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On the Number of Solutions to a Crossing Symmetric Neutral π-π Model

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Abstract. The crossing symmetry of the π–π system has invited many authors to try different versions of the bootstrap hypothesis on it. In the last few years, there has been some hope that the positivity conditions of Martin, Balachandran and Nuyts, Roskies, and, most recently, the Roy physical region constraints, might be sufficient to fix the low π–π partial waves with very little additional information, like the position and width of the ϱ resonance. This hope was recently proved too optimistic by Basdevant, Froggatt and Petersen. A similar result was obtained by the present author in an approximately crossing symmetric, solvable model. In this paper we strengthen this latter result by determining the multiplicity of the exact solution to a crossing symmetric neutral π–π model. We consider only the S wave, but the multiplicity would increase by the addition of other coupled channels. The analysis is not confined to weak coupling only, and includes all solutions, in particular also a class of logarithmically decreasing ones, which are left out by most other authors.

1. Introduction

Some time ago, the present author showed in a simple solvable neutral $\pi\pi$ model [1] that the positivity conditions, derived by Martin [2] and a number of other authors, are not sufficient to fix the $\pi^0\pi^0 S$ wave when, say, the scattering length is given [3]. In fact a number of CDD poles, with arbitrary position and residue, could be added to the expression for $1/f(s)$, without violating seriously the positivity conditions.

One might argue that the model of [1] only approximately satisfies the positivity conditions, and that the result would be different in a model which satisfies the positivity conditions exactly. The obvious answer to this argument is to construct such a model, which satisfies all possible positivity conditions, and prove that its solution is not unique. This is what we shall do in the present work.

The model we shall consider is a neutral, crossing symmetric $\pi\pi$ model, which is known as the Cini-Fubini [4] approximation for the S wave. It is obtained by taking the once subtracted Mandelstam re-

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presentation with a symmetric spectral function $\varrho(s, t)$, that is

$$f(s,t,u)=\frac{s}{\pi}\mathrm{P}\int_4^\infty \frac{g(s',t,u)ds'}{s'(s'-s)}+\frac{t}{\pi}\mathrm{P}\int_4^\infty \frac{g(t',u,s)dt'}{t'(t'-t)} + \frac{u}{\pi}\mathrm{P}\int_4^\infty \frac{g(u',s,t)du'}{u'(u'-u)} \tag{1.1}$$

with the absorptive part

$$g(s',t,u)=h(s')+\frac{t}{2\pi}\mathrm{P}\int_{16}^\infty \frac{\varrho(s',t')dt'}{t'(t'-t)}+\frac{u}{2\pi}\mathrm{P}\int_{16}^\infty \frac{\varrho(u',s')du'}{u'(u'-u)}, \tag{1.2}$$

and assuming that the absorptive part is well approximated by the first few terms of a partial wave expansion, in this case the first term,

$$g(s,t,u)=\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}\,f_l(s)P_l\left(1+\frac{2t}{s-4}\right)\approx \mathrm{Im}\,f(s)\,. \tag{1.3}$$

Adding a unitarity relation for the S wave amplitude $f(s)$, the result is a simple, crossing symmetric model, fulfilling all possible positivity conditions. Of course, it is not a realistic model, since it assumes all higher waves, and in particular the P wave, to be real. However, since crossing and unitarity are exact for the S wave, it is a good testing ground for hypotheses about the usefulness of the positivity conditions. The model is also internally consistent, and believed to have non-trivial solutions, in contrast to the one obtained by taking also the P wave in Eq. (1.3) different from zero, which Lovelace [4] proved to have no solution.

Projecting from Eq. (1.1) the S wave results in the dispersion relation

$$\mathrm{Re}\,f(s)=a_0+\frac{1}{\pi}\mathrm{P}\int_4^\infty\left[\frac{s-4}{(s'-4)(s'-s)}-\frac{2}{s'} + \frac{2}{s-4}\ln\frac{s'+s-4}{s'}\right]\mathrm{Im}\,f(s')ds'\,,$$

or, if no subtraction is necessary,

$$\mathrm{Re}\,f(s)=\frac{1}{\pi}\mathrm{P}\int_4^\infty\left[\frac{1}{s'-s}+\frac{2}{s-4}\ln\frac{s'+s-4}{s'}\right]\mathrm{Im}\,f(s')ds'\,. \tag{A.1}$$

For simplicity we assume that no bound states are present. To the dispersion relation we add the unitarity condition

$$\mathrm{Im}\,f(s)=\varrho(s)R(s)\,|f(s)|^2=\varrho(s)\,|f(s)|^2+\frac{1}{4\varrho(s)}[1-\eta^2(s)] \tag{B}$$

to obtain a non-linear problem for the determination of $\mathrm{Re}\,f$ and $\mathrm{Im}\,f$. The functions ϱ and R or η are assumed given,

$$\varrho(s)=\sqrt{\frac{s-4}{s}},\quad R(s)\geqq 1\,,\quad 0\leqq\eta(s)\leqq 1\,, \tag{1.4}$$

and the inelasticity η and the inelasticity ratio R can be expressed in each other by

$$R(s)=\frac{2-2\eta(s)\cos 2\delta(s)}{1+\eta^2(s)-2\eta(s)\cos 2\delta(s)}, \tag{1.5}$$

or

$$\eta(s)=\frac{R(s)-1}{R(s)}\cos 2\delta(s)\pm\frac{1}{R(s)}\sqrt{1-[R(s)-1]^2\sin^2 2\delta(s)}, \tag{1.6}$$

where $\delta(s)$ is the phase shift.

We shall also discuss the non-crossing symmetric model obtained by modifying the dispersion Relation **A.1** to

$$\operatorname{Re} f(s)=\frac{1}{\pi}\,\mathrm{P}\int_4^\infty\left[\frac{1}{s'-s}+\frac{m}{s'+s-4}\right]\operatorname{Im} f(s')ds', \tag*{\textbf{A.2}}$$

or to a subtracted version of this equation. The parameter $m\geqq 0$ can be varied, which will help us in the analysis. For $m=2$ the left-hand cuts of the dispersion Relations **A.1** and **A.2** have a similar behaviour for large s', a fact that we shall also use.

The multiplicity of a class of exact solutions to a dispersion relation like **A.1** or **A.2** and the unitarity Condition **B** has been studied by Lovelace [5]. He showed that a solution $\delta(s)$ to the present problem, satisfying

$$\delta(s)-\delta(4)=\pi\nu+O(s^{-\mu}),\ \mu>0,\ \text{as}\ s\to\infty, \tag{1.7}$$

contains 2ν arbitrary parameters, corresponding to ν *CDD* poles.

This result of Lovelace already answers, for solutions satisfying Eq. (1.7), the problem posed above. It shows that in order to get a unique solution for the S wave amplitude one must:

(i) fix the total variation of the phase shift (to $\pi\nu$);

(ii) determine a number of parameters (here 2ν) from auxiliary conditions;

besides using analyticity, unitarity and crossing symmetry.

In the present paper, which is a shorter version of an unpublished report [6], we generalize this result to include all solutions, not only those approaching zero like an inverse power when $s\to\infty$, as Eq. (1.7) implies. We begin in Section 2 by classifying all possible solutions to the dispersion Relations **A.1** or **A.2** and the unitarity Condition **B**. We prove that there are only two classes of solutions, one satisfying Eq. (1.7) and the other approaching zero at infinity like an inverse power of the logarithm of s. It is the slow decrease of this latter class of solutions which makes the present study difficult.

In Section 3 we discuss the exactly solvable case $m=0$, and in Section 4 we formulate the general non-linear problem in a suitable Banach space. This Banach space is larger than those used by other authors [5, 7], which allows us to include all solutions. We shall study the local multiplicity, extending the results of [5], but have no new results on the problem of existence of the solutions, which is the main point of interest of [7].

Following Lovelace [5], we take in Section 4 the Fréchet derivative, use the "implicit variable" theorem by Vainberg and Trenogin [8], and reduce the problem of the local multiplicity of the solutions of the non-linear problem to the problem of determining the index of a linear, singular integral operator.

In Section 5 we determine this index in some solvable cases, and in Section 6 we use theorems on ϕ operators by Gokhberg and Krein [9] to include also non-solvable cases. Finally, we present in Section 7 the main results and a short discussion. Two Appendices contain some necessary mathematical details.

2. Classification of Solutions

In this section we shall classify all possible solutions to the dispersion Relations **A.1** or **A.2**, satisfying the unitarity Relation **B**. We begin by proving that no solution exists, which requires any subtraction.

Assume that

$$\operatorname{Im} f(s) > C(\ln s)^{-\alpha} \quad \text{for} \quad s > s_a, C > 0 . \tag{2.1}$$

Then the dispersion Relation **A.2**, subtracted once, calling the subtraction constant C_0, implies that for s large enough

$$|\operatorname{Re} f(s)| > \begin{cases} \left| \dfrac{C}{\pi}(1+m)\ln(\ln s) + C_0 \right| & \text{if} \quad \alpha = 1 \\ \left| \dfrac{C(1+m)}{\pi(1-\alpha)}(\ln s)^{1-\alpha} + C_0 \right| & \text{if} \quad \alpha \neq 1 . \end{cases} \tag{2.2}$$

This follows from the asymptotic analysis of Appendix 1, Eq. (A.1.6). The same result, with $m=2$, can be proved in the case of dispersion Relation **A.1**.

Now, unitarity demands

$$[\operatorname{Re} f(s)]^2 \leqq \frac{1}{\varrho(s)} \operatorname{Im} f(s) \leqq \frac{1}{\varrho(s)} , \tag{2.3}$$

which clearly excludes the growing behaviour of $\operatorname{Re} f(s)$ for $\alpha \leqq 1$.

