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On the Uniqueness of the Newtonian Theory as a Geometric Theory of Gravitation

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Abstract. A study is made of geometric theories of gravitation that are consistent with the local validity of Newtonian dynamics. This involves an analysis of the representations of the Galilean group provided by the curvature tensor of a Newtonian spacetime, and by the contravariant mass-momentum tensor. Subject to certain assumptions that are made also in the foundations of general relativity, it is shown that there exists essentially only one such theory that does not place unacceptable restrictions on the mass density of the source. This is the Newtonian theory, generalized by a cosmological term. Any other theory is weaker and is given by a subset of the geometrical equations of the Newtonian theory.

1. Introduction

The spacetime of Newtonian physics in the absence of gravitation may be described covariantly by a symmetric tensor $g^{\alpha\beta}$, a vector t_α and a symmetric affine connexion $\Gamma^\gamma_{\alpha\beta}$. These satisfy

$$g^{\alpha\beta}t_\beta=0\,, \tag{1.1}$$

$$\nabla_\alpha g^{\beta\gamma}=0 \quad \text{and} \quad \nabla_\alpha t_\beta=0\,, \tag{1.2}$$

and

$$R_{\alpha\beta\gamma}^{\cdot\,\cdot\,\cdot\,\delta}=0\,, \tag{1.3}$$

where $R_{\alpha\beta\gamma}^{\cdot\,\cdot\,\cdot\,\delta}$ and ∇_α are respectively the curvature tensor and covariant derivative operator of the connexion. In addition, $g^{\alpha\beta}$ is required to be positive semi-definite and of matrix rank 3. It follows from these conditions that there exists a family of coordinate systems in which

$$g^{\alpha\beta}=\mathrm{diag}(1,1,1,0) \quad \text{and} \quad t_\alpha=(0,0,0,1)\,, \tag{1.4}$$

and

$$\Gamma^\alpha_{\beta\gamma}=0\,. \tag{1.5}$$

These are related to one another by Galilean transformations, and the connexion with physics is made by identifying them with inertial frames of dynamics.

Newtonian gravitational theory also has a covariant form. This was first shown by Cartan [1, 2] and has been discussed more recently by Trautman [3, 4], Künzle [5], and Misner, Thorne and Wheeler [6]. By suitable manipulation of the usual three-dimensional equations, these authors show that gravitation may be incorporated into the above structure by replacing (1.3) with the set of equations

$$R_{\alpha\beta}=4\pi G\varrho t_\alpha t_\beta\,, \qquad R_{\alpha\beta}{}^{\cdot\cdot\gamma\delta}=0 \quad \text{and} \quad R_{\alpha\cdot\gamma}^{\cdot\beta\cdot\delta}=R_{\gamma\cdot\alpha}^{\cdot\delta\cdot\beta}\,. \tag{1.6}$$

Here, ϱ is the mass density, G is the Newtonian gravitational constant, and indices on the curvature tensor have been raised with $g^{\alpha\beta}$, e.g. $R_{\alpha\beta}{}^{\cdot\cdot\gamma\delta}=g^{\gamma\varepsilon}R_{\alpha\beta\varepsilon}{}^{\cdot\cdot\cdot\delta}$. It should be noted that this raising process is not reversible, since $g^{\alpha\beta}$ is singular. It thus has greater significance than the corresponding operation in relativistic spacetime. Global coordinate systems still exist in which (1.4) holds everywhere, but this is a wider family than the Newtonian inertial frames due to the lack of (1.5). The inertial frames can be picked out uniquely only when the spacetime is asymptotically flat. In that case they are the frames in which (1.5) holds at infinity. Particles which move freely under gravity follow geodesics of the connexion. Continuous matter distributions are described by a contravariant symmetric mass-momentum tensor $T^{\alpha\beta}$ which satisfies

$$\nabla_\beta T^{\alpha\beta}=0\,. \tag{1.7}$$

This is related to the three-dimensional description by

$$T^{44}=\varrho\,, \qquad T^{4a}=T^{a4}=\varrho v^a\,, \qquad T^{ab}=\varrho v^a v^b-\sigma^{ab}\,, \tag{1.8}$$

where $v^a=dx^a/dt$ is the material velocity and σ^{ab} is the Stokes stress tensor.

The field equations (1.6) appear at first sight to be considerably more complicated than the single field equation

$$R^{\alpha\beta}-\tfrac{1}{2}Rg^{\alpha\beta}+\lambda g^{\alpha\beta}=8\pi\kappa T^{\alpha\beta} \tag{1.9}$$

of general relativity. The purpose of the present paper is to show that despite this, they can be based directly on physical considerations analogous to those used in the foundation of general relativity. More precisely, when generalized by a cosmological term they give the strongest theory compatible with the geometrization of gravitation and the local validity of Newtonian dynamics, in that any other compatible theory must consist of a subset of these equations. This is meant in exactly the same sense as that in which (1.9) follows from the corresponding relativistic assumptions. The question is here left open as to whether or not a subset of the Eqs. (1.6) can form a viable generalization of the Newtonian theory. This will be answered in a subsequent paper, where it will be shown that this is not possible if it is required that spacetime be asymptotically flat. It will also be shown there that in the absence of asymptotic flatness, the limit of general relativity as "$c\to\infty$" is not the Newtonian theory itself, but is actually the only possible such generalization. A remark of Künzle [5] provides the key to this rather odd situation. It is that the second of the Eqs. (1.6) becomes redundant in the asymptotically flat case even though it is not an algebraic consequence of the other equations of the set. Asymptotic flatness, of course, also rules out a cosmological term in either the Newtonian or Einsteinian theory. It is thus seen that the Newtonian and Einsteinian theories can be derived from the same postulates as to the nature of gravity, and with the same degree of uniqueness. They differ only in the underlying theories of mechanics in the absence of gravitation.

The mathematical development to justify these claims rests heavily on the representation theory of the general linear and orthogonal groups. A secondary aim of this paper is to demonstrate the usefulness of these aspects of group theory in dealing with tensors having many indices.

In Section 2 a discussion is given of the general form of allowable field equations. A summary of notation is given in Section 3, together with some formulae that will be needed later. As the tensor representations of the Galilean group are not completely reducible, certain problems arise that are not present with the Lorentz group of relativity theory. These are discussed in Section 4. The representations provided by both the curvature and mass-momentum tensors are then analyzed in Sections 5 and 6 respectively, and the results obtained are used in Section 7 to write down all algebraically consistent gravitational field equations. Certain of these have to be eliminated as they impose unacceptable differential constraints on the mass density, as is shown in Sections 7 and 8. If all the equations that remain are taken together, they give (1.6), as required. The paper concludes with a summary and brief discussion in Section 9.

