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Inner*-Automorphisms of Simple C^*-Algebras

Dorte Olesen
Odense University, DK-5000 Odense, Denmark

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Introduction

Given a locally compact abelian group G acting as *-automorphisms α_g on a factor, Connes ([3]) defines a certain subgroup $\Gamma(\alpha)$ of the dual group Γ of G. He shows that under suitable conditions the annihilator of $\Gamma(\alpha)$ is precisely the subgroup of $h \in G$ for which the automorphism α_h is implemented by a unitary element in the centre of the fixed-point algebra of the group. As a corollary it is proved that if a single *-automorphism α has a spectrum (as a bounded operator on the factor) which is not the entire unit circle, then a power of α is inner.

In [2], Borchers proves that on any von Neumann algebra a *-automorphism with a gap in its spectrum has a power which is inner.

Here we generalize the notion of $\Gamma(\alpha)$ to representations of G as *-automorphisms acting on an arbitrary C^*-algebra. We show in 2 that $\Gamma(\alpha)$ is a closed subgroup of Γ, which satisfies $\Gamma(\alpha)+\mathrm{sp}\alpha \subseteqq \mathrm{sp}\alpha$.

In Section 3 we see that for primitive C^*-algebras, the spectra of restricted actions α^B on non-zero α-invariant hereditary C^*-subalgebras B form an approximately filtering family of sets (in a sense made precise in 3.4). The methods of [3] are then applicable to simple C^*-algebras, and we show in 4 that for suitable groups the annihilator of $\Gamma(\alpha)$ is precisely the subgroup of $h \in G$ for which α_h is implemented by a unitary element in the centre of the fixed-point algebra of the bitransposed action on the multiplier algebra. From this it follows that a single *-automorphism with a gap in its spectrum has a power which is given by a multiplier.

Studying a single *-automorphism α on a C^*-algebra we show in Section 5 that the methods of [2] may be generalized to give the result that if $\sigma(\alpha)$ has a gap, then some power α^n is the exponential of a derivation on a non-zero α-invariant hereditary C^*-subalgebra.

When A is a commutative C^*-algebra, this method of proof yields that α^n for a suitable n is the identity operator on A. In fact, it is noted that with slight modifications the arguments given carry over to the case where α is an isometric isomorphism of a commutative semi-simple Banach algebra. This result has earlier been proved in [6] and [7], using different methods.

In Section 6, we return to the group setting in the special case where A is a von Neumann algebra, and reach some generalizations of results obtained for factors in [3]. We also obtain the result in [2] that a *-automorphism with gap in its spectrum has a power which is inner.

The ideas that appear in this paper stem from the reading of the papers by Borchers ([2]) and Connes ([3]), as well as from fruitful conversations with G. K. Pedersen.

1. Notation and Preliminaries

In the following, A denotes a C^*-algebra, Z its centre.

A cone M in the positive part A_+ of A is called *hereditary* if for every y in M the relation $0 \leqq x \leqq y$ implies that $x \in M$. A C^*-subalgebra B of A is *hereditary* if its positive part B_+ is a hereditary cone in A_+.

Assume G to be a locally compact abelian group, 0 its unit element. Let α denote a homomorphism of G into the group aut A of *-automorphisms of A. Given the continuity condition

$$\|\alpha_g x - x\| \to 0 \quad \text{as} \quad g \to 0 \qquad \forall x \in A$$

we call (α, A) a *representation* of G.

The *spectrum* of (α, A) is defined as a subset of the dual group Γ of G:

$$\operatorname{sp}\alpha = \{\gamma \in \Gamma \mid \hat{f}(\gamma) = 0 \quad \forall f \in I\},$$

where $\hat{f}(\gamma) = \int_G f(g)(g, \gamma)dg$ and

$$I = \{f \in L^1(G) \mid \int f(g)\alpha_g x dg = 0 \; \forall x \in A\},$$

the A-valued integral $\alpha(f)x = \int f(g)\alpha_g x dg$ being well-defined on account of the continuity of $g \mapsto \alpha_g x$ (see [1, Proposition 1.4]).

When (α, A) is a representation and B an invariant subspace of A, the restriction of the action of α is denoted (α^B, B).

We define

$$\Gamma(\alpha) = \cap \operatorname{sp}\alpha^B,$$

where the intersection is taken over all non-zero α-invariant hereditary C^*-subalgebras B of A.

Let E be a subset of Γ, define *the spectral subspace*

$$R^\alpha(E) = [\{\alpha(f)x \mid \operatorname{supp}\hat{f} \subset E, x \in A\}],$$

[] denoting closed linear span. We use the symbol $\check{U}(\gamma)$ to denote the family of compact neighbourhoods of γ in Γ.

It is shown e.g. in [9, Proposition 2.4.1] that

$$\gamma \in \operatorname{sp}\alpha \Leftrightarrow R^\alpha(V) \neq \{0\} \qquad \forall V \in \check{U}(\gamma).$$

Assume E_1 and E_2 to be subsets of Γ. Then

$$R^\alpha(E_1)R^\alpha(E_2) \subseteqq R^\alpha(E_1 + E_2)$$

(see e.g. [9, Proposition 2.3.9]).

2. On the Rotation-Group of $\operatorname{sp}\alpha$

2.1. Definition. Let H be a closed subset of Γ.
By the rotation-semigroup of H we mean

$$S(H)=\{\gamma\in\Gamma \mid \gamma+H\subseteqq H\}=\bigcap_{\theta\in H}(H-\theta).$$

Clearly, $S(H)$ is a closed semigroup containing the unit element ω of Γ.

2.2. Lemma. *Let H be a closed symmetric subset of Γ, $\omega\in H$. Then $\omega\in S(H)\subseteqq H$, and $S(H)$ is a group.*

Proof. That $\omega\in H$ implies $S(H)\subseteqq H$ is obvious. To see that $S(H)$ is a group, it suffices to prove that it is symmetric. Let $\theta\in H$, then $-\theta\in H$ by symmetry, and so for a given $\gamma\in S(H)$ there exists $\eta\in H$ such that $\gamma=\eta+\theta$. From this we see that $-\gamma=-\eta-\theta\in H-\theta$. Since θ was arbitrary in H, this implies that $-\gamma\in S(H)$.

2.3. Proposition. *Let (α, A) be a representation of G. Then*

$$\Gamma(\alpha)\subseteqq S(\operatorname{sp}\alpha),$$

and $\Gamma(\alpha)$ is a closed subgroup of Γ.

Proof. Let $\gamma_1\in\operatorname{sp}\alpha$ and $\gamma_2\in\Gamma(\alpha)$. We want to see that

$$\gamma_1+\gamma_2\in\operatorname{sp}\alpha.$$

Let V, V_1, and V_2 be compact neighbourhoods of $\gamma_1+\gamma_2$, γ_1 and γ_2 respectively, such that $V_1+V_2\subset V$. Take $x_1\neq 0$, $x_1\in R^\alpha(V_1)$ and let B denote the smallest hereditary C^*-subalgebra containing the orbit $\{\alpha_g(x_1^*x_1)\mid g\in G\}$.