Assuming also that

$$\operatorname{Im} f(s) < C(\ln s)^{-\alpha+\varepsilon} \quad \text{for} \quad s > s_a, \varepsilon > 0 , \tag{2.4}$$

which means that the asymptotic behaviour is close to the lower limit in Eq. (2.1), we get from Eq. (2.3) that

$$\left[\frac{C(1+m)}{\pi(1-\alpha)} (\ln s)^{1-\alpha} + C_0 \right]^2 < C(\ln s)^{-\alpha+\varepsilon} .$$

Since we know already that $\alpha > 1$, this implies that $C_0 = 0$ and $\alpha > 2 - \varepsilon$. This proves that any solution $\operatorname{Im} f(s)$ to the disperison Relations **A.1** or **A.2**, satisfying the unitarity Relation **B** must be asymptotically smaller than $(\ln s)^{-2+\varepsilon}$, $\varepsilon > 0$. Thus no subtraction is necessary.

From unitarity we also have

$$0 \leqq 1-\eta^2(s) = 4\varrho(s)\{\operatorname{Im} f(s) - \varrho(s)[\operatorname{Re} f(s)]^2 - \varrho(s)[\operatorname{Im} f(s)]^2\} \underset{s\to\infty}{\longrightarrow} -4\varrho^2(s)[\operatorname{Re} f(s)]^2 \leqq 0 \tag{2.5}$$

as soon as $\operatorname{Im} f(s) \to 0$ as $s \to \infty$. Thus, in our model,

$$\operatorname{Re} f(\infty) = 0 \quad \text{and} \quad \eta(\infty) = 1 . \tag{2.6}$$

Next, we shall prove that if a solution exists, behaving asymptotically as an inverse logarithm, then its leading asymptotic behaviour is completely fixed. We thus assume that

$$\operatorname{Im} f(s) \underset{s\to\infty}{\simeq} C(\ln s)^{-\alpha}, \; C \neq 0 , \tag{2.7}$$

where we know from above that $\alpha > 2-\varepsilon$. Then, according to Appendix 1, the dispersion Relation **A.2** implies

$$\operatorname{Re} f(s) \underset{s\to\infty}{\simeq} \frac{C(1+m)}{\pi(\alpha-1)} (\ln s)^{1-\alpha} . \tag{2.8}$$

In the case of **A.1**, the same formula with $m=2$ is valid.

We now employ unitarity, assuming for the inelasticity ratio $R(s)$ the asymptotic form

$$R(s) \underset{s\to\infty}{\simeq} R_0 (\ln s)^{\gamma}, \quad \gamma \geqq 0 . \tag{2.9}$$

This gives a reasonable generality to our results, since experimentally, $R(s)$ seems to stay bounded. For the inelasticity η, the Assumption (2.9) corresponds to an approach to 1 with a rate determined by the relation

$$\frac{1-\eta^2(s)}{4\varrho(s)\operatorname{Im} f(s)} \underset{s\to\infty}{\simeq} 1 - \frac{1}{R_0}(\ln s)^{-\gamma}, \quad \gamma \geqq 0 . \tag{2.9'}$$

This quantity is restricted, by the unitarity relation, to be between 0 and 1.

With the Assumptions (2.9) or (2.9′), unitarity gives

$$C(\ln s)^{-\alpha} \underset{s\to\infty}{\simeq} R_0 (\ln s)^{\gamma} \frac{C^2(1+m)^2}{\pi^2(\alpha-1)^2} (\ln s)^{2-2\alpha} ,$$

which implies

$$\alpha = 2+\gamma , \quad C = \frac{\pi^2(1+\gamma)^2}{R_0(1+m)^2} , \tag{2.10}$$

that is, *a completely fixed leading asymptotic term:*

$$\begin{aligned} \operatorname{Im} f(s) &\underset{s\to\infty}{\simeq} \frac{\pi^2(1+\gamma)^2}{R_0(1+m)^2} (\ln s)^{-2-\gamma} ; \\ \operatorname{Re} f(s) &\underset{s\to\infty}{\simeq} \frac{\pi(1+\gamma)}{R_0(1+m)} (\ln s)^{-1-\gamma} . \end{aligned} \tag{2.11}$$

We shall call any solution fulfilling Eq. (2.11) *a Class* 1 *solution.*

Due to their slow decrease at infinity, the Class 1 solutions are difficult to handle, and have not been studied in detail before (except for some solvable cases [1,4]). In particular, we shall in the following examine their multiplicity. In [6] we also study their asymptotic form in more detail, deriving their asymptotic series in $(\ln s)^{-n}$. We prove there that this asymptotic series is fixed by unitarity and analyticity, except for a single parameter, connected to the scattering length.

In [6] we also prove, that any solution not belonging to Class 1 must belong to Class 2, which is caracterized by the following asymptotic form:

$$\operatorname{Re} f(s)_{s \to \infty} \simeq \begin{cases} \dfrac{2C_0 \ln s}{s} & \text{in the Case A.1} \\ \dfrac{(m-1)C_0}{s} & \text{in the Case A.2 for } m \neq 1 \\ O\left(\dfrac{1}{s^2}\right) & \text{in the Case A.2 for } m = 1\,. \end{cases} \tag{2.12}$$

Here, C_0 is the positive constant

$$C_0 = \frac{1}{\pi} \int_4^{\infty} \operatorname{Im} f(s) ds\,. \tag{2.13}$$

In contrast to the Class 1 solutions, the asymptotic form of the Class 2 solutions thus contains information about the low energy absorptive part.

3. Solution of the Right-Hand Cut Equation

As is well known, our non-linear problem could be easily solved with the Castillejo-Dalitz-Dyson [10] method if the left-hand cut were absent, that is for $m=0$ in the dispersion Relation **A.2**. This exact solution is of great interest to us in this work, since, as we will prove, the gross feature of it cannot be changed by the left-hand cut.

The exact solution for $m=0$ is

$$\begin{aligned} \frac{1}{f(s)} = \frac{1}{a_0} - \beta_0(s-4) - \sum_{n=1}^{\infty} \frac{\beta_n(s-4)}{(\alpha_n-4)(\alpha_n-s)} \\ - \frac{s-4}{\pi} \int_4^{\infty} \frac{\varrho(s')R(s')ds'}{(s'-4)(s'-s)}, \end{aligned} \tag{3.1}$$

where a_0 is the scattering length, and α_n is the position and β_n the residue of the n^{th} *CDD* pole, β_0 corresponding to a *CDD* pole at infinity. These constants have to be real and satisfy

$$\begin{cases} 0 < a_0 < \infty\,, & \\ 4 < \alpha_n < \infty\,, & n = 1, 2, \ldots \\ 0 \leqq \beta_n < \infty\,, & n = 0, 1, 2, \ldots. \end{cases} \tag{3.2}$$

Let us first look at the asymptotic behaviour of the solutions (3.1). If $\beta_0 \neq 0$ it is obvious that the term $-\beta_0 s$ dominates for large s, i.e., that

$$\mathrm{Re}\frac{1}{f(s)} = \frac{\mathrm{Re}\, f(s)}{|f(s)|^2} \underset{s\to\infty}{\simeq} -\beta_0 s\,. \tag{3.3}$$

This clearly corresponds to a Class 2 solution

$$\mathrm{Re}\, f(s) \underset{s\to\infty}{\simeq} -\frac{1}{\beta_0 s}, \quad \mathrm{Im}\, f(s) \underset{s\to\infty}{\simeq} \frac{R_0(\ln s)^\gamma}{\beta_0^2 s^2}, \tag{3.4}$$

according to Eq. (2.12), Case **A.2** with $m=0$.

If $\beta_0 = 0$ the integral will dominate the right-hand side of Eq. (3.1) for large s. According to Appendix 1, Eq. (A.1.6), we have

$$-\frac{s-4}{\pi}\,\mathrm{P}\int_4^\infty \frac{\varrho(s')R(s')ds'}{(s'-4)(s'-s)} \underset{s\to\infty}{\simeq} \frac{R_0}{\pi(1+\gamma)}(\ln s)^{1+\gamma}\,. \tag{3.5}$$

This corresponds to a Class 1 solution with

$$\mathrm{Re}\, f(s) \underset{s\to\infty}{\simeq} \frac{\pi(1+\gamma)}{R_0}(\ln s)^{-1-\gamma}, \quad \mathrm{Im}\, f(s) \underset{s\to\infty}{\simeq} \frac{\pi^2(1+\gamma)^2}{R_0}(\ln s)^{-2-\gamma}, \tag{3.6}$$

in accordance with Eq. (2.11) with $m=0$. Thus the absence or presence of a *CDD* pole at infinity determines whether the solution is of Classes 1 or 2.

For elastic unitarity, we can easily trace the variation of the phase shift for the Solution (3.1) through the formula

$$\cot\delta(s) = \frac{1}{\varrho(s)}\mathrm{Re}\frac{1}{f(s)}\,. \tag{3.7}$$

Assuming first that all $\beta_n = 0$ for $n \geqq 1$, we find from Eq. (3.1) that if $\beta_0 > 0$, $\mathrm{Re}(1/f)$ is positive at threshold, decreases through zero and reaches $-\infty$ at $s=\infty$. This corresponds to an increase of $\delta(s)$ from zero through $\pi/2$ to π at $s=\infty$. If instead $\beta_0 = 0$, $\mathrm{Re}(1/f)$ stays positive and approaches $+\infty$ at $s=\infty$. This is so since for elastic unitarity, the integral can be easily computed, and shown to give a positive contribution for all s. For $\gamma > 0$, on the other hand, the integral may change sign, depending on the actual form of $R(s)$, so we cannot draw such definite conclusions.

For elastic unitarity, and the sum in Eq. (3.1) absent, we thus have

$$\delta(\infty) - \delta(4) = \begin{cases} 0 & \text{for} \quad \beta_0 = 0, \beta_n = 0, n \geqq 1 \\ \pi & \text{for} \quad \beta_0 \neq 0, \beta_n = 0, n \geqq 1\,. \end{cases} \tag{3.8}$$

Adding now *CDD* poles, we find that each pole contributes one zero and one infinity to $\mathrm{Re}(1/f)$. Since the residue is negative, this means that each *CDD* pole adds π to $\delta(\infty)$.