The notation used closely follows that of Schouten [7].

2. General Considerations of Structure

In the foundation of general relativity, it is deduced from the principle of equivalence that spacetime is a pseudo-Riemannian manifold of signature $+2$. This precedes any consideration of possible field equations. The physical arguments needed to support this can also be applied to the corresponding Newtonian assumptions. There, they show that any Newtonian spacetime must possess a positive semi-definite symmetric tensor field $g^{\alpha\beta}$ of matrix rank 3, a covariant vector field t_α, and a symmetric affine connexion $\Gamma^\gamma_{\alpha\beta}$ which satisfy (1.1) and (1.2). In both theories it also follows that particles moving freely under gravity follow geodesics of the connexion, and that the energy-momentum (or, in the Newtonian theory, more properly the mass-momentum) tensor $T^{\alpha\beta}$ of a continuous matter distribution is symmetric and satisfies (1.7). These results form our starting point.

It follows from (1.2) that there exists a scalar function t satisfying

$$t_\alpha = \partial_\alpha t\,. \tag{2.1}$$

The general Newtonian spacetime thus still possesses a well defined absolute time, and $g^{\alpha\beta}$ determines a unique positive definite Riemannian metric on each of the hypersurfaces of constant t. However, it does not follow from the assumptions made so far that this metric is flat. This is a consequence of the gravitational field equations, whose general form will be considered next.

These equations must link the spacetime geometry to $T^{\alpha\beta}$. For the usual reasons, it will be assumed that the geometrical terms involve, *a priori*, derivatives of t_α and $g^{\alpha\beta}$ up to the second order and of $\Gamma^\gamma_{\alpha\beta}$ up to the first order. However, such derivatives of t_α and $g^{\alpha\beta}$ may be algebraically eliminated with the use of (1.2), while covariance considerations show that the connexion and its first derivative can occur only as its curvature tensor $R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}$. The field equations are thus expressible as algebraic relations between the components of the tensors

$$t_\alpha\,, \quad g^{\alpha\beta}\,, \quad R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta} \quad \text{and} \quad T^{\alpha\beta}\,. \tag{2.2}$$

It will be assumed that the latter two tensors occur linearly. A similar simplifying assumption is usually made also in setting up the equations of general relativity.

It is not necessarily true that covariant equations constructed from the tensors (2.2) must be formed according to the usual rules of tensor algebra. But any such equation is equivalent to one formed according to these rules from the tensors (2.2) together with the alternating pseudotensors $\eta_{\alpha\beta\gamma\delta}$ and $\eta^{\alpha\beta\gamma\delta}$. This follows from some powerful general theorems which may be found in Weyl [8]. These pseudotensors are unique up to a scalar multiple, but it is convenient to make a canonical choice for them. The simplest is to choose

$$\eta_{1234}=\Delta\,, \qquad \eta^{1234}=\Delta^{-1}\,, \tag{2.3}$$

where

$$\Delta=[\det(g^{\alpha\beta}+n^\alpha n^\beta)]^{-1/2} \tag{2.4}$$

and n^α is an arbitrary vector satisfying

$$n^\alpha t_\alpha=1\,. \tag{2.5}$$

This is well defined as the pseudoscalar density Δ is easily shown to be independent of the specific choice of n^α.

Suppose now that an arbitrary point P is chosen. There exists a family of coordinate systems defined around P such that (1.4) holds at P, and the matrix of transformation coefficients connecting any two such systems has at P the $(3+1)\times(3+1)$ block form

$$[\partial x^{\alpha'}/\partial x^\alpha]_P=\begin{pmatrix} L & m \\ 0 & 1\end{pmatrix}, \tag{2.6}$$

where L is a 3×3 orthogonal matrix. The field equations reduce at P to the vanishing of a number of linear inhomogeneous combinations of the components of $R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}$ and $T^{\alpha\beta}$. Let these be made homogeneous by the insertion of a factor θ into the constant terms. Then the triples $(R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}, T^{\alpha\beta}, \theta)$ which satisfy these modified equations form a linear subspace V_S of the direct sum of three vector spaces. Two of these are V_R and V_T, which consist respectively of all possible Newtonian curvature tensors and mass-momentum tensors. These are tensor representation spaces for the Galilean group G of all transformations of the form (2.6). The third space is the one-dimensional space R of real scalars θ on which G acts trivially.

The subspace V_S of $V_R\oplus V_T\oplus R$ has two characteristic properties. It is invariant under G, and the canonical projection $V_S\to R$ is surjective. The latter property results from the consistency of the modified equations with the additional condition

$$\theta=1 \tag{2.7}$$

which must be adjoined to them in order to recover the field equations themselves. Conversely, any such subspace determines a mathematically consistent set of field equations. However, not all such equations will be physically acceptable as some of them will imply unacceptable restrictions on $T^{\alpha\beta}$. To avoid algebraic restrictions of this nature, the canonical projection $V_S \to V_T$ also has to be surjective. The possibility of differential restrictions will be considered later.

3. Algebraic Preliminaries

This section gives the notation and conventions that will be used in the subsequent analysis, together with some algebraic results that will be needed. A colon placed before an equals sign indicates that the equation is the definition of the quantity on the left hand side. Symmetrization and antisymmetrization of indices will be denoted by round and square brackets respectively, with indices to be excluded from these operations being enclosed between vertical lines, e.g.

$$A_{[\alpha|\beta|\gamma]} = \tfrac{1}{2}(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}) . \tag{3.1}$$

In this notation, the curvature tensor is defined by

$$R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta} = 2\partial_{[\alpha}\Gamma^{\delta}_{\beta]\gamma} + 2\Gamma^{\delta}_{[\alpha|\varepsilon|}\Gamma^{\varepsilon}_{\beta]\gamma} . \tag{3.2}$$

Since the connexion is symmetric, this identically satisfies

$$R_{(\alpha\beta)\gamma}^{\cdot\cdot\cdot\delta} = 0 , \qquad R_{[\alpha\beta\gamma]}^{\cdot\cdot\cdot\delta} = 0 . \tag{3.3}$$

The convention of using $g^{\alpha\beta}$ to raise tensor indices, which was discussed in the Introduction, will be used throughout on all tensors except $\eta_{\alpha\beta\gamma\delta}$. Any index so raised gives zero on contraction with t_α, in virtue of (1.1). Application of the Ricci identity to t_α and $g^{\alpha\beta}$, with the use of (1.2), gives the additional identities

$$R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta} t_\delta = 0 , \qquad R_{\alpha\beta}^{\cdot\cdot(\gamma\delta)} = 0 . \tag{3.4}$$