Then $B\neq\{0\}$, so by the definition of $\Gamma(\alpha)$ we can find $x_2\in B\cap R^\alpha(V_2)$, $x_2\neq 0$.

If $\alpha_g(x_1)x_2=0$ for every $g\in G$, then $bx_2=0$ for every b in B. But if $(u_\lambda)_\Lambda$ is an approximate unit for B, then $\|u_\lambda x_2-x_2\|\to 0$, whence $x_2=0$, a contradiction. So we can find $g_1\in G$ such that $\alpha_{g_1}(x_1)x_2\neq 0$, and as $\alpha_{g_1}(x_1)\in R^\alpha(V_1)$ and $x_2\in R^\alpha(V_2)$ this product belongs to $R^\alpha(V_1+V_2)\subset R^\alpha(V)$. It follows that $\gamma_1+\gamma_2\in\operatorname{sp}\alpha$. So we have shown that

$$\Gamma(\alpha)\subseteqq S(\operatorname{sp}\alpha).$$

The same relation obviously holds when we restrict the action to a non-zero α-invariant hereditary C^*-subalgebra B,

$$\Gamma(\alpha^B)\subseteqq S(\operatorname{sp}\alpha^B),$$

and as $\Gamma(\alpha)\subseteqq\Gamma(\alpha^B)$ we see that

$$\Gamma(\alpha)\subseteqq S(\operatorname{sp}\alpha^B),$$

thus

$$\Gamma(\alpha)\subseteqq S(\cap\operatorname{sp}\alpha^B)=S(\Gamma(\alpha))$$

so $\Gamma(\alpha)$ is a semigroup. Since $\Gamma(\alpha)$ is closed, symmetric and contains ω, it is a closed subgroup of Γ.

2.4. Remark. $\Gamma(\alpha)$ is often smaller than $S(\operatorname{sp}\alpha)$. In fact, let u be a unitary operator on the Hilbert space H with spectrum equal to the unit circle $\mathbb{T}$. The automorphism of $B(H)$ defined by

$$\alpha(x) = uxu^*$$

has spectrum equal to the unit circle $\mathbb{T}$ (see e.g. [3, Lemma 2.3.10 (a)]). Thus

$$S(\mathrm{sp}\alpha) = \mathbb{T}.$$

Obviously, u is in the centre of the fixed-point algebra of α, thus so is each of its spectral projections p. Let p be chosen so that

$$\|up - \lambda p\| < \varepsilon$$

then $\mathrm{sp}\alpha^{pAp} \subset \{z \mid |z| = 1 \text{ and } |z-1| < 2\varepsilon\}$ and so we see that $\Gamma(\alpha) = \{1\}$.

3. $\Gamma(\alpha)$ for Primitive C^*-Algebras

Recall that A is called primitive if $\{0\}$ is a primitive ideal. In this section, we want to see that $\Gamma(\alpha)$ for a primitive C^*-algebra can be viewed as the intersection of a filter-basis.

The following lemma, which is presumably well-known, is included for the sake of completeness.

3.1. Lemma. *Let A be a C^*-algebra, L a closed left ideal, R a closed right ideal, I_L (resp. I_R) the smallest closed 2-sided ideal containing L (resp. R).*

The following conditions are equivalent:

(i) $L \cap R = \{0\}$,

(ii) $I_L \cap I_R = \{0\}$.

Proof. Assume $I_L \cap I_R = \{0\}$, then trivially $L \cap R = \{0\}$.

Conversely, assume $L \cap R = \{0\}$, and set $\mathrm{An}(R) = \{x \in A \mid Rx = \{0\}\}$. $\mathrm{An}(R)$ is a closed ideal containing L, so $\mathrm{An}(R) \supseteq I_L$.

Let $\mathrm{An}(\mathrm{An}(R)) = \{x \in A \mid x\mathrm{An}(R) = \{0\}\}$, then $\mathrm{An}(\mathrm{An}(R)) \supseteq R$ and is a closed 2-sided ideal. Thus $I_R I_L = \{0\}$, from which it follows that $I_R \cap I_L = \{0\}$ (here we use that A is a C^*-algebra, thus $I_R I_L = I_R \cap I_L$).

3.2. Corollary. *Let L be a non-zero closed left ideal and R a non-zero closed right ideal of the primitive C^*-algebra A. Then $L \cap R \neq \{0\}$*

Proof. Immediate from 3.1, since no non-zero closed 2-sided ideals of a primitive algebra have zero intersection.

3.3. Lemma. *Let A be a primitive C^*-algebra, (α, A) a representation of G. Let A_1 and A_2 be non-zero α-invariant hereditary C^*-subalgebras of A, V a compact neighbourhood of ω in Γ.*

There exist non-zero α-invariant hereditary C^*-subalgebras $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ such that

$$\mathrm{sp}\alpha^{B_i} \subset V + \mathrm{sp}\alpha^{B_j}, \ i, j = 1, 2.$$

Proof. Let $L_i = \{x \in A \mid x^* x \in A_i\}$ for $i = 1, 2$.

It follows from 3.2 that we can find $x \neq 0$, $x \in L_1 \cap L_2^*$. Choose $f \in L^1(G)$ such that

$$\operatorname{supp} \hat{f} - \operatorname{supp} \hat{f} \subset V$$

and take $y = \alpha(f)x$. Then

$$y \in (L_1 \cap L_2^*) \cap R^\alpha(\operatorname{supp} \hat{f}) \ .$$

Let $L(\alpha_g(y))$ denote the smallest closed left ideal containing the orbit $\{\alpha_g(y)\}_{g \in G}$. Then

$$L(\alpha_g(y)) \subseteqq L_1 \quad \text{and} \quad L(\alpha_g(y^*)) \subseteqq L_2 \ .$$

Put

$$B_1 = L(\alpha_g(y)) \cap (L(\alpha_g(y)))^*$$

and

$$B_2 = L(\alpha_g(y^*)) \cap (L(\alpha_g(y^*)))^* \ ,$$

then for $i = 1, 2$

$$B_i \subseteqq L_i \cap L_i^* = A_i \ .$$

Take $\gamma \in \operatorname{sp}\alpha^{B_2}$, $V_2 \in \check{U}(\gamma)$ and $x_2 \in R^\alpha(V_2) \cap B_2$, $x_2 \neq 0$. As $x_2 \in B_2$ there is an approximate unit $(u_\lambda) \subset (B_2)_+ = (L(\alpha_g(y^*)))_+$ such that

$$\|x_2 u_\lambda - x_2\| \to 0$$

and if now $x_2 \alpha_g(y) = 0$ for all g in G we had that $u_\lambda x_2^* = 0$, thus $x_2^* = x_2 = 0$. So there exists g_2 in G such that $x_2 \alpha_{g_2}(y) \neq 0$.