Consider now a particular solution, for which

$$\delta(4) = 0, \delta(\infty) = \pi\nu, \quad \nu \text{ integer}\,. \tag{3.9}$$

Then the analysis above shows that this solution has exactly ν *CDD* poles (counting also the one at infinity), and that its multiplicity $\varkappa$, that is the number of arbitrary constants in it, is given by

$$\varkappa = \begin{cases} 2\nu + 1 & \text{for Class 1 solutions} \\ 2\nu & \text{for Class 2 solutions}. \end{cases} \tag{3.10}$$

This result is now proved for elastic unitarity and $m = 0$, and we go on to generalize it to inelastic unitarity, to $m \neq 0$ and to the case of dispersion Relation **A.1**.

4. Banach Space Formulation of the Non-linear Problem

The non-linear problem of finding solutions to the dispersion Relations **A.1** or **A.2** and the unitarity Condition **B** is best discussed within the framework of mappings of Banach spaces with suitable norms [5, 7].

We have chosen to work in a Banach space of Hölder continuous functions on $4 \leqq s < \infty$, which, however, in order to allow for Class 1 solutions, do not have to obey the Hölder condition at $s = \infty$. As $s \to \infty$ they are only requested to approach a limiting value at least as fast as $(\ln s)^{-k}$, $k \geqq 1$.

To be more explicit, we put

$$u = 1 - \frac{4}{s} \tag{4.1}$$

and let $H(\mu, k)$ be the space of Hölder continuous functions $\varphi(u)$ on $u \in [0, 1)$, including $u = 0$ but excluding $u = 1$, with the norm

$$\|\varphi\| = \sup_{u \in [0,1]} |\varphi(u)| + \sup_{u_1, u_2 \in [0,1]} \frac{|\varphi(u_1) - \varphi(u_2)|}{\left|\frac{u_1 - u_2}{1 - u_<}\right|^{\mu}} \left(\ln \frac{4}{1 - u_<}\right)^k, \tag{4.2}$$

where $0 < \mu < 1$, $k \geqq 1$ and $u_< = \min(u_1, u_2)$. That this is a Banach space, and even a normed ring, is proved in Appendix 2.

We shall also use the subspace $H_0(\mu, k)$, consisting of all the functions of $H(\mu, k)$ that are zero at $u = 1$, $\varphi(1) = 0$.

In $H_0(\mu, k)$ we use the simpler norm

$$\|\varphi\| = \sup_{u_1, u_2 \in [0,1]} \frac{|\varphi(u_1) - \varphi(u_2)|}{\left|\frac{u_1 - u_2}{1 - u_<}\right|^{\mu}} \left(\ln \frac{4}{1 - u_<}\right)^k, \tag{4.3}$$

since if this norm is finite and $\varphi(1) = 0$, it follows that $\sup |\varphi(u)|$ is finite.

Let finally $C(k)$ be the space of continuous functions on $[0, 1]$ with the norm

$$\|\varphi\| = \sup_{u \in [0,1]} \left\{ |\varphi(u)| \left(\ln \frac{4}{1 - u}\right)^k \right\}, \quad k \geqq 0. \tag{4.4}$$

For $k=0$ this is the usual supremum norm, for $k>0$ it also demands φ to approach zero as $u\to 1$ at least as fast as $(\ln 4/(1-u))^{-k}$.

Since Hölder continuity of the scattering amplitude follows from unitarity and analyticity, the discussion of Section 2 shows that any solution to the dispersion Relations **A.1** or **A.2** and the unitarity Condition **B** will have

$$\operatorname{Re} f(u)\in H_0(\mu, 1+\gamma)\,, \quad \operatorname{Im} f(u)\in H_0(\mu, 2+\gamma)\,, \tag{4.5}$$

with $\mu \leqq \frac{1}{2}$ and γ defined by Eq. (2.9).

Let us now write the dispersion Relations in the form

$$\varrho(u)\operatorname{Re} f(u)=[(K_R+K_L)\operatorname{Im} f](u)\,, \tag{4.6}$$

where

$$(K_R\varphi)(u)=\frac{\varrho(u)}{\pi}\,\mathrm{P}\int_0^1 \frac{(1-u)\varphi(u')du'}{(1-u')(u'-u)} \tag{4.7}$$

and

$$(K_L\varphi)(u)=(K_2\varphi)(u)=\frac{\varrho(u)}{\pi}\int_0^1 \frac{2(1-u)}{u(1-u')^2}\ln\left[1+\frac{u}{1-u}(1-u')\right]\varphi(u')du' \tag{4.8}$$

for the case of dispersion Relation **A.1**, and

$$(K_L\varphi)(u)=(\bar{K}_m\varphi)(u)=\frac{\varrho(u)}{\pi}\int_0^1 \frac{m(1-u)\varphi(u')du'}{(1-u')(1-u'u)}\,, \quad m\geqq 0\,, \tag{4.9}$$

for the case of dispersion Relation **A.2**.

K_R is a singular linear integral operator from $H_0(\mu, k+1)$ to $H_0(\mu, k)$, any $k>0$. It is bounded in the norm (4.3), as shown in Appendix 2. K_2 and $\bar{K}_m$ are non-singular linear integral operators, defining mappings of $C(k+1)$ into $H_0(\mu, k)$, which are also bounded in the norm (4.3).

K_2 and $\bar{K}_2$ have a similar behaviour near $u=1$, so that their difference

$$\Delta K = K_2-\bar{K}_2 \tag{4.10}$$

is more well-behaved than either of them. It is shown in Appendix 2 that ΔK defines a bounded map of $C(k)$ into $H_0(\mu, k)$.

Now $H(\mu, k)$ is a normed ring, multiplication being a well-defined operation in it. Therefore, the unitarity Condition **B** is a non-linear relation in $H(\mu, 1+\gamma)$, provided we assume

$$\eta(u)\in H(\mu, 1+\gamma)\,, \quad R(u)\left(\ln\frac{4}{1-u}\right)^{-\gamma}\in H(\mu, 1)\,. \tag{4.11}$$

Introduce the phase shift $\delta(u)\in H(\mu, 1+\gamma)$, and put

$$\operatorname{Re} f(u)=\frac{\eta(u)}{2\varrho(u)}\sin 2\delta(u)\,, \quad \operatorname{Im} f(u)=\frac{1}{2\varrho(u)}\{1-\eta(u)\cos 2\delta(u)\}\,. \tag{4.12}$$

Assuming also $\delta(0)=0$ and $\eta(0)=1$, so that we get normal threshold behaviour, Eq. (4.12) are well-defined relations in the normed ring $H(\mu, 1+\gamma)$.

Since further $\operatorname{Re} f(1)=0$, $\operatorname{Im} f(1)=0$, $\eta(1)=1$ for any solution according to Section 2, we must have $\delta(1)=\pi\nu$, $\nu=0, \pm 1, \pm 2, \ldots$. In fact we have as $u \to 1$

$$\delta(u)=\begin{cases}\pi\nu+\pi\sigma\left(\ln\dfrac{4}{1-u}\right)^{-1-\gamma}+O\left(\left(\ln\dfrac{4}{1-u}\right)^{-2-\gamma}\right) & \text{for Class 1 solutions}\\ \pi\nu+O((1-u)^{\mu}) & \text{for Class 2 solutions,}\end{cases} \tag{4.13}$$

where

$$\sigma=\frac{1+\gamma}{R_0(1+m)}. \tag{4.14}$$

With the phase shift, the dispersion relation can now be written

$$P(\delta(u)) \equiv \frac{1}{2}\eta(u)\sin 2\delta(u)-\left[(K_R+K_L)\frac{1-\eta\cos 2\delta}{2\varrho}\right](u)=0. \tag{4.15}$$

This a non-linear relation in $H(\mu, 1+\gamma)$, which is equivalent to the unitarity Condition **B** combined with the dispersion Relations **A.1** or **A.2**. In the following, it is the solutions of this equation that we shall study.

Since $H(\mu, 1+\gamma) \subset H(\mu, 1)$ for $\gamma \geqq 0$, we shall treat Eq. (4.15) as a relation in $H(\mu, 1)$, thereby including all cases.

We are interested in the local multiplicity of the solutions. Therefore, we assume $\delta(u)$ to be a solution to Eq. (4.15), of Classes 1 or 2, and make a small variation $\delta\delta(u)$ of it. Let P'_η be the Fréchet derivative of P, that is

$$P'_\eta(m, \delta)\delta\delta(u)=\eta(u)\cos 2\delta(u)\delta\delta(u)-\left[(K_R+K_L)\frac{\eta\sin 2\delta}{\varrho}\delta\delta\right](u). \tag{4.16}$$

The index η is to indicate that η has been held constant during the differentiation. The m dependence comes from K_L of Eq. (4.9). When we occasionally use Eq. (4.8) for K_L we just leave out the argument m. The Fréchet derivative is a linear operator in $H(\mu, 1)$, or in $H_0(\mu, 1)$ if we assume $\delta\delta(1)=0$.

At this point it should be observed, that the multiplicity of the solutions to Eq. (4.15) may depend on how the inelasticity is given. If $\eta(s)$ is a given function in the unitarity Condition **B**, then P'_η of Eq. (4.16) in the Fréchet derivative to be studied, but if $R(s)$ is the given function, we should instead define a Fréchet derivative with constant R. Here, we shall content ourselves with a study of the constant η case. Details about the constant R case can be found in [6].

Provided we can show that $P'_\eta(m, \delta)$ is a ϕ operator, i.e., that it is closed, has a closed range, and that its null space and defect space are finite dimensional, we can follow Lovelace [5] and employ the "implicit variable" theorem of Vainberg and Trenogin [8]. Putting α and β equal to the dimensions of the null space and the defect space, respectively, of $P'_\eta(m, \delta)$, the implicit variable theorem says that in the neighbourhood of the solution $\delta(u)$, the Eq. (4.15) is exactly equivalent to β non-linear scalar equations in α real variables.

The index $\varkappa = \alpha - \beta$ of the ϕ operator $P'_\eta(m, \delta)$ is thus the excess of the number of variables over the number of equations, or, in general, the dimensionality of the manifold of solutions to Eq. (4.15) containing $\delta(u)$. Thus, $\delta(u)$ should contain $\varkappa$ arbitrary parameters.