If n^α is an arbitrary vector satisfying (2.5), and A^α_β is the unit tensor, then a projection operator B^α_β may be defined by

$$B^\alpha_\beta := A^\alpha_\beta - n^\alpha t_\beta . \tag{3.5}$$

A tensor $h_{\alpha\beta}$ will also be defined as a function of n^α, by requiring $h_{\alpha\beta} + t_\alpha t_\beta$ to be the inverse of the nonsingular matrix $g^{\alpha\beta} + n^\alpha n^\beta$. This satisfies

$$h_{\alpha\beta} n^\beta = 0 \quad \text{and} \quad g^{\alpha\gamma} h_{\beta\gamma} = B^\alpha_\beta , \tag{3.6}$$

and is given explicitly by

$$h_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\kappa\lambda\mu} \eta_{\beta\nu\varrho\sigma} g^{\kappa\nu} g^{\lambda\varrho} n^\mu n^\sigma . \tag{3.7}$$

It also satisfies

$$h_{\alpha[\gamma} h_{\delta]\beta} = \tfrac{1}{2}\, \eta_{\alpha\beta\kappa\lambda} \eta_{\gamma\delta\mu\nu} g^{\kappa\mu} n^\lambda n^\nu . \tag{3.8}$$

Both (3.7) and (3.8) are easily verified in a coordinate system in which

$$n^\alpha = \delta^\alpha_4 , \qquad t_\alpha = \delta^4_\alpha . \tag{3.9}$$

Another useful result concerning the η-pseudotensors is that

$$g^{\alpha\kappa}g^{\beta\lambda}g^{\gamma\mu}\eta_{\kappa\lambda\mu\nu}=\eta^{\alpha\beta\gamma\delta}t_\delta t_\nu\,, \tag{3.10}$$

which can be proved similarly.

It can be seen from (3.5) and (3.6) that

$$R_{\alpha\beta\gamma}{}^{\gamma}=R_{\alpha\beta}{}^{\gamma\delta}h_{\gamma\delta}+R_{\alpha\beta\gamma}{}^{\delta}n^\gamma t_\delta \tag{3.11}$$

identically. But both terms of the right hand side are zero, from (3.4), and hence

$$R_{\alpha\beta\gamma}{}^{\gamma}=0\,. \tag{3.12}$$

In consequence of this and (3.3), the Ricci tensor

$$R_{\beta\gamma}:=R_{\alpha\beta\gamma}{}^{\alpha} \tag{3.13}$$

is symmetric. A curvature scalar can be defined from it by

$$R:=g^{\alpha\beta}R_{\alpha\beta}=h_{\alpha\beta}R^{\alpha\beta} \tag{3.14}$$

and used to construct a tensor $S^{\alpha\beta}$ by

$$S^{\alpha\beta}:=R^{\alpha\beta}-\tfrac{1}{3}Rg^{\alpha\beta}\,. \tag{3.15}$$

This is both symmetric and tracefree, in the sense that $h_{\alpha\beta}S^{\alpha\beta}=0$ for arbitrary n^α.

4. Description of Group Representations

In contrast to most of the group representations of interest in physics, the tensor representations of the Galilean group are, in general, not completely reducible. This complicates their invariant subspace structure considerably. The general situation for finite-dimensional representations of an arbitrary group H over the real field needs a fairly discriminating terminology to deal with the various possibilities. In the following account, a representation space for a group H will be called an H-space, and a subspace invariant under H will be called an H-subspace.

An H-space is said to be irreducible if it has no proper H-subspaces, and to be indecomposable if none of its proper H-subspaces possesses an invariant complement. It is possible for an indecomposable H-space W to be expressible as the join, W_1+W_2, of two proper H-subspaces W_1 and W_2, but only if $W_1\cap W_2\neq 0$, so that the join is not a direct sum. If W cannot be so expressed, it is said to be join-irreducible. Any H-subspace of an H-space V is expressible as an irredundant join of join-irreducible H-subspaces. This mode of expression is unique, for all H-subspaces of V, if and only if the number of join-irreducible H-subspaces of V is finite. If this is not the case, then the several expressions for a given H-subspace all contain the same number of elements. V itself is decomposable as a direct sum of indecomposable H-subspaces in a manner which is always unique up to H-isomorphism, and which is absolutely unique when V has only a finite number of join-irreducible H-subspaces.

The relationships between the various H-subspaces of a given H-space V are most naturally considered within the framework of lattice theory. Under the operations of intersection $\cap$ and join $+$, these subspaces form a modular lattice of finite length. This lattice is distributive if and only if it contains only a finite number of join-irreducible elements, and it is complemented if and only if V is completely reducible. An irreducible subspace of V is an atom of the lattice. The statements made above follow from standard results of lattice theory, such as may be found in Birkhoff [9].

The above remarks all concern the relationships connecting invariant subspaces under the join operation. When subspaces are specified by linear relations, such as the field equations that are being sought, the natural way to combine two subspaces is by uniting the sets of linear equations which specify them. This corresponds to forming the intersection, rather than the join, of the corresponding subspaces. Since the lattice properties of being modular, distributive and complemented are self-dual, the duals of the above results all hold, with join-irreducibility being replaced by intersection-irreducibility, etc. Alternatively, use may be made of the correspondence between subspaces of V and of its dual space V^*. It is this method that will be adopted below. If U is a subspace of V, the subspace

$$j(U) := \{f;\, f \in V^*,\, f(u) = 0 \text{ for all } u \in U\} \tag{4.1}$$

of V^* is said to be *incident* with U. If U is invariant under a group H, then so is $j(U)$ under the contragredient action of H on V^*. If U, $W \subset V$, the incidence relation satisfies

$$j(U \cap W) = j(U) + j(W)\,. \tag{4.2}$$

Intersection-irreducibility in V thus corresponds to join-irreducibility in V^*, and so the more usual terminology may be retained by studying V^* rather than V.

Some specific features of the Galilean group G will be studied next. If the element (2.6) of G is denoted by (L, m), then the subgroups of elements having $m = 0$ and $L = I$ will be denoted by S and T respectively. S is isomorphic to $O(3)$, while T is the subgroup of translations. Any irreducible representation ϱ of S generates an irreducible representation $\bar{\varrho}$ of G by

$$\bar{\varrho}(L, m) = \varrho(L)\,, \tag{4.3}$$

and all the irreducible tensorial representations of G have this form. The complete reducibility of all representations of S provides the following method for determining the join-irreducible subspaces of any G-space V. Its irreducible S-subspaces are first determined. The action of G on each of these then generates a G-subspace. Different S-subspaces may generate the same G-subspace, and the G-subspaces so obtained may not all be join-irreducible. However, all the join-irreducible G-subspaces will be among the set of spaces so generated, and so it remains only to discard those of this set which can be expressed as the join of other members of the set.

