O. A.: Proceedings of the Steklov Institute of Math. N 110 (1970)

Communicated by W. Hunziker

Received September 15, 1975

Note Added in Proof. Fortunately Condition A which cannot be effectively controlled turns out to be not so important. It can be proved (I. M. Sigal, unpublished) that for any potential $V = \Sigma V_\alpha$ satisfying (I, 2) there exists a number $\delta > 0$ such that Condition A is satisfied for Hamiltonians $H_0 + (1+\varepsilon) V$, $0 < |\varepsilon| \leqq \delta$.