We now set out to prove that $P'_\eta(m, \delta)$ is a ϕ operator, and to determine its index $\varkappa$. For a smaller Banach space, containing only the Class 2 solutions, this has already been done by Lovelace [5]. We shall try to extend his proof to include all solutions. Because of the very slow decrease of the Class 1 solutions this is a very delicate task, as we shall see.

We decompose P'_η into a sum of a right-hand cut contribution

$$P'_R(\delta) = P'_\eta(0, \delta) \tag{4.17}$$

and a left-hand cut contribution

$$P'_L(m, \delta) = P'_\eta(m, \delta) - P'_\eta(0, \delta)\,. \tag{4.18}$$

The right-hand cut operator is examined in the next section and in Section 6 the influence of the left-hand cut contribution is analyzed.

5. The Index of the Right Hand Cut Fréchet Derivative

Combining the result of Section 3 with the implicit variable theorem immediately tells us that for elastic unitarity and $m = 0$, that is for the right-hand cut Fréchet derivative $P'_R(\delta)$, the index is $\varkappa$, where

$$\varkappa = \begin{cases} 2\nu + 1 & \text{if } \delta \text{ is a Class 1 solution with } \quad m = 0 \\ 2\nu & \text{if } \delta \text{ is a Class 2 solution with } \quad m = 0\,. \end{cases} \tag{5.1}$$

However, in order to be allowed to use the implicit variable theorem, we should first prove that $P'_R(\delta)$ is a ϕ operator. We would also like to cover inelastic unitarity, and study $P'_R(\delta)$ when δ is a solution with $m > 0$. Therefore, we shall now solve explicitly the equation

$$P'_R(\delta)\delta\delta(u) = \delta f(u)\,, \tag{5.2}$$

that is the singular integral equation

$$A(u)\delta\delta(u) - \frac{1}{i\pi} \mathrm{P} \int_0^1 \frac{(1-u)B(u')}{(1-u')(u'-u)} \delta\delta(u')du' = \frac{\delta f(u)}{\varrho(u)}\,. \tag{5.3}$$

Here, A and B are the functions

$$\begin{cases} A(u) = \dfrac{\eta(u)}{\varrho(u)} \cos 2\delta(u) \\[2ex] B(u) = i\dfrac{\eta(u)}{\varrho(u)} \sin 2\delta(u) \end{cases} \tag{5.4}$$

which are Hölder continuous on (0, 1), excluding the end points. But then Eq. (5.3) is of the type studied in Muskhelishvili's book [11], except that for Class 1 solutions we must relax the Hölder condition at $u=1$. The function ϕ,

$$\phi(z)=\frac{1}{2\pi i}\int_0^1 \frac{(1-z)B(u')\delta\delta(u')du'}{(1-u')(u'-z)}, \tag{5.5}$$

is analytic, except for a cut (0, 1). On the cut its limiting values $\phi^{\pm}(u)$, from above and below, respectively, belong to $H_0(\mu, 1)$ if $\delta\delta \in H_0(\mu, 1)$. This follows from the proof of the boundedness of K_R in Appendix 2 and the fact that $B(0)=0$ and

$$B(u)_{u \simeq 1}\, 2i\pi\sigma\left(\ln\frac{4}{1-u}\right)^{-1-\gamma} \tag{5.6}$$

when δ is a Class 1 solution with

$$\delta(u)_{u \simeq 1}\, \pi\nu+\pi\sigma\left(\ln\frac{4}{1-u}\right)^{-1-\gamma}. \tag{5.7}$$

Equation (5.3) can now be written

$$A(u)\delta\delta(u)-\phi^{+}(u)-\phi^{-}(u)=\frac{\delta f(u)}{\varrho(u)}, \tag{5.8}$$

which, since $\phi^{+}-\phi^{-}=B\delta\delta$, is equivalent to the Hilbert problem

$$\phi^{+}(u)=\frac{A(u)+B(u)}{A(u)-B(u)}\phi^{-}(u)+\frac{B(u)}{A(u)-B(u)}\frac{\delta f(u)}{\varrho(u)}. \tag{5.9}$$

Here

$$\frac{A(u)+B(u)}{A(u)-B(u)}=e^{4i\delta(u)}, \tag{5.10}$$

which is finite and non-zero on $0 \leqq u \leqq 1$. Taking the logarithm thus gives a finite function with the asymptotic form

$$\begin{aligned}\Delta(u)&=\frac{1}{2i}\ln\frac{A+B}{A-B}=2\delta(u)\\ &=2\pi\nu+2\pi\sigma\left(\ln\frac{4}{1-u}\right)^{-1-\gamma}+O\left(\left(\ln\frac{4}{1-u}\right)^{-2-\gamma}\right)\end{aligned} \tag{5.11}$$

as $u\to 1$, for any Class 1 solution with the asymptotic form (5.7).

For Class 2 solutions one gets instead

$$\Delta(u)=2\pi\nu+O((1-u)^{\mu}), \tag{5.11'}$$

that is, a Hölder condition also at $u=1$.

Introduce now the function

$$\Gamma(z)=\frac{1}{\pi}\int_0^1 \frac{\Delta(u')du'}{u'-z}, \tag{5.12}$$

which is analytic except for a cut (0, 1), on which $\Gamma^{\pm}(u)$ is Hölder continuous, excluding $u=1$, where

$$\Gamma^{\pm}(u)=\begin{cases} 2\nu\ln(1-u)-2\sigma\ln\left(\ln\dfrac{4}{1-u}\right)+O(1) & \text{for } \delta \text{ Class 1 and } \gamma=0 \\ 2\nu\ln(1-u)+O(1) \quad \text{otherwise}. \end{cases} \tag{5.13}$$

This follows as for $\phi^{\pm}$ above, using for the asymptotic form of the integral (5.12) also the results of Appendix 1.

The function $\exp(\Gamma)$ is now a solution to the homogeneous Hilbert problem, and a solution to the inhomogeneous problem (5.8) can be constructed as in [11]. However, we shall follow Lovelace, and use the Dashen-Frautschi method [12]. The D function is, since we have assumed no bound states, given by

$$D(u)=e^{-\frac{1}{2}\Gamma^{+}(u)}. \tag{5.14}$$

It has the asymptotic form

$$D(u)\underset{u\to 1}{\simeq}\begin{cases} \text{const}\,(1-u)^{-\nu}\left(\ln\dfrac{4}{1-u}\right)^{\sigma} & \text{for } \delta \text{ Class 1 and } \gamma=0 \\ \text{const}\,(1-u)^{-\nu} \quad \text{otherwise}, \end{cases} \tag{5.15}$$

or in the s variable

$$D(s)\underset{s\to\infty}{\simeq}\begin{cases} \text{const}\, s^{\nu}(\ln s)^{\sigma} & \text{for } \delta \text{ Class 1 and } \gamma=0 \\ \text{const}\, s^{\nu} \quad \text{otherwise}, \end{cases} \tag{5.15'}$$

where the constants ν and σ are defined by Eqs. (5.7) or (4.13) and (4.14).

According to the Dashen-Frautschi method, we next write a dispersion relation for the function $D^2\phi$, which is analytic, except for a cut (0, 1) in u or (4, ∞) in s. On the cut

$$\operatorname{Im}\{D^2(s)\phi^{+}(s)\}=|D(s)|^2\sin\Delta(s)\frac{\delta f(s)}{\varrho(s)}, \tag{5.16}$$

as follows from Eqs. (5.8), (5.12), and (5.14). The asymptotic form of this is

$$\operatorname{Im}\{D^2(s)\phi^{+}(s)\}\underset{s\to\infty}{\simeq}\begin{cases} \text{const}\, s^{2\nu}(\ln s)^{-2+2\sigma} & \text{for } \delta \text{ Class 1 and } \gamma=0 \\ \text{const}\, s^{2\nu}(\ln s)^{-2-\gamma} & \text{for } \delta \text{ Class 1 and } \gamma\neq 0 \\ \text{const}\, s^{2\nu-\mu} & \text{for } \delta \text{ Class 2}, \end{cases} \tag{5.17}$$

provided we assume the perturbation to go to zero as $s\to\infty$, that is $\delta f\in H_0(\mu, 1)$.

If δ is Class 1, $\gamma=0$, and $2\sigma\geqq 1$ we obviously have to make $2\nu+1$ subtractions in the dispersion relation for $D^2\phi$, otherwise it is sufficient with 2ν subtractions. Since the non-linear problem of Section 3 had no solutions with $\nu < 0$, we assume here that $\nu \geqq 0$. Putting

$$\varkappa = \begin{cases} 2\nu + 1 & \text{for Class 1, } \gamma = 0 \text{ and } 2\sigma \geqq 1\,, \\ 2\nu & \text{otherwise}\,, \end{cases} \tag{5.18}$$

the dispersion relation for $D^2\phi$ can be written

$$D^2(s)\phi(s) = Q_{\varkappa-1}(s) + \frac{s^\varkappa}{\pi} \int_4^\infty \frac{|D(s')|^2 \sin \Delta(s')\delta f(s')ds'}{s'^\varkappa(s'-s)2\varrho(s')}\,, \tag{5.19}$$

where $Q_{\varkappa-1}(s)$ is an arbitrary polynomial of degree $\leqq \varkappa - 1$. This is obviously the most general analytic function fulfilling Eqs. (5.16), (5.17), and the bound

$$|D^2(s)\phi(s)| \underset{|s|\to\infty}{\lesssim} \begin{cases} \text{const}\,|s|^{2\nu}(\ln|s|)^{-1+2\sigma} & \text{for } \delta \text{ Class 1 and } \gamma = 0 \\ \text{const}\,|s|^{2\nu}(\ln|s|)^{-1} & \text{otherwise}\,, \end{cases} \tag{5.20}$$

which follows from Eq. (5.15′) and the fact that $\phi^\pm \in H_0(\mu, 1)$.