To carry out this process for the V_R and V_T of Section 2, it is necessary to have a systematic labelling for the irreducible tensor representations of the groups $GL(4)$, $GL(3)$ and $O(3)$. The method that has been developed by Weyl [8] will be used here. This assigns to any such representation of either $GL(n)$ or $O(n)$ a *signature* $[f_1, \ldots, f_n]$ which consists of a non-increasing sequence of n integers. Any zeroes in the sequence are normally omitted. All such sequences describe possible representations of $GL(n)$, but there are certain restrictions for $O(n)$, one of which is that $f_n \geqq 0$. The notation is an extension of the Young diagram method for describing tensor symmetries, in which an irreducible symmetry of m tensor indices is labelled by a partition of m. For full details, reference should be made to Weyl [8], but the main features for $GL(n)$ are as follows. If $f_n \geqq 0$, the representation corresponds to a covariant tensor with Σf_i indices having the irreducible symmetry given by the partition $(f_1, \ldots, f_n)$. If $f_1 \leqq 0$, it corresponds to a contravariant tensor with $-\Sigma f_i$ indices whose symmetry is described by the partition $(-f_n, \ldots, -f_1)$. If both positive and negative integers occur in the signature, the tensor is a mixed one which is tracefree between any contravariant and any covariant index.

For $O(3)$, the only allowed signatures are $[s]$ for $s \geqq 0$, $[s, 1]$ for $s \geqq 1$, and $[1, 1, 1]$. The representations given by $[s]$ and $[s, 1]$, $s \geqq 1$, both have spin s in the language of quantum theory. They are given respectively by a tensor and pseudotensor which has s indices, is totally symmetric, and is tracefree on all index pairs. The scalar and pseudoscalar representations are $[0]$ and $[1, 1, 1]$ respectively.

It should be noted that a given representation can have several different tensorial forms, so that the above descriptions are not unique. To distinguish between the various groups that will be used, signatures for $GL(4)$, $GL(3)$ and $O(3)$ will be denoted by $\langle\ \rangle$, $\{\ \}$ and $[\ ]$ respectively.

5. Decomposition of the Curvature Tensor

The representation of $GL(4)$ provided by the curvature tensor may be specialized to give one of $GL(3)$ by restricting consideration to those transformations which preserve (3.9). The first step in the scheme outlined above is to decompose this representation into its irreducible constituents. Of the properties of the curvature tensor deduced in Section 3, those invariant under $GL(3)$ are (3.3), the first of the conditions (3.4), and (3.12). For the moment the last of these will be ignored. The corresponding $GL(3)$-space is then the tensor product of the space of covariant tensors $u_{\alpha\beta\gamma}$ satisfying

$$u_{(\alpha\beta)\gamma} = 0\,, \qquad u_{[\alpha\beta\gamma]} = 0\,, \tag{5.1}$$

with that of contravariant vectors v^α satisfying

$$v^\alpha t_\alpha = 0\,. \tag{5.2}$$

The first of these is an irreducible $GL(4)$-space of signature $\langle 2, 1\rangle$, which decomposes under $GL(3)$ into the direct sum $\{2, 1\} \oplus \{2\} \oplus \{1, 1\} \oplus \{1\}$. The second is already irreducible under $GL(3)$, with signature $\{-1\}$. Because of the tensor product decompositions

$$\left.\begin{aligned} \{2, 1\} \otimes \{-1\} &= \{2, 1, -1\} \oplus \{2\} \oplus \{1, 1\}\,, \\ \{2\} \otimes \{-1\} &= \{2, -1\} \oplus \{1\}\,, \\ \{1, 1\} \otimes \{-1\} &= \{1, 1, -1\} \oplus \{1\}\,, \\ \{1\} \otimes \{-1\} &= \{1, -1\} \oplus \{0\}\,, \end{aligned}\right\} \tag{5.3}$$

the curvature tensor would thus decompose into nine irreducible constituents under $GL(3)$ were it not for the condition (3.12) which must still be taken into account. Now $R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\gamma}$ is irreducible under $GL(4)$ with signature $\langle 1,1\rangle$, which decomposes under $GL(3)$ into $\{1,1\}\oplus\{1\}$. The effect of (3.12) is thus to make the $\{1,1\}$ part of $R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}$ vanish identically and to impose a linear relation between the two parts of signature $\{1\}$, so that one of them may be dropped from the decomposition. The remaining seven parts may be labelled uniquely by their signatures and taken as

$$\left.\begin{aligned}
&\{2,1,-1\}: B^{\kappa\lambda\mu}_{\alpha\beta\gamma}(R_{\kappa\lambda\mu}^{\cdot\cdot\cdot\delta} - A^{\delta}_{[\kappa}R_{\lambda]\mu})\\
&\{1,1,-1\}: B^{\kappa\lambda}_{\alpha\beta}n^{\mu}(R_{\kappa\lambda\mu}^{\cdot\cdot\cdot\gamma} - A^{\gamma}_{[\kappa}R_{\lambda]\mu})\\
&\{2,-1\}: B^{\kappa\lambda}_{\alpha\beta}n^{\mu}(R_{\mu(\kappa\lambda)}^{\cdot\cdot\cdot\gamma} + \tfrac{1}{4}R_{\mu(\kappa}A^{\gamma}_{\lambda)})\\
&\{1,-1\}: B^{\kappa}_{\alpha}n^{\lambda}n^{\mu}(R_{\lambda\kappa\mu}^{\cdot\cdot\cdot\beta} + \tfrac{1}{3}R_{\lambda\mu}A^{\beta}_{\kappa})\\
&\{2\}: B^{\kappa\lambda}_{\alpha\beta}R_{\kappa\lambda} \qquad \{1\}: B^{\kappa}_{\alpha}n^{\lambda}R_{\kappa\lambda}\,, \qquad \{0\}: n^{\kappa}n^{\lambda}R_{\kappa\lambda}\,,
\end{aligned}\right\}\tag{5.4}$$

where $B^{\kappa\lambda}_{\alpha\beta} := B^{\kappa}_{\alpha}B^{\lambda}_{\beta}$, etc. The Ricci tensor terms occurring in the first four of these have been determined by the zero trace condition.