From $x \in B_2$ it follows that

$$\|u_\lambda x_2 \alpha_{g_2}(y) - x_2 \alpha_{g_2}(y)\| \to 0 \ .$$

We see as above that this implies the existence of g_1 in G so

$$x_1 = \alpha_{g_1}(y^*) x_2 \alpha_{g_2}(y) \neq 0 \ .$$

Now inspection reveals that

$$x_1 \in L(\alpha_g(y)) \cap (L(\alpha_g(y)))^* = B_1$$

and

$$x_1 \in R^\alpha(-\operatorname{supp} \hat{f}) R^\alpha(V_2) R^\alpha(\operatorname{supp} \hat{f}) \subseteqq R^\alpha(V_2 - V)$$

which shows that

$$(V_2 - V) \cap \operatorname{sp}\alpha^{B_1} \neq \emptyset$$

i.e.

$$\gamma \in V + \operatorname{sp}\alpha^{B_1}$$

so we have

$$\operatorname{sp}\alpha^{B_2} \subseteqq V + \operatorname{sp}\alpha^{B_1} \ .$$

The other inclusion follows from symmetry.

3.4. Lemma. *Let A be a primitive C^*-algebra, (α, A) a representation of G. The family $F(\alpha)$ of subsets of Γ of the form*

$$V + \operatorname{sp}\alpha^B,$$

where V runs through the compact neighbourhoods $\check{U}(\omega)$ of ω in Γ and B through the non-zero α-invariant hereditary C^*-subalgebras of A form a filter-basis with intersection $\Gamma(\alpha)$.

Proof. That $\Gamma(\alpha) = \bigcap_{F(\alpha)}(V + \operatorname{sp}\alpha^B)$ is obvious. To see that $F(\alpha)$ is a filter-basis take two elements $V_1 + \operatorname{sp}\alpha^{A_1}$ and $V_2 + \operatorname{sp}\alpha^{A_2}$, and take $V \in \check{U}(\omega)$ such that

$$V \subset V_1 \quad \text{and} \quad V + V \subset V_2 .$$

Lemma 3.3 applied to A_1, A_2, and V shows that we can find $B_1 \subseteqq A_1$ and $B_2 \subseteqq A_2$ such that $V + \operatorname{sp}\alpha^{B_1} \subset V_1 + \operatorname{sp}\alpha^{A_1}$ and $V + \operatorname{sp}\alpha^{B_1} \subset V + V + \operatorname{sp}\alpha^{B_2} \subset V_2 + \operatorname{sp}\alpha^{A_2}$.

Here we have used that sp is a monotone increasing function of the subalgebras, i.e. that $B_i \subseteqq A_i \Rightarrow \operatorname{sp}\alpha^{B_i} \subseteqq \operatorname{sp}\alpha^{A_i}$.

Thus we have found

$$V + \operatorname{sp}\alpha^{B_1} \subset (V_1 + \operatorname{sp}\alpha^{A_1}) \cap (V_2 + \operatorname{sp}\alpha^{A_2}) .$$

3.5. Remark. In case A is any C^*-algebra, 3.4 shows that the family $\{\operatorname{sp}\alpha^B + V\}$ is a filter-basis if we allow only such non-zero α-invariant hereditary C^*-subalgebras B which generate essential ideals – i.e. ideals that have no non-zero orthogonal closed two-sided ideals in A.

4. Automorphisms of Simple C^*-Algebras

By a *simple* C^*-algebra A we mean one whose only closed two-sided ideals are $\{0\}$ and A. Each hereditary C^*-subalgebra B of a simple C^*-algebra A is itself simple (see e.g. [11, Theorem 1.6]).

When A is embedded in its second dual A'' (itself a von Neumann algebra, see [4, 12.1]), we denote by the multiplier algebra of A the C^*-algebra $M(A) = \bigcap_{a \in A}\{x \in A'' | xa \in A \text{ and } ax \in A\}$ (see [12]). If α is a *-automorphism of A, its bitransposed α'' is a *-automorphism of A''.

If δ is a derivation of A, δ'' is a derivation of A''.

By the annihilator θ^0 of a subset θ in Γ we mean the group of $g \in G$ such that $(g, \gamma) = 1$ for every $\gamma \in \theta$.

4.1. Proposition. *Let A be a simple C^*-algebra. Let (α, A) be a representation of G. Assume that for every compact neighbourhood V of ω in Γ there is a non-zero α-invariant hereditary C^*-subalgebra A_0 in A such that*

$$\operatorname{sp}\alpha^{A_0} \subset \Gamma(\alpha) + V .$$

Then α_h is implemented by a unitary element u in the multiplier algebra $M(A)$ for every h in $(\Gamma(\alpha))^0$.

Proof. Let $h \in (\Gamma(\alpha))^0$, let V_ε be the neighbourhood of 1 on the unit circle $\{e^{i\theta} | \theta \in [-\varepsilon, \varepsilon]\}$. Choose a neighbourhood V of ω in Γ such that $(h, V) \subset V_\varepsilon$. For this V, take A_0 as a non-zero α-invariant hereditary C^*-subalgebra of A, such that

$$\mathrm{sp}\,\alpha^{A_0} \subset \Gamma(\alpha) + V .$$

Then

$$\begin{aligned}\sigma(\alpha_h^{A_0}) &= \{(h, \gamma) | \gamma \in \mathrm{sp}\,\alpha^{A_0}\} \\ &\subseteqq \{(h, \gamma) | \gamma \in \Gamma(\alpha) + V\} \\ &= \{(h, \gamma) | \gamma \in V\} \subseteqq V_\varepsilon\end{aligned}$$

and so by a classical result (see e.g. [5, III, 9, 4]) the principal branch of the logarithm of $\alpha_h^{A_0}$ is a derivation δ_0. As δ_0 derives A_0, δ_0'' derives A_0'' and since A_0 is simple we know from [8, Theorem 2.1] that $i\delta_0''$ has a minimal positive generator h_0 in $M(A_0)$, $\mathrm{ad}(h_0) = i\delta_0''$, where the spectrum of $h_0 \subset [0, \varepsilon]$. Thus $e^{-ih_0} \cdot e^{ih_0} = \alpha_h''$ on A_0''. [That a derivation of a simple C^*-algebra is always given by some multiplier was originally proved by Sakai [13]. Here we need the stronger result for later use (4.2).]