The expression for $\delta\delta$ corresponding to Eq. (5.19) can now be obtained with the help of Eq. (5.8):

$$\begin{aligned} \delta\delta(s) = {} & \frac{\delta f(s)}{\eta(s)} \cos 2\delta(s) + \frac{2\varrho(s)}{\eta(s)|D(s)|^2} \\ & \cdot \left[Q_{\varkappa-1}(s) + \frac{s^\varkappa}{\pi} \mathrm{P} \int_4^\infty \frac{|D(s')|^2 \sin \Delta(s')\delta f(s')ds'}{s'^\varkappa(s'-s)2\varrho(s')} \right]. \end{aligned} \tag{5.21}$$

This is the explicit solution of Eq. (5.3). It contains exactly $\varkappa$ arbitrary parameters, the coefficients of the polynomial $Q_{\varkappa-1}(s)$. Note that

$$\frac{2\varrho(s)}{\eta(s)|D(s)|^2} Q_{\varkappa-1}(s) \underset{s\to\infty}{\simeq} \begin{cases} \text{const}\, s^{\varkappa-2\nu-1}(\ln s)^{-2\sigma} & \text{for } \delta \text{ Class 1 and } \gamma = 0 \\ \text{const}\, s^{\varkappa-2\nu-1} & \text{otherwise}\,, \end{cases} \tag{5.22}$$

so that if we, e.g., had chosen $\varkappa = 2\nu + 1$ for $2\sigma < 1$, $\delta\delta$ would not have belonged to $H_0(\mu, 1)$. Therefore, we are forced to choose $\varkappa$ according to Eq. (5.18).

Using the asymptotic form (5.11) for Δ it is easy to see (cf. Appendix 2) that the last term of Eq. (5.21), like the first, gives contributions to $\delta\delta \in H_0(\mu, 1)$ for any $\delta f \in H_0(\mu, 1)$. [Provided $\eta(s) \neq 0$, which we assume.] Thus the defect space of $P_R'(\delta)$ is empty, and its range is the whole Banach space $H_0(\mu, 1)$ thus closed. Since $P_R'(\delta)$ is bounded, it is a closed operator. Since further, according to Eq. (5.21), its null space has dimension $\varkappa$, we have now proved $P_R'(\delta)$ to be a ϕ operator with index $\varkappa$ given by Eq. (5.18).

Let $\mathcal{N}_\varkappa$ be the null space of $P_R'(\delta)$, and define the factor space

$$\mathscr{B} = H_0(\mu, 1)/\mathcal{N}_\varkappa\,. \tag{5.23}$$

Let furhter G_R be the restriction of $P'_R(\delta)$ to $\mathscr{B}$. Then G_R defines a 1 to 1 mapping of $\mathscr{B}$ onto $H_0(\mu, 1)$, and its inverse G_R^{-1} exists. From Eq. (5.21) we obtain

$$G_R^{-1}(\delta)\delta f(s) = \frac{\delta f(s)}{\eta(s)} \cos 2\delta(s) + \frac{\varrho(s)s^{\varkappa}}{\pi\eta(s)|D(s)|^2} P \int_4^{\infty} \frac{|D(s')|^2 \sin \Delta(s')\delta f(s')ds'}{s'^{\varkappa}(s'-s)\varrho(s')} . \tag{5.24}$$

This is a bounded operator form $H_0(\mu, 1)$ to $\mathscr{B}$, [provided $\eta(s) \neq 0$] as proved in Appendix 2. G_R and G_R^{-1} are ϕ operators with index zero.

We shall finally take advantage of the fact that for $m=1$ the left-hand cut discontinuity is just the mirror image in $s=2$ of the one of the right-hand cut. This means that in the variable

$$t = (s-2)^2 , \tag{5.25}$$

the amplitude is a real analytic function with just a right-hand cut $(4, \infty)$. As a result of this, we can solve the equation

$$P'_\eta(1, \delta)\delta\delta(u) = \delta f(u) \tag{5.26}$$

with the method used above for $P'_R(\delta)$.

In the variable t we have

$$P'_\eta(1, \delta)\delta\delta(t) = \varrho(t)A(t)\delta\delta(t) - \frac{\varrho(t)}{\pi i} \mathrm{P} \int_4^{\infty} \frac{B(t')\delta\delta(t')dt'}{t'-t} , \tag{5.27}$$

where A and B are given in Eq. (5.4) as functions of

$$u = 1 - \frac{4}{s} = 1 - \frac{4}{2+\sqrt{t}} . \tag{5.28}$$

The asymptotic form of a Class 1 solution $\delta(t)$ in the t variable is

$$\delta(t)_{t \to \infty}^{\simeq} \pi\nu + 2^{1+\gamma}\pi\sigma(\ln t)^{-1-\gamma} , \tag{5.29}$$

since $s \sim \sqrt{t}$. Thus the asymptotic form of the D function for $\gamma = 0$ gets an extra factor 2 in the exponent, as compared with Eq. (5.15'):

$$D(t)_{t \to \infty}^{\simeq} \begin{cases} \text{const}\, t^{\nu}(\ln t)^{2\sigma} & \text{for } \delta \text{ Class 1 and } \gamma = 0 \\ \text{const}\, t^{\nu} & \text{otherwise} . \end{cases} \tag{5.30}$$

This implies the following rule for the number of subtractions, instead of Eq. (5.18):

$$\varkappa = \begin{cases} 2\nu + 1 & \text{for } \delta \text{ Class 1, } \gamma = 0 \text{, and } 4\sigma \geqq 1 \\ 2\nu & \text{otherwise} . \end{cases} \tag{5.31}$$

6. The Left-Hand Cut Contribution

In this section we shall discuss whether or not the addition of the left-hand cut operator $P'_L(m, \delta)$ to the ϕ operator $P'_R(\delta)$ may change its index. The tools we have at disposal for this task are some theorems on ϕ operators from [9].

First we shall employ Theorem 2.3 of [9] (= Theorem 5A of Lovelace [5]), which says that if $P'_L(m, \delta)$ is a compact operator, the sum

$$P'_\eta(m, \delta) = P'_R(\delta) + P'_L(m, \delta) \tag{6.1}$$

is a ϕ operator with the same index as $P'_R(\delta)$.

We recall from Section 4 that we had two different forms of K_L, Eqs. (4.8) and (4.9), respectively, and that their difference ΔK, defined in Eq. (4.10), is asymptotically smaller than either of them, so that ΔK defines a bounded map from $C(k)$ to $H_0(\mu, k)$ (cf. Appendix 2).

For the corresponding Fréchet derivatives P'_L we correspondingly define the difference

$$\begin{aligned} \Delta P'_\eta(\delta)\delta\delta(u) &= [P'_\eta(\delta) - P'_\eta(2, \delta)]\delta\delta(u) \\ &= -\left(\Delta K \frac{\eta \sin 2\delta}{\varrho} \delta\delta\right)(u). \end{aligned} \tag{6.2}$$

This is a compact operator in $H_0(\mu, 1)$, as is seen from the following decomposition into a series of successive mappings:

(i) a compact map from $H_0(\mu, 1)$ to $C(0)$: this embedding is compact by the Arzelà-Ascoli theorem (see Appendix 2);

(ii) a bounded map from $C(0)$ to $C(1)$: multiplication by

$$\frac{\eta \sin 2\delta}{\varrho};$$

(iii) a bounded map from $C(1)$ to $H_0(\mu, 1)$: the action of the operator ΔK.

Since the product of a compact operator and a bounded operator is compact [13, VI.5.4], it follows that the entire mapping is compact.

That the operator (7.2) is compact implies by the above theorem that the Fréchet derivative P'_η has the same index for the case of dispersion Relation **A.1** as for dispersion Relation **A.2** with $m = 2$, and that if P'_η is a ϕ operator for one of those cases it is so for the other also. Therefore, we can in the following concentrate on the case of dispersion Relation **A.2**.

Next, let us discuss the operator P'_L with the explicit form

$$\begin{aligned} P'_L(m, \delta)\delta\delta(u) &= -\left(K_L \frac{\eta \sin 2\delta}{\varrho} \delta\delta\right)(u) \\ &= -m\frac{\varrho(u)}{\pi} \int_0^1 \frac{(1-u)\eta(u')\sin 2\delta(u')}{(1-u')(1-u'u)\varrho(u')} \delta\delta(u')du'. \end{aligned} \tag{6.3}$$

If δ is a Class 2 solution, or Class 1 solution when $\gamma > 0$, we can follow Lovelace and decompose P'_L as above for $\Delta P'$. We write

$$P'_L(m,\delta)\delta\delta(u) = -\left[K_L\left(\ln\frac{4}{1-u'}\right)^{\gamma-1}\frac{\eta(u')\sin 2\delta(u')}{\varrho(u')}\cdot\left(\ln\frac{4}{1-u'}\right)^{1-\gamma}\delta\delta(u')\right](u)\,. \tag{6.4}$$

The successive mappings are:

(i) a bounded map from $H_0(\mu, 1)$ to $H_0(\mu,\gamma)$: multiplication with $[\ln(4/(1-u'))]^{1-\gamma}$;

(ii) a compact map from $H_0(\mu,\gamma)$ to $C(0)$: this embedding is compact by the Arzelà-Ascoli theorem, provided $\gamma > 0$;

(iii) a bounded map from $C(0)$ to $C(2)$: multiplication with [cf. Eqs. (5.5) and (5.12)]

$$\left(\ln\frac{4}{1-u'}\right)^{\gamma-1}\frac{\eta(u')\sin 2\delta(u')}{\varrho(u')}\,.$$

(iv) a bounded map from $C(2)$ to $H_0(\mu, 1)$: the action of the operator K_L.

The net result is thus a compact map of $H_0(\mu, 1)$ into itself, proving that for $\gamma > 0$ $P'_L(m,\delta)$ is a compact operator in $H_0(\mu, 1)$.

For $\gamma = 0$ and δ a Class 2 solution, the same result follows by simply replacing γ by $\varepsilon > 0$ in Eq. (6.4) and repeating the argument. However, if δ is a Class 1 solution, this trick would not work, since the map (iii) would not be bounded.

Employing again Theorem 2.3 of [9], we now have that for δ a Class 2 solution, or a Class 1 solution for $\gamma > 0$, $P'_\eta(m,\delta)$ is a ϕ operator with the index fixed by Eq. (5.18) to $\varkappa = 2\nu$.