The next step is to determine what further decomposition is possible when the allowed transformations are restricted still further, to those preserving both (1.4) and (3.9). The group then specializes from $GL(3)$ to $O(3)$. Before this is considered explicitly, it should be noted that this introduction of $g^{\alpha\beta}$ enables a simplification to be made in the table (5.4). This arises from equation (3.6), which shows that the projection of a covariant index orthogonally to n^{κ} using B^{κ}_{α} is equivalent to the raising of that index with $g^{\kappa\alpha}$.

The only identity of the curvature tensor that has not yet been fully taken into account is the second of the conditions (3.4). This too needs consideration before the tensors (5.4) are decomposed further. It has been allowed for in part since it was needed in the proof of (3.12). In order to see what further restrictions it places on the curvature tensor, it is thus necessary to decompose it into its parts irreducible under $O(3)$. The tensor $R_{\alpha\beta}^{\cdot\cdot(\gamma\delta)}$ belongs to the tensor product of the space of skew tensors $u_{\alpha\beta}$ with the space of symmetric tensors $v^{\alpha\beta}$ satisfying $v^{\alpha\beta}t_{\beta}=0$. The former is irreducible under $GL(4)$ with signature $\langle 1,1\rangle$, which decomposes under $O(3)$ into $[1,1]\oplus[1]$. The latter is irreducible under $GL(3)$ with signature $\{-2\}$, which decomposes under $O(3)$ into $[2]\oplus[0]$. The $[0]$ part of $v^{\alpha\beta}$, namely $h_{\alpha\beta}v^{\alpha\beta}$, corresponds to that part of $R_{\alpha\beta}^{\cdot\cdot(\gamma\delta)}$ whose vanishing has already been accounted for in (3.12). The part of (3.4) that remains thus corresponds to the representations of $O(3)$ given by

$$\left.\begin{aligned}
&[1,1]\otimes[2]=[3,1]\oplus[2]\oplus[1,1]\\
\text{and}&\\
&[1]\otimes[2]=[3]\oplus[2,1]\oplus[1]\,.
\end{aligned}\right\}\tag{5.5}$$

The further decomposition of (5.4) under $O(3)$ will be performed in the reverse order to the listing in that table. The $\{0\}$ and $\{1\}$ tensors are already irreducible under $O(3)$ and have the same signatures under that group. The $\{2\}$ term is essentially $R^{\alpha\beta}$, as discussed above, and this decomposes into the $S^{\alpha\beta}$ and R of (3.15) and (3.14) whose signatures under $O(3)$ are [2] and [0] respectively. The $\{1,-1\}$ term, which may now be taken as

$$n^\lambda n^\mu (R_{\lambda\cdot\mu}^{\cdot\alpha\cdot\beta} + \tfrac{1}{3} R_{\lambda\mu} g^{\alpha\beta}),$$

separates into its symmetric and antisymmetric parts of signatures [2] and [1, 1] respectively under $O(3)$.

If no further restrictions were imposed, the remaining three tensors would decompose thus:

$$\{2,-1\} \to [3] \oplus [2,1] \oplus [1], \tag{5.6}$$

$$\{1,1,-1\} \to [2,1] \oplus [1,1,1], \tag{5.7}$$

$$\{2,1,-1\} \to [3,1] \oplus [2] \oplus [1,1]. \tag{5.8}$$

However, six tensors irreducible under $O(3)$, whose signatures are given by (5.5), are constrained to vanish. It may be shown that these include the [3] part of (5.6) and the [3, 1] and [1, 1] parts of (5.8). The remaining three constraints show that the [1] part of (5.6) repeats the $\{1\}$ term, the [2, 1] part of (5.7) repeats that of (5.6), and the [2] part of (5.8) repeats $S^{\alpha\beta}$. Hence only two new tensors arise from (5.6) to (5.8). These are the [2, 1] part of (5.6) and the [1, 1, 1] part of (5.7), which may be taken as

$$n^\kappa (R_\kappa^{\cdot(\alpha\beta)\gamma} + \tfrac{1}{2} R_\kappa^{\cdot(\alpha} g^{\beta)\gamma} - \tfrac{1}{2} g^{\alpha\beta} R_\kappa^{\cdot\gamma}) \tag{5.9}$$

and

$$n^\kappa R_\kappa^{\cdot[\alpha\beta\gamma]} \tag{5.10}$$

respectively.

The next step is to form the subspaces of V_R generated by the action of G on its irreducible S-subspaces. But before this is done, some thought should be given as to the precise nature of the tensors such as (5.9). They have been treated above as irreducible S-subspaces of V_R, but more specifically they are homomorphic images of V_R isomorphic to such irreducible S-subspaces. Since such an image cannot be identified with a unique subspace of V_R, the scheme of Section 4 cannot be applied directly. However, it can be identified with a unique subspace U^* of V_R^*, namely that subspace incident with the null space of the homomorphism, as defined by (4.1). The corresponding tensor homomorphic image will be said to *describe* U^*. The scheme will thus be applied to V_R^*, rather than to V_R itself.

The group G preserves (1.4) but acts transitively on the set of all n^α satisfying (2.5). Since (2.5) is a normalization condition, the G-subspace of V_R^* generated by any irreducible S-subspace U^* may thus be found as follows. First, express the tensor describing U^* as a homogeneous polynomial in n^α, say of degree p. This removes the need for n^α to be normalized, and may always be done by the appropriate addition of factors $n^\alpha t_\alpha \equiv 1$. Then take the tensor coefficient of the n's in this expression, symmetrizing over the p covariant indices that were contracted with n's. The subspace of V_R^* described by this coefficient tensor is the desired subspace GU^*.

This has to be applied to *every* irreducible S-subspace. When the direct sum decomposition of the whole space contains two or more equivalent S-subspaces, the general S-subspace of this signature is described by a general linear combination of the tensors describing those occurring in this decomposition. For V_R, this is the situation for the signatures [2] and [0]. The case of signature [0] will be treated as an example; the conclusion for signature [2] is similar but involves more tedious algebra. The tensors concerned are $n^\alpha n^\beta R_{\alpha\beta}$ and R. The general linear combination, made homogeneous in n^α, is

$$(aR_{\alpha\beta}+bRt_\alpha t_\beta)n^\alpha n^\beta\,, \tag{5.11}$$

for which the coefficient tensor is

$$aR_{\alpha\beta}+bRt_\alpha t_\beta\,. \tag{5.12}$$

On contraction with $g^{\alpha\beta}$, this yields aR. Hence if $a\neq 0$, R may be eliminated from (5.12) by a nonsingular transformation to give simply $R_{\alpha\beta}$. If $a=0$ but $b\neq 0$, it similarly gives just R. The infinite number of irreducible S-subspaces described by (5.11) thus yield only two distinct G-subspaces, one of which contains the other.