We claim that $A_0'' = pA''p$ for some α''-invariant projection p in A'', when A_0'' is viewed in the canonical embedding in A''. To see this, let (u_λ) be an approximate unit in A_0 so that $u_\lambda \nearrow p$. Clearly, $pA''p \supseteqq A_0$. Take $x \in A_+''$, then by the Kaplansky density theorem ([5, I.3.5]) $x^{\frac{1}{2}}$ is the strong limit of a net (x_i) of self-adjoint operators with norm less than or equal to $\|x^{\frac{1}{2}}\|$, thus

$$x_i^2 \xrightarrow{s} x$$

and so

$$u_\lambda x_i^2 u_\lambda \xrightarrow{s} pxp .$$

Since $u_\lambda x_i^2 u_\lambda \leqq \|x_i^2\| u_\lambda^2 \leqq \|x\| u_\lambda^2 \in A_0$ we have that $u_\lambda x_i^2 u_\lambda \in A_0$. From this we conclude that $pA''p \subseteqq A_0^s = A''_0$. Thus we have shown that α_h'' is inner on $pA''p$, and by a simple computation, see e.g. [2, 5.7 Lemma] this implies that α_h'' is inner on $c(p)A''$, $c(p)$ denoting the central support of p in A''. Here $c(p) = 1$, since $c(p)A'' \cap A$ is a non-zero ideal in A which contains A_0 and is norm-closed, thus equal to A itself. This means there is a unitary u extending e^{ih_0} such that

$$\alpha_h''(x) = uxu^* \forall x \in A'' .$$

We claim that u multiplies A. To see this, note that u multiplies A_0, and that the set $I = \{x \in A | ux \in A\}$ is a closed right ideal of A. Take $y \in A$, $x \in I$, then $uyx = (uyu^*)(ux) = \alpha(y)ux \in A$.

Thus I is also a left ideal, so it becomes a 2-sided ideal. As $I \supseteqq A_0$, this means $I = A$.

4.2. Theorem. *Let (α, A) be a representation of G on the simple C^*-algebra A. If $\mathrm{sp}\,\alpha/\Gamma(\alpha)$ is compact, then $(\Gamma(\alpha))^0$ is the subgroup of elements h in G such that*

$$\alpha_h(x) = uxu^* \qquad \forall x \in A$$

for some unitary element u in $M(A)$ which belongs to the centre of the fixed-point algebra for (α'', A'').

Proof. Assume $\alpha_h = u \cdot u^*$, $u \in M(A) \cap Z(A''^{\alpha''})$. There exists a $\lambda \in \sigma(u)$ such that whenever f is a continuous function on the circle which is non-vanishing on a compact neighbourhood of λ we have $f(u)Af(u)^* \neq \{0\}$.

Taking p to be the spectral projection of u corresponding to an interval V around λ we see that $B = pA''p \cap A$ is a non-zero hereditary subalgebra of A, invariant under α. Since $\|up - \lambda p\| < \varepsilon$ for some $\varepsilon > 0$ we have that the spectrum of α'' on $pA''p$ is contained in a 2ε-neighbourhood of 1 on the circle. Thus $\sigma(\alpha_h^B) = \{(h, \gamma) | \lambda \in \mathrm{sp}\,\alpha^B\}$ is contained in this neighbourhood, and as the preceding argument is valid for any $\varepsilon > 0$ this shows that $(h, \gamma) = 1$ when

$$\gamma \in \cap\, \mathrm{sp}\,\alpha^B = \Gamma(\alpha).$$

Let V_0 be a compact neighbourhood of ω in Γ. With k denoting the quotient map $\Gamma \mapsto \Gamma/\Gamma(\alpha)$ we have that $k^{-1}(k(V_0)) = V_0 + \Gamma(\alpha)$.

Let $F(\alpha)$ be the filter-basis $\{V + \mathrm{sp}\,\alpha^B)$ as defined in 3.4. The image sets $k(F_1)$, $F_1 \in F(\alpha)$, form a filter-basis of compact sets such that $k^{-1}(k(F_1)) = F_1$ (by 2.3 above) and $\cap k(F_1) = \{0\}$. There exists $F_0 \in F(\alpha)$ such that

$$k(F_0) \subset k(V_0)$$

i.e.

$$F_0 \subset V_0 + \Gamma(\alpha).$$

But this implies that a non-zero invariant hereditary C^*-subalgebra A_0 of A can be found such that

$$\mathrm{sp}\,\alpha^{A_0} \subset F_0 \subset V_0 + \Gamma(\alpha)$$

and so by Proposition 4.1 we have that for every $h \in (\Gamma(\alpha))^0$, α_h'' is given by a unitary element u of the multiplier algebra $M(A)$.

We claim that u is fixed under the action of α_g'' for every g in G. Note that $\alpha_h''(x) = \alpha_g''(u)x\alpha_g''(u^*)$ for every $x \in A''$ – indeed,

$$\alpha_g''(u)x\alpha_g''(u^*) = \alpha_g''(u\alpha_{-g}''(x)u^*) = \alpha_g''(\alpha_{h-g}''(x)) = \alpha_h''(x).$$

So $\alpha_g''(u)\, u^*$ belongs to the centre of $M(A)$ which consists of the scalars since A is primitive, [12, Proposition 2.7]. Thus

$\alpha_g''(u) = \gamma_g u$, $|\gamma_g| = 1$. The map $g \mapsto \gamma_g$ is a character on G since

$$\gamma_{g+h}(u) = \alpha_{g+h}''(u) = \alpha_g''(\alpha_h''(u)) = \alpha_g''(\gamma_h u) = \gamma_h(\alpha_g'' u) = \gamma_h \gamma_g u.$$

Recall how the element u was constructed in 4.1 as an extension of a partial isometry e^{ih_0} with spectrum in $V_\varepsilon = \{e^{i\theta} | -\varepsilon \leqq \theta \leqq \varepsilon\}$. Since by 4.1 $up = e^{ih_0}$ with p an α''-invariant projection we have that for every g in G

$$\|\alpha_g''(u)p - p\| = \|\alpha_g''(up - p)\| = \|up - p\| \leqq e^{ih_0} - 1\| < \varepsilon$$

thus

$$\begin{aligned} |1 - \gamma_g| &= \|(1 - \gamma_g)up\| \\ &= \|(up - p) + (p - \gamma_g up)\| \\ &\leqq \|up - p\| + \|\alpha_g''(u)p - p\| \\ &< 2\varepsilon. \end{aligned}$$

Since this evaluation is independent of g, we have that $\gamma_g = 1$ for every g in G.

4.3. Corollary. *Let α be a *-automorphism of the simple C^*-algebra A.*

(i) *If the spectrum $\sigma(\alpha)$ of α as a bounded operator on A satisfies that $S(\sigma(\alpha)) = \{1\}$, then α is implemented by a unitary in $M(A)$.*

(ii) *If $\sigma(\alpha)$ is not the unit circle, then for some natural number n the power α^n is implemented by a unitary in $M(A)$.*

Proof. (i) Let (α, A) denote the representation of $\mathbb{Z}$.