For δ a Class 1 solution and $\gamma = 0$, however, we cannot prove the compactness of P'_L. If the limit $\gamma \to 0$ is taken in Eq. (4.16) we formally get

$$P'_\eta(m,\gamma,\delta) \xrightarrow[\gamma\to 0]{} P'_\eta(m,0,\delta)\,, \tag{6.5}$$

but since this limit in Eq. (4.16) is not approached uniformly in u, Eq. (6.5) must be understood in a weak sense, not in the uniform operator topology. (If the limit had been uniform, P'_L would have been compact by [13], VI.5.3.)

Because of the non-uniformness of the limit (6.5), the proof that P'_η is a ϕ operator also for $\gamma = 0$ becomes rather long. We give such a proof, for almost all m [1], in [6]. For shortness, we omit that proof here.

What is the index of the ϕ operator $P'_\eta(m,0,\delta)$? From Section 5 we have that if

$$\delta(s) = \pi\nu + \pi\sigma(\ln s)^{-1} + O((\ln s)^{-2}) \tag{6.6}$$

[1] To prove this, we made in [6] a technical assumption about the size of the continuous spectrum of a certain bounded operator. This assumption is probably valid, but could not be rigorously proved to be so.

then the index of

$$P_\eta'(0,0,\delta) \quad \text{is} \quad \varkappa = \begin{cases} 2\nu+1 & \text{if} \quad 2\sigma \geqq 1 \\ 2\nu & \text{if} \quad 2\sigma < 1 \end{cases} \tag{6.7}$$

and the index of

$$P_\eta'(1,0,\delta) \quad \text{is} \quad \varkappa = \begin{cases} 2\nu+1 & \text{if} \quad 4\sigma \geqq 1 \\ 2\nu & \text{if} \quad 4\sigma < 1 . \end{cases} \tag{6.8}$$

From Section 2 we further know that in order for δ to be a solution to the non-linear problem, we must couple σ and m. But then the conditions of Eqs. (6.7) and (6.8) on $\varkappa$ become

$$\sigma = \frac{1+\gamma}{R_0(1+m)} \Rightarrow \varkappa = \begin{cases} 2\nu+1 & \text{if} \quad R_0 \leqq 2 \\ 2\nu & \text{if} \quad R_0 > 2 . \end{cases} \tag{6.9}$$

We shall now prove that this rule for $\varkappa$ is valid not only for $m=0$ and $m=1$, but for (almost) all m. We can see the mechanism: if m is increased, the phase shift δ decreases faster as $s\to\infty$, tending to decrease the index of $P_\eta'(0,0,\delta)$, but adding the (non-compact!) left-hand cut contribution to get $P_\eta'(m,0,\delta)$ tends to increase the index, so that the final rule for the index, Eq. (6.9), does not contain m!

Gokhberg and Krein [9] have some theorems on the stability of the index, which we shall use. Their Theorem 2.2 tells us that adding a bounded operator with sufficiently small norm to a ϕ operator does not change its index. Thus there is a neighbourhood of $m=0$ and $m=1$, where the index is given by Eq. (6.9).

Theorem 3.5 of [9] says that if $P_\eta'(m,0,\delta)$ is a holomorphic operator function for m in a certain region Ω, and is a ϕ operator for each $m\in\Omega$, then the index is constant throughout the region Ω.

Take the non-linear Eq. (4.15):

$$P(m,\delta(u)) = \frac{1}{2}\eta(u)\sin 2\delta(u) - \left[(K_R + m\bar{K}_1)\frac{1-\eta\cos 2\delta}{2\varrho}\right](u) = 0 . \tag{6.10}$$

The only m dependence of $P(m,\delta)$ is the factor m in front of the left-hand cut operator. Thus it can be trivially extended to a holomorphic operator function, by just allowing complex multipliers in the space $H(\mu,1)$, so that it consists of complex-valued Hölder-continuous functions on $u\in(0,1)$.

But then it can be proved from the implicit variable theorem that each solution $\delta(u)$ of Eq. (6.10) has an analytical dependence on the complex variable m. The region of analyticity, Ω, is moreover only bounded by points where the Fréchet derivative $P_\eta'(m,\delta)$ ceases to be continuous, or to be a ϕ operator, or possibly where $\varkappa$, α or β for this ϕ operator change. Since the last points are isolated, and $P_\eta'(m,\delta)$ is a ϕ operator for almost all m, we can conclude that Ω is a connected region in the complex m plane containing the whole real positive m axis, except possibly for some isolated points.

The Fréchet derivative $P'_\eta(m, \delta)$ depends on m explicitly, and implicitly through δ. As just discussed, both these dependences are analytical in the region Ω, so we can employ the theorem 3.5 given above. Thus $\varkappa$ is given by Eq. (6.9) for almost all real positive m. Q.E.D.

7. Results and Discussion

In this paper, and in [6], we have studied the solutions to the dispersion Relation **A.1** or **A.2** and the unitarity Condition **B**, given in the Introduction. The main results about those solutions are the following.

I. The solutions fall into two distinct classes, according to their large s behaviour. The asymptotic behaviour is like an inverse logarithm for Class 1 solutions [Eq. (2.11)] and like an inverse power for Class 2 solutions [Eqs. (2.12)].

The two classes can also be characterized with the integral

$$I = \int_4^\infty \operatorname{Im} f(s) ds . \tag{7.1}$$

Class 2 consists of those solutions for which this integral is convergent. They are well behaved, and their asymptotic form contains information about the absorptive part at low energy. Class 1, on the other hand, consists of those solutions for which the integral (7.1) is divergent. They are dominated for large s by their own asymptotic form, so that this form can be determined self-consistently, and does not contain any information about the absorptive part at low energy.

The next result is [6]:

II. Class 1 solutions have a complete asymptotic expansion in inverse powers of $\ln s$. This expansion has exactly one free parameter, which does not enter the leading term, however. The parameter is related to the scattering length, but contains in general also other contributions.

About the local multiplicity of the solutions to our non-linear problem we get for Class 2 solutions the same result as Lovelace, presented in the Introduction. For Class 1 solutions, we get:

III. If in our non-linear problem we change dispersion Relation **A.1** to dispersion Relation **A.2** with $m=2$, all other characteristics of the problem remaining unchanged, the local multiplicity does not change.

IV. The local multiplicity may depend on whether we ask for solutions with the inelasticity $\eta(s)$ given, or the inelasticity ratio $R(s)$ given in the unitarity Condition **B**.

For

$$R(s) \underset{s\to\infty}{\simeq} R_0 (\ln s)^\gamma, \ \gamma \geqq 0, \ R_0 > 0 \, (R_0 \geqq 1 \text{ for } \gamma = 0) \tag{7.2}$$

and $\delta(s)$ a Class 1 solution satisfying

$$\delta(s) \underset{s\to\infty}{\simeq} \pi\nu + O((\ln s)^{-1-\gamma}), \tag{7.3}$$

we get the following more specific results:

V. If $\eta(s)$ is prescribed, the local multiplicity of the Class 1 solution $\delta(s)$ to our non-linear problem is, for almost all $m \geqq 0$, given by

$$\varkappa = \begin{cases} 2\nu+1 & \text{if} \quad \gamma=0 \quad \text{and} \quad R_0 \leqq 2 \\ 2\nu & \text{if} \quad \gamma=0 \quad \text{and} \quad R_0 > 2 \\ 2\nu & \text{if} \quad \gamma>0 . \end{cases} \tag{7.4}$$

VI. If $R(s)$ is prescribed, the local multiplicity of the Class 1 solution $\delta(s)$ to our non-linear problem is, for almost all $m \geqq 0$, given by [6]

$$\varkappa = 2\nu + 1 \quad \text{for any } \gamma \text{ and } R_0 . \tag{7.5}$$

We have not been able to exclude the possibility that for some isolated m values the above result might not hold. This fact is related to the possibility of spontaneous symmetry breaking, discussed by Lovelace [5]. We propose to resolve this difficulty for each particular interesting case by numerical calculations.

A numerical solution of our non-linear problem with the Newton-Kantorovich [14] method would, at the same time as it resolved this difficulty, also provide an existence proof for the solution of that particular case.

As a final result we formulate what we have to do, besides using analyticity, unitarity and crossing symmetry, in order to get a unique solution for the S wave amplitude:

(i) decide whether we want a Class 1 or a Class 2 solution;

(ii) fix the total variation of the phase shift (to $\pi\nu$);

(iii) determine a number of parameters [2ν for Class 2 solutions, 2ν or $2\nu+1$ according to Eqs. (7.4) and (7.5) for Class 1 solutions] from auxiliary conditions.

If other partial waves or coupled channels are included, the number of arbitrary parameters increases, as discussed by Lovelace [5].

About the question of the existence of solutions to our non-linear problem we have, unfortunately, very little to say. For Class 2 solutions the existence has been proved (under certain restrictions on the input function) by Atkinson *et al.* [7], but their analysis does not apply to Class 1 solutions. Moreover, since the Class 1 solutions are not connected to the zero solution, because of their fixed asymptotic form, it seems to us extremely difficult to modify the method of [7] to include these solutions. Therefore, we believe the Newton-Kantorovich method [14] to be the best to prove the existence of Class 1 solutions. The present work contains some necessary prerequisite for such a study, since one must know the multiplicity, and work in the factor space $\mathscr{B}$ of Eq. (5.23) in order to have a well-defined inverse operator G^{-1}, as necessary in the Newton-Kantorovich method.

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Appendix 1

The aim of this Appendix is to derive the asymptotic form for large s of integrals of the type

$$I(A)=\int_4^\infty \frac{ds'}{(s'+A)(\ln s')^{j+2}} \tag{A.1.1}$$

for $A=O(s)$ as $s\to\infty$.