All the G-subspaces obtained by this procedure are found to be join-irreducible. There are eight of them, described by the following tensors:

$$\left.\begin{array}{l}
A\,[2]R_{(\gamma}{}^{\cdot(\alpha}{}_{\cdot\delta)}{}^{\beta)}+\frac{1}{3}\,R_{\gamma\delta}g^{\alpha\beta}\\
B\,[0]R_{\alpha\beta}\\
C\,[2,1]R_\alpha{}^{\cdot(\beta\gamma)\delta}+\frac{1}{2}\,R_\alpha{}^{\cdot(\beta}g^{\gamma)\delta}-\frac{1}{2}\,R_\alpha{}^{\cdot\delta}g^{\beta\gamma}\\
D\,[1]R_\alpha{}^{\cdot\beta}\\
E\,[2]R^{\alpha\beta}-\frac{1}{3}\,Rg^{\alpha\beta}\\
F\,[0]R\\
G\,[1,1]R_{(\gamma}{}^{\cdot[\alpha}{}_{\cdot\delta)}{}^{\beta]}\\
H[1,1,1]R_\alpha{}^{\cdot[\beta\gamma\delta]}\,.
\end{array}\right\} \tag{5.13}$$

They are here labelled and preceded by the signature of the irreducible S-subspace which generates them. They form a partially ordered set under the inclusion relation for subspaces of V_R^*, whose Hasse diagram may be shown to be

$$\begin{tikzcd}
A \arrow[d, dash] \arrow[dr, dash] & B \arrow[d, dash] & G \arrow[d, dash] \\
C \arrow[d, dash] & D \arrow[d, dash] \arrow[dl, dash] & H \\
E & F &
\end{tikzcd} \tag{5.14}$$

In this, two elements a and b are comparable, with $a\supset b$, if and only if a is connected to b by a chain of descending lines. Thus $A\supset F$ but C and F are not comparable. Since the join-irreducible subspaces are finite in number, the lattice of all G-subspaces of V_R^* is distributive. V_R^* itself has a unique decomposition as a direct sum of the indecomposable subspaces $A+B$ and G, which are described by the tensors $R_{(\gamma}{}^{\cdot(\alpha}{}_{\cdot\delta)}{}^{\beta)}$ and $R_{(\gamma}{}^{\cdot[\alpha}{}_{\cdot\delta)}{}^{\beta]}$ respectively. This implies the surprising result that $R_{(\gamma}{}^{\cdot\alpha}{}_{\cdot\delta)}{}^{\beta}$ determines $R_{\alpha\beta\gamma}{}^{\cdot\cdot\cdot\delta}$ completely, which can indeed be verified explicitly. Note that (3.3) and (3.4) together imply

$$R_{[\gamma}{}^{\cdot(\alpha\cdot}{}_{\cdot\delta]}{}^{\beta)}=0\,, \tag{5.15}$$

so that the symmetrizing brackets around γ and δ in tensor A may be omitted.

6. Decomposition of the Mass-Momentum Tensor

The same technique must now be applied to $T^{\alpha\beta}$, to determine the join-irreducible G-subspaces of V_T^*. This is irreducible under $GL(4)$ with signature $\langle -2\rangle$, and so decomposes under restriction to $GL(3)$ into

$$\{-2\}\oplus\{-1\}\oplus\{0\}\,.$$

Its three constituents may be taken as

$$\{-2\}:B^{\alpha\beta}_{\kappa\lambda}T^{\kappa\lambda}\,,\qquad \{-1\}:B^{\alpha}_{\kappa}t_\lambda T^{\kappa\lambda}\,,\qquad \{0\}:t_\kappa t_\lambda T^{\kappa\lambda}\,. \tag{6.1}$$

Under the further reduction to $O(3)$, the $\{0\}$ and $\{-1\}$ tensors remain irreducible but acquire the signatures $[0]$ and $[1]$ respectively, while the $\{-2\}$ tensor splits into the two parts

$$[0]:h_{\kappa\lambda}T^{\kappa\lambda}\quad\text{and}\quad [2]:(B^{\alpha\beta}_{\kappa\lambda}-\tfrac{1}{3}\,g^{\alpha\beta}h_{\kappa\lambda})T^{\kappa\lambda}\,. \tag{6.2}$$

These must next be written as homogeneous polynominals in n^κ. It follows from (2.5) and (3.5) that

$$B^{\alpha}_{\kappa}t_\lambda T^{\kappa\lambda}=2n^\beta A^{\alpha}_{[\kappa}t_{\beta]}t_\lambda T^{\kappa\lambda}\,, \tag{6.3}$$

and from (3.7) that

$$h_{\kappa\lambda}T^{\kappa\lambda}=\tfrac{1}{2}\,U_{\alpha\beta}n^\alpha n^\beta\,, \tag{6.4}$$

where

$$U_{\alpha\beta}:=\eta_{\alpha\kappa\lambda\mu}\eta_{\beta\nu\varrho\sigma}g^{\kappa\nu}g^{\lambda\varrho}T^{\mu\sigma}\,. \tag{6.5}$$

To treat the second tensor of (6.2) it is first rewritten as

$$(2g^{\alpha\kappa}g^{\beta\lambda}h_{\kappa[\mu}h_{\lambda]\nu}+\tfrac{2}{3}\,g^{\alpha\beta}h_{\mu\nu})T^{\mu\nu}\,. \tag{6.6}$$

With the use of (3.7) and (3.8), this may be expressed as $\frac{1}{2}\,V^{\alpha\beta}_{\cdot\cdot\gamma\delta}n^\gamma n^\delta$, where

$$V^{\alpha\beta}_{\cdot\cdot\gamma\delta}:=\eta_{\gamma\kappa\lambda\mu}\eta_{\delta\nu\varrho\sigma}g^{\lambda\varrho}T^{\mu\sigma}(\tfrac{2}{3}\,g^{\alpha\beta}g^{\kappa\nu}-2g^{\alpha(\kappa}g^{\nu)\beta})\,. \tag{6.7}$$