Then $\operatorname{sp}\alpha = \sigma(\alpha)$ (see [3, Lemma 2.3.8] or [10, Lemma 2.3]) and by 2.2 we have that $\Gamma(\alpha) \subsetneqq S(\sigma(\alpha)) = \{1\}$.

Thus 4.2 gives the desired conclusion.

(ii) If α has a gap in $\sigma(\alpha)$, then we must have $S(\sigma(\alpha)) = \left\{e^{i\frac{m}{n}\cdot 2\pi}\right\}_{m=0}^{n-1}$ for some natural number n. Thus 4.2 yields that α^n is unitarily implemented.

4.4. Corollary. *Let (α, A) be a representation of G on the simple C^*-algebra A such that*

$$\|\alpha_g - \iota\| \to 0 \quad \text{as} \quad g \to 0$$

ι denoting the identity automorphism of A. Then α_h is implemented by a unitary in $M(A)$ for every $h \in (\Gamma(\alpha))^0$.

Proof. This is immediate from 4.2 and the fact that the continuity condition above implies that $\operatorname{sp}\alpha$ is compact ([10, Proposition 2.1]).

5. Single *-Automorphisms of C^*-Algebras

When dealing with a single *-automorphism α on the C^*-algebra A, we often think of the associated representation of $\mathbb{Z}$, $n \mapsto \alpha^n$, and note that $\operatorname{sp}\alpha$ as defined for the representation coincides with the spectrum $\sigma(\alpha)$ of α as an element of $B(A)$, the bounded linear operators on A (see [3, Lemma 2.3.8] or [10, Lemma 2.3]). Throughout this section, $\mathbb{T}$ denotes the unit circle.

5.1. Proposition. *Let A be a C^*-algebra, let α be a *-automorphism of A. Either $\Gamma(\alpha) = \mathbb{T}$ or there exists a smallest natural number m such that on a canonically associated non-zero α-invariant ideal J_m we have*

$$\Gamma'(\alpha^{J_m}) = \left\{e^{i\frac{s}{m}2\pi}\right\}_{s=0}^{m-1}.$$

defining

$$\Gamma'(\alpha^{J_m}) = \bigcap\nolimits_{I(B)\,\text{essential}} \operatorname{sp}\alpha^B,$$

where the intersection is taken over all non-zero α-invariant hereditary C^*-subalgebras B that generate essential ideals of J_m.

Proof. By $l(B)$ we mean the maximal length of a connected component in $\mathbb{T} \setminus \sigma(\alpha^B)$, B a non-zero α-invariant hereditary C^*-subalgebra of A. Let

$$\sup l(B) = l,$$

where B runs through all such. Either $l=0$, in which case $\Gamma(\alpha)=\mathbb{T}$. Or $l>0$, in which case we claim that $l=\frac{2\pi}{m}$ for some m. To see this, take B_ε so $l(B_\varepsilon)>l-\varepsilon$, and define

$$L_\varepsilon=\{x\in A|x^*x\in B_\varepsilon\}.$$

Let $a_0\in\mathbb{R}$ be so that [by (a,b) we mean $\{e^{it}|a<t<b\}$]

$$\mathbb{T}\setminus\sigma(\alpha^{B_\varepsilon})\supset\left(a_0+\frac{\varepsilon}{2},a_0+l-\frac{\varepsilon}{2}\right).$$

We want to prove $\mathbb{T}\setminus\sigma(\alpha^{B_\varepsilon})\supset(\varepsilon,l-\varepsilon)$. If not, we could find $\gamma\in(\varepsilon,l-\varepsilon)\cap\sigma(\alpha^{B}\varepsilon)$, and $\varepsilon_1>0$ so that $(\gamma-\varepsilon_1,\gamma+\varepsilon_1)\subset(\varepsilon,l-\varepsilon)$.

$$R^{\alpha^{B_\varepsilon}}(\gamma-\varepsilon_1,\gamma+\varepsilon_1)\neq\{0\}$$

since $\gamma\in\sigma(\alpha^{B_\varepsilon})$, and the smallest right ideal L_1^* in L_ε^* containing this set is spanned by elements of the form $(\alpha(f)x)y$, where supp $\hat{f}\subset(\gamma-\varepsilon_1,\gamma+\varepsilon_1)$ and $x\in B_\varepsilon$, $y\in L_\varepsilon^*$.

$$B_1=L_1\cap L_1^*\subseteqq L_\varepsilon\cap L_\varepsilon^*=B_\varepsilon$$

is non-zero since $(\alpha(f)x)y\in L_1^*$ implies $y^*(\alpha(\bar{f})x^*)\in L_1$ and thus

$$(\alpha(f)x)yy^*(\alpha(\bar{f})x^*)\in L_1^*L_1=L_1^*\cap L_1=B_1$$

and if this element is zero, so is $(\alpha(f)x)y$. If all elements $(\alpha(f)x)y$ were zero, then since $(B_\varepsilon)_+=(L_\varepsilon^*)_+$, $\alpha(f)x=0$ for all x in B_ε, a contradiction.

B_1 is also α-invariant, and it is spanned by elements of the form

$$(\alpha(f)x)yz(\alpha(\bar{g})v),$$

where x and v are in B_ε, $y\in L_\varepsilon^*$, $z\in L_\varepsilon$. Since then $yz\in B_\varepsilon$ we have that all elements belong to

$$R^\alpha(\gamma-\varepsilon_1,\gamma+\varepsilon_1)R^\alpha(\mathrm{sp}\,\alpha^{B_\varepsilon})\subseteqq R^\alpha((\gamma-\varepsilon_1,\gamma+\varepsilon_1)+\mathrm{sp}\,\alpha^{B_\varepsilon}).$$

Thus

$$\mathrm{sp}\,\alpha^{B_1}\subseteqq(\gamma-\varepsilon_1,\gamma+\varepsilon_1)+\mathrm{sp}\,\alpha^{B_\varepsilon}$$

and so

$$\mathbb{T}\setminus\mathrm{sp}\,\alpha^{B_1}\supset\left(a_0+\frac{\varepsilon}{2}+\gamma+\varepsilon_1,a_0+l-\frac{\varepsilon}{2}+\gamma-\varepsilon_1\right)$$

$$\supset\left(a_0+l-\frac{\varepsilon}{2},a_0+l+\frac{\varepsilon}{2}\right)$$

but $B_1\subseteqq B_\varepsilon$ implies $\mathbb{T}\setminus\mathrm{sp}\,\alpha^{B_1}\supseteqq\mathbb{T}\setminus\mathrm{sp}\,\alpha^{B_\varepsilon}$ thus

$$\mathbb{T}\setminus\mathrm{sp}\,\alpha^{B_1}\supset\left(a_0+\frac{\varepsilon}{2},a_0+l+\frac{\varepsilon}{2}\right)$$

a contradiction to the definition of l.