Beginning with positive A and $j>-1$, $I(A)$ is obviously convergent, approaches zero as $A\to\infty$, and we have

$$I(A)=\frac{1}{A}\int_4^A \frac{ds'}{(\ln s')^{j+2}}\left(1+\frac{s'}{A}\right)^{-1}+\int_A^\infty \frac{ds'}{s'(\ln s')^{j+2}}\left(1+\frac{A}{s'}\right)^{-1},$$

which gives

$$I(A)=\sum_{i=0}^\infty \frac{(-1)^i}{A^{i+1}}\int_4^A \frac{s'^i ds'}{(\ln s')^{j+2}}+\sum_{i=0}^\infty (-A)^i\int_A^\infty \frac{ds'}{s'^{i+1}(\ln s')^{j+2}}. \tag{A.1.2}$$

Since we only need the leading asymptotic behaviour, it is here sufficient to note that

$$\int_A^\infty \frac{ds'}{s'(\ln s')^{j+2}}=\frac{1}{(j+1)(\ln A)^{j+1}}$$

and that

$$\int_4^A \frac{s'^i ds'}{(\ln s')^{j+2}}=O(A^{i+1}(\ln A)^{-j-2}),$$

$$\int_A^\infty \frac{ds'}{s'^{i+1}(\ln s')^{j+2}}=O(A^{-i}(\ln A)^{-j-2}).$$

Thus, we can conclude that

$$I(A)=\frac{1}{(j+1)(\ln A)^{j+1}}+O((\ln A)^{-j-2}) \quad \text{as } A\to\infty. \tag{A.1.3}$$

For negative $A<-4$ the integral (A.1.1) is defined by its Cauchy principal value. The contribution to the integral from near the singularity is then

$$
\begin{aligned}
&\mathrm{P}\int_{|A|-\varepsilon}^{|A|+\varepsilon}\frac{ds'}{(s'+A)(\ln s')^{j+2}}=\mathrm{P}\int_{-\varepsilon}^{\varepsilon}\frac{dx}{x\left[\ln|A|+\ln\left(1-\frac{x}{A}\right)\right]^{j+2}}\\
&=(\ln|A|)^{-j-2}\left[\mathrm{P}\int_{-\varepsilon}^{\varepsilon}\frac{dx}{x}-(j+2)\mathrm{P}\int_{-\varepsilon}^{\varepsilon}\frac{dx}{x}\right.\\
&\qquad\left.\cdot\left\{\frac{\ln\left(1-\frac{x}{A}\right)}{\ln|A|}+O\left(\frac{\ln^2\left(1-\frac{x}{A}\right)}{\ln^2|A|}\right)\right\}\right]\\
&=(\ln|A|)^{-j-2}\left[\mathrm{P}\int_{-\varepsilon}^{\varepsilon}\frac{dx}{x}+\frac{j+2}{A\ln|A|}\mathrm{P}\int_{-\varepsilon}^{\varepsilon}dx\left\{1+O\left(\frac{x}{A}\right)\right\}\right]\\
&\doteq O\left(\frac{\varepsilon}{A(\ln|A|)^{j+3}}\right)
\end{aligned}
\tag{A.1.4}
$$

as $\varepsilon/A\to 0$. This proves that the contribution from the region near the singularity can be included in the rest term of expressions like Eq. (A.1.3). But this also means that it is unnecessary to subtract the integral (A.1.4) explicity from $I(A)$ before making the expansion in Eq. (A.1.2). Thus Eq. (A.1.3) is valid also for $A\to-\infty$.

For $j\leqq -1$ we must introduce a subtraction in $I(A)$, and define

$$I(A)=\int_4^{\infty}\frac{A\,ds'}{s'(s'+A)(\ln s')^{j+2}}\,. \tag{A.1.5}$$

For this integral the above method gives the asymptotic form

$$I(A)=\begin{cases}\ln(\ln|A|)+O(1) & \text{for } j=-1\\ \dfrac{1}{-j-1}(\ln|A|)^{-j-1}+O((\ln|A|)^{-j-2}) & \text{for } j<-1\end{cases} \tag{A.1.6}$$

as $A\to\pm\infty$.

Appendix 2

In this Appendix we have collected some proofs of Banach space properties. We begin by proving that $H_0(\mu,k)$, defined in Section 4, is a complete space.

If the norm (4.3) is finite, it follows that

$$|\varphi(u)-\varphi(1)|\leqq\|\varphi\|\left(\ln\frac{4}{1-u}\right)^{-k}<\|\varphi\|\,, \tag{A.2.1}$$

which, since $\varphi(1)=0$ for $\varphi\in H_0(\mu,k)$, implies that

$$H_0(\mu,k)\subset C(k)\subset C(0)\,,\quad k>0\,. \tag{A.2.2}$$

Assume that $\{\varphi_n\}$ is a Cauchy sequence in $H_0(\mu, k)$, that is

$$\|\varphi_{n+p}-\varphi_n\| \leqq \varepsilon_n, \quad \text{any } p, \quad \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty\,. \tag{A.2.3}$$

Then Eq. (A.2.1) implies that it is a Cauchy sequence also in $C(0)$, which is known to be complete. But then there exists a φ in $C(0)$, such that

$$|\varphi(u)-\varphi_n(u)| \leqq \varepsilon_n\,. \tag{A.2.4}$$

Moreover, Eq. (A.2.3) implies that

$$|\varphi_{n+p}(u_1)-\varphi_{n+p}(u_2)-\varphi_n(u_1)+\varphi_n(u_2)| \leqq \varepsilon_n \left|\frac{u_1-u_2}{1-u_<}\right|^{\mu} \left(\ln \frac{4}{1-u_<}\right)^{-k}. \tag{A.2.5}$$

Since the right-hand side of this equation is independent of p, we can take the limit $p \to \infty$. Then according to Eq. (A.2.4) we get

$$|\varphi(u_1)-\varphi(u_2)-\varphi_n(u_1)+\varphi_n(u_2)| \leqq \varepsilon_n \left|\frac{u_1-u_2}{1-u_<}\right|^{\mu} \left(\ln \frac{4}{1-u_<}\right)^{-k}, \tag{A.2.6}$$

which implies that

$$\|\varphi\| < \infty\,, \quad \|\varphi-\varphi_n\| \leqq \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty\,. \tag{A.2.7}$$

Thus $H_0(\mu, k)$ is complete.

The proof above is almost identical to the one, for a smaller space, of Muskhelishvili (p. 132 of [11]). It works equally well for $H(\mu, k)$ and $C(k)$.

Next we prove that the Banach space $H(\mu, k)$ is also a normed ring. This follows if we can prove that the product of two elements φ and ψ of $H(\mu, k)$ is again an element of $H(\mu, k)$, and that

$$\|\varphi\psi\| \leqq \|\varphi\|\,\|\psi\|\,. \tag{A.2.8}$$

Let us distinguish the two parts of the norm (4.2) by indices

$$\|\varphi\| = \|\varphi\|_C + \|\varphi\|_0\,. \tag{A.2.9}$$

Then $\varphi\psi$ is Hölder continuous, and

$$\begin{aligned} \|\varphi\psi\|_0 &\leqq \sup_{u_1 u_2} \frac{[|\varphi(u_1)-\varphi(u_2)|\,|\psi(u_1)| + |\varphi(u_2)|\,|\psi(u_1)-\psi(u_2)|]}{\left|\dfrac{u_1-u_2}{1-u_<}\right|^{\mu} \left(\ln \dfrac{4}{1-u_<}\right)^{-k}} \\ &\leqq \|\varphi\|_0\,\|\psi\|_C + \|\varphi\|_C\,\|\psi\|_0\,. \end{aligned} \tag{A.2.10}$$

Thus

$$\begin{aligned} \|\varphi\psi\| &\leqq \|\varphi\|_C\,\|\psi\|_C + \|\varphi\|_0\,\|\psi\|_C + \|\varphi\|_C\,\|\psi\|_0 \\ &\leqq \|\varphi\|\,\|\psi\|\,. \quad \text{Q.E.D.} \end{aligned} \tag{A.2.11}$$

In Section 6 we need the fact that the embedding (A.2.2) of $H_0(\mu, k)$ in $C(0)$ is a compact map. To prove this, we prove that the bounded set $\|\varphi\| \leqq C$ in $H_0(\mu, k)$ is a compact set in $C(0)$.

Any function of this set if uniformly bounded by Eq. (A.2.1), and further satisfies

$$|\varphi(u_1)-\varphi(u_2)| \leqq C\frac{\delta^\mu}{(1-u_<)^\mu}\left(\ln\frac{4}{1-u_<}\right)^{-k}, \quad \delta=|u_1-u_2|\,. \tag{A.2.12}$$

Now

$$\underset{0\leqq u_<\leqq 1-\delta}{\mathrm{Max}} \frac{\delta^\mu}{(1-u_<)^\mu}\left(\ln\frac{4}{1-u_<}\right)^{-k} = \left(\ln\frac{4}{\delta}\right)^{-k} \underset{\delta\to 0}{\longrightarrow} 0 \tag{A.2.13}$$

for any $k>0$. Thus for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$|\varphi(u_1)-\varphi(u_2)| \leqq \varepsilon \quad \text{for} \quad |u_1-u_2|\leqq \delta = 4\exp\left[-\left(\frac{C}{\varepsilon}\right)^{\frac{1}{k}}\right]>0 \tag{A.2.14}$$

for any function of the set $\|\varphi\|\leqq C$. But this is the statement that this set of functions is equicontinuous. Then the Arzelà-Ascoli theorem ([13], IV.6.7) tells us that the set is compact. Q.E.D.