It may be shown from (3.10) and (6.5) that

$$U^{\alpha}_{\cdot\beta}=4A^{\alpha}_{[\beta}t_{\mu]}t_\nu T^{\mu\nu}\,,\qquad U^{\alpha\beta}=2\varrho g^{\alpha\beta}\,,\qquad U^{\alpha}_{\cdot\alpha}=6\varrho\,, \tag{6.8}$$

where the mass density ϱ is defined by

$$\varrho:=T^{\alpha\beta}t_\alpha t_\beta\,. \tag{6.9}$$

It also follows from (6.5) and (6.7) that

$$\left.\begin{aligned} &V^{\alpha\beta}_{\cdot\cdot\gamma\delta}=V^{(\alpha\beta)}_{\cdot\cdot(\gamma\delta)}\,,\qquad t_\alpha V^{\alpha\beta}_{\cdot\cdot\gamma\delta}=0\,,\qquad g^{\gamma\delta}V^{\alpha\beta}_{\cdot\cdot\gamma\delta}=0\,,\\ &h_{\alpha\beta}V^{\alpha\beta}_{\cdot\cdot\gamma\delta}=0\,,\qquad V^{\alpha\beta\gamma}_{\cdot\cdot\cdot\delta}=g^{\gamma(\alpha}U^{\beta)}_{\cdot\delta}-\tfrac{1}{3}\,g^{\alpha\beta}U^{\gamma}_{\cdot\delta}\,.\end{aligned}\right\} \tag{6.10}$$

The join-irreducible subspaces of V_T^* are thus described, in the notation of (5.13), by the tensors

$$\left.\begin{array}{ll} J[2]V^{\alpha\beta}_{\cdot\cdot\gamma\delta}\,, & K[0]U_{\alpha\beta}\,, \\ L[1]U^{\alpha}_{\cdot\beta}\,, & M[0]\varrho\,, \end{array}\right\} \tag{6.11}$$

and their Hasse diagram is

$$\begin{tikzcd} J \arrow[dr, dash] & & K \arrow[dl, dash] \\ & L \arrow[d, dash] & \\ & M\,. & \end{tikzcd} \tag{6.12}$$

V_T^* is indecomposable, and its lattice of G-subspaces is distributive.

For completeness, the single trivial join-irreducible subspace of R^* will be written in this notation as

$$N[0]\theta\,. \tag{6.13}$$

7. Possible Field Equations

It is now a simple exercise to repeat the procedure for $V_R^*\oplus V_T^*\oplus R^*$, as a decomposition into a direct sum of irreducible S-subspaces follows immediately from the corresponding decompositions of V_R^*, V_T^* and R^* separately. The intersection-irreducible G-subspaces of $V_R\oplus V_T\oplus R$ are then given by equating to zero one of the homomorphic image tensors thus obtained, and every other G-subspace is expressible by the vanishing of several of these tensors. The discussion of Section 2 shows that the equations so obtained, when augmented by (2.7), contain all possible gravitational field equations. However, to these must be applied the consistency and acceptability criteria of the surjectivity of the projections $V_S\to R$ and $V_S\to V_T$. Eight individual equations are left after this selection. They express the eight tensors A to H of (5.13) in terms of the five tensors J to N of (6.11) and (6.13). If (5.15) is used to simplify tensor A, they are found to be

$$\left.\begin{array}{lll} (1) & (A=J) & R_{\gamma\cdot\delta}^{\cdot(\alpha\cdot\beta)}+\frac{1}{3}R_{\gamma\delta}g^{\alpha\beta}=aV^{\alpha\beta}_{\cdot\cdot\gamma\delta} \\ (2) & (B=K+M+N) & R_{\alpha\beta}=2aU_{\alpha\beta}+(b\varrho+c)t_\alpha t_\beta \\ (3) & (D=L) & R^{\alpha}_{\cdot\beta}=2aU^{\alpha}_{\cdot\beta} \\ (4) & (F=M+N) & R=12a\varrho+d\,, \end{array}\right\} \tag{7.1}$$

together with the vanishing of C, E, G and H. In these, a, b, c and d are absolute constants, and a symbolic form has also been given to indicate the tensors involved.

If several of these equations are taken together, the surjectivity criteria require that the constant a be the same in all those taken from the set (7.1), and that if (4) is included then $d=0$. With these conditions, it follows from (6.8) and (6.10) that the values given for A to H are mutually consistent, and so the relationships between the various equations may be read off from the Hasse diagram (5.14).

At least one of the four equations (7.1) must hold, as otherwise there would be no coupling between the matter and the spacetime geometry. However, the contracted Bianchi identity

$$\nabla_\beta(R^{\alpha\beta} - \tfrac{1}{2} R g^{\alpha\beta}) = 0 \tag{7.2}$$

is still valid in a Newtonian spacetime. It shows that if $E=0$ then $\nabla^\alpha R = 0$, so that R is constant on each hypersurface of constant t, t being defined by (2.1). This is incompatible with equation (4) of (7.1) unless $a=0$, as the mass density ϱ cannot be constrained to be purely a function of time. But it may be seen from (5.14) that both of these equations are implied by each of (1), (2) and (3) of (7.1), which are thus physically acceptable only if $a=0$. Since (1) and (3) do not then couple the matter to the geometry, one of (2) and (4) must hold. If (2) does not hold, then (4) must have $a \neq 0$, and so is inconsistent with any equation that implies $E=0$. This leaves $G=0$ or $H=0$ as the only equations that could supplement (4). Of these two, $G=0$ is the stronger. It contains just four linearly independent conditions on the components of $R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\delta}$, calculated as the sum of the dimensions of the representations of $O(3)$ into which it decomposes. The strongest theory in which (2) does not hold is thus given by

$$R = 12a\varrho + d\,, \qquad R_{\alpha\cdot\gamma}^{\cdot\beta\cdot\delta} = R_{\gamma\cdot\alpha}^{\cdot\delta\cdot\beta}\,, \tag{7.3}$$

where (5.15) has been used to simplify the condition $G=0$. With only five constraints on the curvature tensor, it is too weak to form a viable gravitational theory.

This leaves (2) with $a=0$ as the only possible equation involving $T^{\alpha\beta}$. It is algebraically consistent to supplement it with the vanishing of any of A, C, G and H, of which only two are independent since $A \supset C$ and $G \supset H$. It will now be shown that despite this algebraic consistency, $A=0$ is not acceptable as it implies a differential restriction on ϱ similar to, but of a higher order than, the restriction $\nabla^\alpha \varrho = 0$ considered above.