Repeating the above argument we find that for every k in $\mathbb{Z}$

$$\mathbb{T}\setminus\sigma(\alpha^{B_\varepsilon})\supset(kl+(k+1)\varepsilon,(k+1)l-(k+1)\varepsilon)$$

from which we conclude that $l=\frac{2\pi}{m}$, m in $\mathbb{N}$.

Let J_m be the sum of all ideals I which are generated by non-zero α-invariant hereditary C^*-subalgebras B for which $l(B)>\frac{2\pi}{m+1}$. Given $a\in\left(\frac{2\pi}{m+1},\frac{2\pi}{m}\right)$ define F_a to be the set of pairs (I,B) with I a two-sided ideal generated by the non-zero α-invariant hereditary C^*-subalgebra B which satisfies

$$\bigcup_{k=1,3,\ldots,2m-1}\left(k\frac{\pi}{m}-\frac{a}{2},k\frac{\pi}{m}+\frac{a}{2}\right)\subset\mathbb{T}\setminus\sigma(\alpha^B).$$

Given the ordering $(I_1,B_1)\leqq(I_2,B_2)$ if $I_1\subseteqq I_2$ and $B_1=I_1\cap B_2$, F_a has a maximal element (I_a,B_a). This is based on the relation

$$\mathbb{T}\setminus\sigma(\alpha^B)=\left(\bigcap_\mu\mathbb{T}\setminus\sigma(\alpha^{B_\mu})\right)^0$$

when (I_μ,B_μ) is a totally ordered family and $I=\Sigma I_\mu$, $B_\mu=L_\mu\cap L_\mu^*$ with $L_\mu=\{x\in A|x^*x\in B_\mu\}$ and $B=L\cap L^*$ with $L=\Sigma L_\mu$. Note that

$$L_\mu=\{x\in A|x^*x\in B_\mu=B\cap I_\mu\}=L\cap I_\mu$$

so $B_\mu=L\cap L_\mu^*=L_\mu\cap L^*$.

Let $\gamma\in(\cap\mathbb{T}\setminus\sigma(\alpha^{B_\mu}))^0$ and $\varepsilon>0$ be so that $(\gamma-\varepsilon,\gamma+\varepsilon)\subset(\bigcap_\mu\mathbb{T}\setminus\sigma(\alpha^{B_\mu}))$, then $\alpha(f)x=0$ when $\operatorname{supp}\hat{f}\subset(\gamma-\varepsilon,\gamma+\varepsilon)$ and $x\in L\cap L_\mu^*$. Since for $y\in L\cap L^*$, $y\geqq0$ we have that $y^{\frac{1}{2}}$ is the norm-limit of sums Σx_μ, $x_\mu\in L_\mu^*$ we see that $(\Sigma x_\mu)y^{\frac{1}{2}}\to y$ in norm, and $x=(\Sigma x_\mu)y^{\frac{1}{2}}\in L_\mu^*L$. Thus $\alpha(f)y=0$ when $\operatorname{supp}\hat{f}\subset(\gamma-\varepsilon,\gamma+\varepsilon)$, so $\gamma\in\mathbb{T}\setminus\sigma(\alpha^B)$. The other inclusion

$$\mathbb{T}\setminus\sigma(\alpha^B)\subseteqq\left(\bigcap_\mu\mathbb{T}\setminus\sigma(\alpha^{B_\mu})\right)^0$$

is obvious.

Since the ideal generated by B contains all I_μ, it must contain I. However, I contains all I_μ, thus all L_μ, thus L, so $I\supseteqq B$. It follows that $(I,B)\in F_a$, $(I,B)=\sup(I_\mu,B_\mu)$.

So F_a has a maximal element (I_a,B_a). We claim that I_a is an essential ideal in J_m. If not, we could find $I\subseteqq J_m$ with $I\cap I_a=\{0\}$. By the same reasoning as above we see that I must contain a non-zero α-invariant hereditary subalgebra B^I so that

$$\mathbb{T}\setminus(\sigma(\alpha^{B^I}))\supset\bigcup_{k=1,3,\ldots,2m-1}\left(\frac{k\cdot\pi}{m}-\frac{a}{2},\frac{k\cdot\pi}{m}+\frac{a}{2}\right)$$

and so $(I_a+I,B_a+B^I)\geqq(I_a,B_a)$ in F_a.

Since the argument given above was valid for all $a\in\left(\frac{2\pi}{m+1},\frac{2\pi}{m}\right)$ we see that the intersection over all B that generate essential ideals must be contained in the m'th roots of unity. However, since for any non-zero B, $l(B) \leqq \frac{2\pi}{m}$, we have $\Gamma'(\alpha^{J_m}) \supseteqq \left\{e^{i\frac{s}{m}2\pi}\right\}_{s=0}^{m-1}$. From this the desired equality follows.

5.2. Lemma. *Let A be a C^*-algebra, α a $*$-automorphism of A. If the spectrum of α has a gap, then a power of α is the exponential of a derivation when restricted to a non-zero α-invariant hereditary C^*-subalgebra.*

Proof. With notation as in the proof of 5.1 we have that $l = \frac{2\pi}{m} > 0$, and since the spectra $\mathrm{sp}\,\alpha^B$ where B generates an essential ideal in J_m are approximately filtering by 3.5 we see that given a compact neighbourhood V of ω in Γ we can find such a B so that

$$\mathrm{sp}\,\alpha^B \subset \Gamma'(\alpha^{J_m}) + V = \left\{e^{i\frac{s}{m}2\pi}\right\}_{s=0}^{m-1} + V .$$

Thus by the argument given in the proof of 4.1 we have for a suitably chosen V_0 a B_0 such that $(\alpha^m)^{B_0} = e^{\delta}$, with δ a derivation of B_0.

5.3. Corollary. *Let A be a commutative C^*-algebra, α a $*$-automorphism of A with a gap in its spectrum. Then α^{m_0} is the identity operator for some natural number m_0.*

Proof. In the commutative case, the hereditary C^*-subalgebras are two-sided ideals. Using the notation from above, we see that on $B_a = I_a$ the power α^m is the identity. Taking the quotient A/I_a we see that no ideal I in this C^*-algebra can have $l(I) > \frac{2\pi}{m+1}$. If such an ideal existed, it would contain non-zero subideals with gaps containing $\cup\left(\frac{k\cdot\pi}{m} - \frac{a}{2}, \frac{k\cdot\pi}{m} + \frac{a}{2}\right)$, and taking inverse images we would have larger ideals than I_a in F_a. Thus we have found a new C^*-algebra (A/I_a) where

$$l = \sup l(I) = \frac{2\pi}{m+p}, \qquad p > 0 .$$

Repeating the argument we get that α^{m+p} is the identity on an ideal I_{a_p} in A/I_a, thus $\alpha^{m\cdot(m+p)}$ is the identity on $I_{a_p} + I_a$. Continuing in this fashion, we can make but a finite number of steps, since $l(A) > 0$ means that $l(A) > \frac{2\pi}{n+1}$ for some n. Thus we have found that at least $\alpha^{n!}$ is the identity on A – in fact we can do with the power $m\cdot(m+p)\ldots n$ which may be strictly smaller than $n!$.