8. A Differential Inconsistency

Any tensor that is totally antisymmetric on three contravariant indices, and is orthogonal to t_α on each of them, must be a multiple of $\eta^{\beta\gamma\delta\varepsilon} t_\varepsilon$. There thus exists a vector field v_α such that

$$R_\alpha^{\cdot[\beta\gamma\delta]} = v_\alpha t_\varepsilon \eta^{\beta\gamma\delta\varepsilon}\,. \tag{8.1}$$

But it follows from (3.3) and (3.4) that

$$R^{\alpha[\beta\gamma\delta]} = 0 \tag{8.2}$$

identically. Hence $v_\alpha = \phi t_\alpha$ for some scalar field ϕ, which gives

$$R_\alpha^{\cdot[\beta\gamma\delta]} = \phi t_\alpha t_\varepsilon \eta^{\beta\gamma\delta\varepsilon}\,. \tag{8.3}$$

The field equation finally adopted in the preceding section is

$$R_{\alpha\beta} = (b\varrho + c) t_\alpha t_\beta\,, \tag{8.4}$$

with $b \neq 0$. It will be shown that this and $C=0$ together imply $\phi=\phi(t)$. As (8.4) implies $D=0$, it follows from (5.13) that $C=0$ is equivalent to

$$R_{\alpha}^{\cdot(\beta\gamma)\delta}=0. \tag{8.5}$$

When this holds, it may be seen from (3.3) and (3.4) that

$$R_{\alpha}^{\cdot\beta\gamma\delta}=-\tfrac{1}{2}R^{\beta\gamma\cdot\delta}_{\cdot\cdot\alpha}=R_{\alpha}^{\cdot[\beta\gamma\delta]}. \tag{8.6}$$

But if the Bianchi identity

$$\nabla_{[\alpha}R_{\beta\gamma]\delta}^{\cdot\cdot\cdot\varepsilon}=0 \tag{8.7}$$

is contracted on α and ε, the result may be combined with $D=0$ to give

$$\nabla_{\alpha}R^{\beta\gamma\cdot\alpha}_{\cdot\cdot\delta}=0. \tag{8.8}$$

Substitution into this from (8.6) and (8.3) yields

$$t_{[\alpha}\nabla_{\beta]}\phi=0, \tag{8.9}$$

from which the required result follows immediately.

Since $A=0$ implies $C=0$, this result also holds when $A=0$. But in this case one can go further. It may be deduced from (8.6) and (8.7) that

$$\nabla^{[\alpha}R_{\delta\cdot\varepsilon}^{\cdot\beta]\cdot\gamma}+\nabla_{\delta}R_{\varepsilon}^{\cdot[\alpha\beta\gamma]}=0. \tag{8.10}$$

If (8.10) is combined with both (5.15) and $A=0$, it implies that

$$\nabla^{\alpha}R_{\delta\cdot\varepsilon}^{\cdot[\beta\cdot\gamma]}=\tfrac{2}{3}g^{\alpha[\beta}\nabla^{\gamma]}R_{\delta\varepsilon}-\nabla_{\delta}R_{\varepsilon}^{\cdot[\alpha\beta\gamma]}, \tag{8.11}$$

which may be simplified with the use of (8.3) and (8.4) to give

$$\nabla^{\alpha}R_{\delta\cdot\varepsilon}^{\cdot[\beta\cdot\gamma]}=t_{\delta}t_{\varepsilon}(\tfrac{2}{3}bg^{\alpha[\beta}\nabla^{\gamma]}\varrho-\phi'\eta^{\alpha\beta\gamma\lambda}t_{\lambda}), \tag{8.12}$$

where $\phi' := d\phi/dt$. This must now be operated on with ∇^{ζ} and the result antisymmetrized on ζ and α. Use of the Ricci identity, (8.6), (8.3) and (3.10) enables the left hand side to be simplified thus:

$$\nabla^{[\zeta\alpha]}R_{\delta\cdot\varepsilon}^{\cdot[\beta\cdot\gamma]}=6\phi^{2}t_{\delta}t_{\varepsilon}g^{\alpha[\beta}g^{\gamma]\zeta}. \tag{8.13}$$

Contraction of the resulting equation with $h_{\alpha\beta}h_{\gamma\zeta}$ then yields

$$bg^{\alpha\beta}\nabla_{\alpha\beta}\varrho=27\phi^{2}(t). \tag{8.14}$$

As $b \neq 0$ and the right hand side of (8.14) depends only on t, this is a physically unacceptable constraint on the mass density. The equation $A=0$ is thus physically inconsistent with (8.4), as claimed above.

9. Summary and Conclusions

This paper has followed up the consequences of the joint adoption of two main hypotheses. These are that gravitation is a manifestation of spacetime geometry, and that Newtonian mechanics is valid in the absence of gravitation. To them has been added only the same slight extra assumptions as are needed to set up the field equations of general relativity from similar hypotheses. It has been shown that the only acceptable field equation consistent with these hypotheses which couples the matter to the geometry is (8.4), which with a relabelling of the constants is

$$R_{\alpha\beta} = (4\pi G\varrho - \Lambda) t_\alpha t_\beta \,. \tag{9.1}$$

There are also three constraints which may be imposed on the curvature tensor in addition to (9.1) and consistently with it. In the notation of (5.13) these are the vanishing of the tensors C, G and H. The last of these gives

$$R_\alpha^{\cdot[\beta\gamma\delta]} = 0 \,. \tag{9.2}$$

The equation $G=0$ can be simplified, with the use of (5.15), to

$$R_{\alpha\cdot\gamma}^{\cdot\beta\cdot\delta} = R_{\gamma\cdot\alpha}^{\cdot\delta\cdot\beta} \,. \tag{9.3}$$

Finally, $C=0$ can be simplified, when (9.1) holds, to the form (8.5). With the use of (3.4), this can be expressed as

$$R_\alpha^{\cdot\beta\gamma\delta} = R_\alpha^{\cdot[\beta\gamma\delta]} \,. \tag{9.4}$$

Of these equations, (9.3) implies (9.2) but (9.4) is independent of them both. Equations (9.2) and (9.4) together are equivalent to

$$R_{\alpha\beta}^{\cdot\cdot\gamma\delta} = 0 \,. \tag{9.5}$$

To see this, observe first that (3.5) and (3.6) imply

$$R_{\alpha\beta}^{\cdot\cdot\gamma\delta} = (h_{\alpha\kappa} g^{\varepsilon\kappa} + t_\alpha n^\varepsilon)(h_{\beta\lambda} g^{\zeta\lambda} + t_\beta n^\zeta) R_{\varepsilon\zeta}^{\cdot\cdot\gamma\delta} \,. \tag{9.6}$$

The required result follows on expansion of the brackets with the use of (3.3). The Eqs. (9.1) to (9.4) taken together are thus equivalent to (9.1) with (9.3) and (9.5). If $\Lambda=0$, this gives the set (1.6). A proof that (1.6) is equivalent to the usual formulation of the Newtonian theory may be found in the references mentioned earlier. The constant Λ is analogous to the cosmological constant of general relativity. If it is nonzero, then the Poisson equation of the Newtonian theory is generalized to

$$\nabla^2 \phi = 4\pi G\varrho - \Lambda \,. \tag{9.7}$$

The justification of the claims made in the Introduction is complete.

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