5.4. Remark. It follows from the construction in 5.1 that the spectrum of α on each of the ideals we have found is exactly the roots of unity for the natural number q such that $\alpha^q = \mathrm{id}$. The spectrum of α thus becomes the finite union of these subgroups of the circle.

5.5. Remark. Note that the argument from 5.1 in the commutative case carries over to isometric isomorphisms acting on arbitrary semi-simple Banach algebras. We take ideals instead of hereditary subalgebras, and then the arguments do not depend on specific C^*-algebra properties. This result has earlier been proved in [6] and [7] by different methods.

6. Automorphisms of von Neumann Algebras

In this section we prove the von Neumann algebra analogue of 4.1. We show that this leads to a new proof of the result in [2] which states that a *-automorphism α acting on a von Neumann algebra either has its spectrum equal to the unit circle or has some power α^n which is inner. One should note, however, that we also derive a new proof of the theorem in [1] that every one-parameter group of automorphisms satisfying a spectrum condition is (at least pointwise) inner. That the unitary implementing operators may be chosen to form a group then follows from cohomology considerations.

In this section, A denotes a von Neumann algebra, A_* its predual, (α, A) a representation of G which is $\sigma(A, A_*)$ – continuous, i.e.

$$\phi(\alpha_g x - x) \to 0 \quad \text{as} \quad g \to 0 \qquad \forall \phi \in A_* \qquad \forall x \in A .$$

6.1. Lemma. *Let E be an α-invariant norm-closed subspace of A, F its $\sigma(A, A_*)$ – closure in A.*

$$\mathrm{sp}\alpha^E = \mathrm{sp}\alpha^F .$$

Proof. As $E \subseteqq F$ we have that $\mathrm{sp}\alpha^E \subseteqq \mathrm{sp}\alpha^F$. Let $\gamma \in \mathrm{sp}\alpha^F$ and $V \in \check{U}(\gamma)$. Choose $f \in L^1(G)$ with supp $\hat{f} \subset V$ and take $x \in F$ such that

$$\alpha(f)x \neq 0 .$$

We know that there is a net $(x_\lambda) \subset E$ which converges σ-weakly to x. This implies that

$$\alpha(f)x_\lambda \xrightarrow{\sigma} \alpha(f)x$$

(see [1, Proposition 1.4] or [8, Proposition 1.3]).

Thus we can find λ such that $\alpha(f)x_\lambda \neq 0$, which implies that $\gamma \in \mathrm{sp}\alpha^E$.

6.2. Remark. Let as in [2] P_0 denote the set of projections in the centre of the fixed-point algebra for (α, A). If follows from 6.1 that

$$\Gamma(\alpha) = \bigcap\nolimits_{p \in P_0 \setminus \{0\}} \mathrm{sp}\alpha^{pAp} .$$

In the following, we write $\mathrm{sp}\alpha^p$ for $\mathrm{sp}\alpha^{pAp}$.

6.3. Proposition. *Let A be a von Neumann algebra, Z its centre. Let (α, A) be a representation of G. Let θ be a closed subset of the dual group Γ. Assume that for every $V \in \check{U}(\omega)$ and $0 \neq q \in Z \cap P_0$ there exists $0 \neq p \in P_0$ such that $p \leqq q$ and*

$$\mathrm{sp}\alpha^p \subset \theta + V .$$

Then α_h is implemented by a unitary u in the centre of the fixed-point algebra for every h in the orthogonal of θ.

Proof. Fix $V \in \check{U}(\omega)$ and choose p as above. As in 4.1 we get $\alpha_h^p = e^\delta$, with δ a derivation of pAp. Since pAp is a von Neumann algebra, δ is inner, thus α_h^p is implemented by a unitary $u \in pAp$. It follows from [2, 5.7 Lemma] that $\alpha_h^{c(p)}$ is unitarily implemented on $c(p)A$. Take $q = 1 - c(p)$ and repeat this argument with $p_1 \leqq q$. Let q_0 denote the maximal element of $Z \cap P_0$ such that α_h is unitarily implemented on $q_0 A$. If $1 - q_0 \neq 0$ we could find $p_0 \leqq 1 - q_0$ such that α_h was unitarily implemented on $c(p_0)A$, thus on $(q_0 + c(p_0))A$, a contradiction to the maximality of q_0. Thus $q_0 = 1$.

6.4. Corollary. *Let $A \subset B(H)$, H a Hilbert space, and let (u, H) be a strongly continuous unitary representation of* $\mathbb{R}$ *with* $\mathrm{sp}\, u \subset [0, \infty)$. *Let (α, A) denote the representation defined by*

$$\alpha_t(x) = u_t x u_{-t} \qquad \forall x \in A .$$

Then α_t is inner for every t in $\mathbb{R}$.

Proof. Let $s \in \mathbb{R}$. Define

$$p[s, \infty)H = \bigcap\nolimits_{t<s} [R^\alpha[t, \infty)H] ,$$

$[]$ denoting closed linear span in H. (This family is identical to the one defined in [1, Theorem 3.1].) Let q denote the projection-valued measure of u_t, i.e.

$$u_t = \int_0^\infty e^{itx} dq(x) .$$

If $q[\delta, \infty) = 1$ for some $\delta > 0$, the translated measure $q_\delta[s, \infty) = q[s+\delta, \infty)$ belongs to the group $e^{-it\delta} u_t$ which also implements α and has positive spectrum. Now we have that for any unitary group v_t implementing α_t with $\mathrm{sp}\, v \subset [0, \infty)$

$$R^\alpha[s, \infty)H = R^\alpha[s, \infty)R^v[-\alpha, \infty) \subseteqq R^v[s-\alpha, \infty) \qquad \forall \alpha > 0 .$$

From this it follows that $[R^\alpha[\varepsilon, \infty)H] \subsetneqq H$ for every $\varepsilon > 0$. Applying the same argument to $\alpha_t^e = u_t^e \cdot u_{-t}^e$, e a fixed-projection in the centre of A, we see that $[R^{\alpha^e}[\varepsilon, \infty)eH] \subsetneqq eH$.

Thus $ep[\varepsilon, \infty) < e$ for very $\varepsilon > 0$. This shows that $(1 - p[\varepsilon, \infty))e \neq 0$.

[As we know (e.g. using that every derivation of A is inner) that Z is pointwise fixed under α_t, we have that $c(1 - p[\varepsilon, \infty)) = 1$.] It follows from the definition above that

$$\mathrm{sp}\, \alpha^{1 - p[\varepsilon, \infty)} \subset [-\varepsilon, \varepsilon]$$

thus $\Gamma(\alpha) = \{0\}$ and 6.3 can be applied with $\theta = \Gamma(\alpha) = \{0\}$.

6.5. Corollary. *Let A be a von Neumann algebra, α a *-automorphism with a spectrum which is not the entire circle. Then some power of α is inner.*

Proof. From 5.2 we know that with

$$l = \sup l(p) = \frac{2\pi}{m} ,$$

α^m is inner on a subalgebra $q_m A q_m$ (the σ-closure of the hereditary subalgebra B_a). Thus α^m is inner on $c(q_m)A$, and $c(q_m)A$ must be the σ-closure of J_m in 5.1. Taking $(1 - c(q_m))$ A as our new von Neumann algebra, we get that

$$\sup_{0 \neq p \leqq 1 - c(q_m)} l(p) = \frac{2\pi}{m_1}, \qquad m_1 > m .$$

So we can repeat the argument, obtaining that α^{m_1} is inner on $c(q_{m_1})A \subset (1 - c(q_m))A$. After a finite number of steps we have finished, since the gap in $\sigma(\alpha)$ has a connected component of strictly positive length. Adding the relevant powers of the implementing operators we get that $\alpha^{m \cdot m_1 \cdots m_k}$ is inner on A.

6.6. Remark. For α as above we see that on each central summand $c(q_{m_l})A$, $\Gamma'(\alpha)$ is of order m_l – and this is also true for $\Gamma'(\alpha^e)$, e a central projection in $c(q_{m_l})A$. Thus the intersection over $\mathrm{sp}\,\alpha^q$ where $c(q)=1$ in A and $\alpha(q)=q$ yields the finite union of the subgroups of order m_l, $l=1, \ldots, k$. This generalizes the commutative result that the spectrum of α is the finite union of subgroups (5.4).

As a final application of 5.3 we show that if $(\alpha, A) \in K_n$ in the terminology of [2] (see Definition 6.8 below), then α^n is implemented by a unitary in the centre of the fixed-point algebra of α, and α^k is not so for any $k \neq 0 \bmod n$.
By P_0 we mean the set of projections in the centre of the fixed-point algebra of α.

6.7. Definition ([2, 3.6 Definition]). Let α be a *-automorphism of A, let (α, A) denote the representation of $\mathbb{Z}$ defined by $n \mapsto \alpha^n$. (α, A) is said to belong to class K_n if

(i) For every pair p_1 and p_2 in P_0,

$$\left\{e^{i\frac{m}{n}2\pi}\right\}_{m=0}^{n-1} \subseteqq S(\mathrm{sp}\,\alpha^{p_1 A p_2}).$$

(ii) For every non-zero f in $P_0 \cap Z$ and every $V \in \grave{U}(\omega)$ there exists a non-zero $p \leqq f$ in P_0, such that

$$\mathrm{sp}\,\alpha^p \subseteqq \left\{e^{i\frac{m}{n}2\pi}\right\}_{m=0}^{n-1} \cdot V.$$

6.8. Lemma. *Let (α, A) be a representation of G.*
Let p_1 and p_2 be in P_0. Then

$$\Gamma(\alpha) \subseteqq S(\mathrm{sp}\,\alpha^{p_1 A p_2}).$$

Proof. Let p_1 and p_2 be in P_0. In case $c(p_1)c(p_2)=0$ we have $S(\mathrm{sp}\,\alpha^{p_1 A p_2})=\mathbb{T}$, and so the statement is obviously correct. Assume $c(p_1)c(p_2) \neq 0$, or equivalently that $p_1 A p_2 \neq \{0\}$. Let $\gamma \in \mathrm{sp}\,\alpha^{p_1 A p_2}$ (which is non-empty when $p_1 A p_2 \neq \{0\}$) and let $V \in \grave{U}(\omega)$.

Let p be the range projection of $R^{\alpha^{p_1 A p_2}}\{\gamma + V\}$, then $p \neq 0$. Now pAp is the smallest hereditary σ-closed subalgebra of A containing $R^{\alpha^{p_1 A p_2}}\{\gamma + V\}$, thus generated by elements of the form

$$(\alpha(f)p_1 x p_2) y z^* (\alpha(\bar{f}) p_2 v p_1) = p_1(\alpha(f) x p_2) y z^* p_2(\alpha(\bar{f}) v p_1),$$

from which we see that

$$\mathrm{sp}\,\alpha^p \subset \mathrm{sp}\,\alpha^{p_1 A p_2} - (\gamma + V).$$

We know that

$$S(\mathrm{sp}\,\alpha^{p_1 A p_2}) = \bigcap_{\gamma \in \mathrm{sp}\,\alpha^{p_1 A p_2}} \bigcap_{V \in \grave{U}(0)} (\mathrm{sp}\,\alpha^{p_1 A p_2} - (\gamma + V))$$

and so $\Gamma(\alpha) = \bigcap_{p \in P_0 \setminus \{0\}} \mathrm{sp}\,\alpha^p$ implies that

$$\Gamma(\alpha) \subseteqq S(\mathrm{sp}\,\alpha^{p_1 A p_2}).$$

6.9. Proposition. $(\alpha, A) \in K_n$ *if and only if*

(i) $\Gamma(\alpha)$ *is of order* n,

(ii) *for every non-zero f in $P_0 \cap Z$ and every V in $\grave{U}(\omega)$ there is a non-zero $p \leqq f$ in P_0 such that* $\mathrm{sp}\,\alpha^p \subset \Gamma(\alpha) + V$.

Proof. If $(\alpha, A) \in K_n$ we only need to prove (i) above. Now it follows from 6.7 (i) that $\Gamma(\alpha) = \bigcap_{p \in P_0 \setminus \{0\}} \mathrm{sp}\,\alpha^p \supseteqq \bigcap_{p \in P_0} S(\mathrm{sp}\,\alpha^p) \supseteqq \left\{ e^{i\frac{m}{n}2\pi} \right\}_{m=0}^{n-1}$. Since 6.7 (ii) implies that $\Gamma(\alpha) \subseteqq \left\{ e^{i\frac{m}{n}2\pi} \right\}_{m=0}^{n-1}$, we have (i).

Conversely, assume (i) and (ii) above, then 6.8 shows that $(\alpha, A) \in K_n$.

6.10. Corollary. *Let $(\alpha, A) \in K_n$. Then α is implemented by a unitary in the centre of the fixed-point algebra of α, and this is not the case for α^k when $k \not\equiv 0$ mod n.*

Proof. This is an immediate consequence of 6.3, and the fact that if $\alpha^k = u \cdot u^*$, with u in the centre of the fixed-point algebra, then $k \in (\Gamma(\alpha))^0$ (see the argument in the proof of 4.2).

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