Next, we want to prove that the operator K_R,

$$(K_R\varphi)(u) = -\frac{\varrho(u)}{\pi}\,\mathrm{P}\int_0^1 \frac{(1-u)\varphi(u')du'}{(1-u')(u'-u)}\,, \tag{A.2.15}$$

is a bounded operator from $H_0(\mu, k+1)$ to $H_0(\mu, k)$, $k>0$, $0<\mu<\frac{1}{2}$. We write

$$(K_R\varphi)(u) = I_1(u) - I_2(u)\,, \tag{A.2.16}$$

where

$$I_1(u) = \frac{\varrho(u)}{\pi}\int_0^1 \frac{\varphi(u')du'}{u'-1}\,, \quad I_2(u) = \frac{\varrho(u)}{\pi}\,\mathrm{P}\int_0^1 \frac{\varphi(u')du'}{u'-u}\,. \tag{A.2.17}$$

Since $\varrho(u)=\sqrt{u}$ and the integral is convergent for any $\varphi\in H_0(\mu, k+1)$, $k>0$, $I_1(u)$ is obviously a Hölder continuous function on $0\leqq u\leqq 1$ with $\mu\leqq\frac{1}{2}$. Further, as proved in Muskhelishvili ([11], pp. 46–48), the integral in $I_2(u)$ is a Hölder continuous function on the open interval $0<u<1$, with the same Hölder index μ as φ. At $u=0$ the integral has a logarithmic singularity, but since it is to be multiplied with $\sqrt{u}$, we obtain that $I_2(u)$ is Hölder continuous on $0\leqq u<1$ with index $\mu<\frac{1}{2}$. This implies that

$$|(K_R\varphi)(u_1)-(K_R\varphi)(u_2)| \leqq C_R(\varepsilon)C\left|\frac{u_1-u_2}{1-u_1}\right|^\mu, \quad 0\leqq u_1\leqq u_2\leqq 1-\varepsilon,\ \varepsilon>0 \tag{A.2.18}$$

as soon as $\varphi\in H_0(\mu, k+1)$, $\mu<\frac{1}{2}$, $k>0$, with norm $\|\varphi\|\leqq C$. For the constant $C_R(\varepsilon)$, which may be ε dependent, one can, with some extra work, following Muskhelishvili, obtain a finite, explicit value. However, here we only need the fact that it is finite for any fixed $\varepsilon>0$.

What happens when $\varepsilon\to 0$? For the behaviour of the integral (A.2.15) when $u\to 1$ we can use the analysis of Appendix 1. Equation (A.1.6) gives

$$(K_R\varphi)(u)_{u \to 1} \simeq \frac{C}{\pi k}\left(\ln \frac{4}{1-u}\right)^{-k} \tag{A.2.19}$$

if

$$\varphi(u)_{u \to 1} \simeq C\left(\ln \frac{4}{1-u}\right)^{-k-1}. \tag{A.2.20}$$

But then

$$|(K_R\varphi)(u_1)-(K_R\varphi)(u_2)| \leqq C'_R C\left(\ln \frac{4}{1-u_1}\right)^{-k}, \quad 0 \leqq u_1 \leqq u_2 \leqq 1, \tag{A.2.21}$$

for some finite constant C'_R, since, besides fulfilling Eq. (A.2.19), $(K_R\varphi)(u)$ is bounded on $0 \leqq u < 1$. This shows that if we let $\varepsilon \to 0$, $u_2 \to 1$ and hold u_1 constant in Eq. (A.2.18), the equation is still valid, with a finite $C_R(0)$. Finally, we put

$$\bar{C}_R = \text{Max}\,\{C_R(0)(\ln 4)^k, C'_R\} \tag{A.2.22}$$

and combine Eqs. (A.2.18) and (A.2.21) to get

$$|(K_R\varphi)(u_1)-(K_R\varphi)(u_2)| \leqq \bar{C}_R C\left|\frac{u_1-u_2}{1-u_1}\right|^{\mu}\left(\ln \frac{4}{1-u_1}\right)^{-k},$$
$$0 \leqq u_1 \leqq u_2 \leqq 1, \tag{A.2.23}$$

which is to say that

$$\|K_R\| = \sup_{\varphi} \frac{\|K_R\varphi\|_k}{\|\varphi\|_{k+1}} \leqq \bar{C}_R. \quad \text{Q.E.D.} \tag{A.2.24}$$

The operator K_L has, for the case of dispersion Relation **A.2**, the form

$$(K_L\varphi)(u) = -\frac{m\varrho(u)}{\pi}\int_0^1 \frac{(1-u)\varphi(u')du'}{(1-u')(1-u'u)} = \frac{m\varrho(u)}{\pi} I_3(u). \tag{A.2.25}$$

For any $\varphi \in C(k+1)$, $k>0$, the integral $I_3(u)$ is convergent for $0 \leqq u < 1$. As $u \to 1$, we obtain from Appendix 1 the asymptotic form

$$I_3(u)_{u \to 1} \simeq \frac{C}{k}\left(\ln \frac{4}{1-u}\right)^{-k} \quad \text{if} \quad \varphi(u)_{u \to 1} \simeq C\left(\ln \frac{4}{1-u}\right)^{-k-1}, \tag{A.2.26}$$

which implies that in analogy with Eq. (A.2.21) we have

$$|I_3(u_1)-I_3(u_2)| \leqq C'_L C\left(\ln \frac{4}{1-u_1}\right)^{-k}, \quad 0 \leqq u_1 \leqq u_2 \leqq 1, \tag{A.2.27}$$

for some finite constant C'_L. Moreover, we get directly from Eq. (A.2.25) that

$$|I_3(u_1)-I_3(u_2)| = \left|(u_1-u_2)\int_0^1 \frac{\varphi(u')du'}{(1-u_2u')(1-u_1u')}\right|$$
$$\leqq C_L(\varepsilon)C\left|\frac{u_1-u_2}{1-u_1}\right| \tag{A.2.28}$$
$$\text{for} \quad 0 \leqq u_1 \leqq u_2 \leqq 1-\varepsilon, \; \varepsilon > 0.$$

Combining the last two equations gives that $I_3(u)$ has a finite norm in $H_0(1,k)$ for any finite element φ of $C(k+1)$. This implies for K_L that its norm as an operator from $C(k+1)$ to $H_0(\mu,k)$, $\mu\leqq\frac{1}{2}$, is bounded by $\bar{C}_L$, where

$$\bar{C}_L=\frac{m}{\pi}\operatorname{Max}\{C_L(0)(\ln 4)^k, C_L'\}. \tag{A.2.29}$$

For the case of dispersion Relation **A.1**, the form of K_L is

$$\begin{aligned}(K_L\varphi)(u) &= -\frac{2\varrho(u)}{\pi}\int_0^1\frac{1-u}{u(1-u')^2}\ln\left[1+\frac{u}{1-u}(1-u')\right]\varphi(u')du' \\ &= \frac{2\varrho(u)}{\pi}I_4(u).\end{aligned} \tag{A.2.30}$$

This integral is also convergent for $0\leqq u<1$ for any $\varphi\in C(k+1)$, $k>0$. As $u\to 1$, the integrand has the same asymptotic form as the integrand of Eq. (A.2.25), and the asymptotic analysis of Appendix 1 shows that

$$I_4(u)\underset{u\to 1}{\simeq}\frac{C}{k}\left(\ln\frac{4}{1-u}\right)^{-k} \quad \text{if} \quad \varphi(u)\underset{u\to 1}{\simeq}C\left(\ln\frac{4}{1-u}\right)^{-k-1}. \tag{A.2.31}$$

This implies that Eq. (A.2.27) is valid also for $I_4(u)$. Moreover, $I_4(u)$ has a continuous derivative on $\varepsilon<u<1-\varepsilon$, since

$$I_4'(u)=\frac{1}{u}\int_0^1\left\{\frac{1}{1-uu'}-\frac{1}{u(1-u')}\ln\left[1+\frac{u}{1-u}(1-u')\right]\right\}\frac{\varphi(u')du'}{1-u'} \tag{A.2.32}$$

is a uniformly convergent integral on this interval. Thus Eq. (A.2.28) is valid for $I_4(u)$, provided we cut off the interval at $u_1=\varepsilon$. However, this cut-off is not necessary, since from the fact that

$$x-\tfrac{1}{2}x^2<\ln(1+x)<x\,,\quad 0<x<1\,, \tag{A.2.33}$$

it follows that

$$|I_4(u_1)-I_4(u_2)|\leqq\frac{u_2}{2(1-u_2)}\int_0^1|\varphi(u')|du'<\text{const}\,\varepsilon \quad \text{for} \quad 0\leqq u_1\leqq u_2\leqq\varepsilon. \tag{A.2.34}$$

But then the boundedness of K_L follows just as for the previous case.

Consider now the difference ΔK,

$$(\Delta K\varphi)(u)=\frac{2\varrho(u)}{\pi}\int_0^1\left\{\frac{1}{1-uu'}-\frac{1}{u(1-u')}\ln\left[1+\frac{u}{1-u}(1-u')\right]\right\}\cdot\frac{(1-u)\varphi(u')du'}{1-u'}, \tag{A.2.35}$$

for $\varphi \in C(k)$, $k>0$. Let first $u \leqq 1-\varepsilon$, $\varepsilon>0$, and divide the integration interval into two, expanding the logarithm in the second term:

$$(\Delta K\varphi)(u)=\frac{2\varrho(u)}{\pi}\int_0^{1-\varepsilon}\left\{\frac{1}{1-uu'}-\frac{1}{u(1-u')}\ln\left[1+\frac{u}{1-u}(1-u')\right]\right\}\cdot\frac{(1-u)\varphi(u')du'}{1-u'}+\frac{2\varrho(u)}{\pi}\int_{1-\varepsilon}^{1}\left\{\frac{u}{1-uu'}+O\left(\frac{u}{1-u}\right)\right\}\varphi(u')du'. \quad (A.2.36)$$

This shows that there appear no convergence problems at $u'=1$ when we take $\varphi \in C(k)$ instead of $C(k+1)$, as long as $u \leqq 1-\varepsilon$. Thus $(\Delta K\varphi)(u)$ is Hölder continuous on $0 \leqq u \leqq 1-\varepsilon$, just as $K_L\varphi$.

To see what happens when $u\to 1$ we note that the asymptotic form of the discontinuity on the left-hand cut is the same to leading order for dispersion Relations **A.1** and **A.2** with $m=2$, but differs in the next order, Since in Eq. (A.2.35) we take the difference between these two cases, the leading term in $K_L\varphi$ is cancelled, but the next appears, that is

$$(\Delta K\varphi)(u)=O\left(\left(\ln\frac{4}{1-u}\right)^{-k-1}\right) \quad \text{if} \quad \varphi(u)\underset{u\to 1}{\simeq} C\left(\ln\frac{4}{1-u}\right)^{-k-1}. \quad (A.2.37)$$

It follows from Appendix 1, by analogy with Eq. (A.1.6), that this equation is in fact valid for all $k \neq -1$. Thus

$$(\Delta K\varphi)(u)\in H_0(\mu, k) \quad \text{for any} \quad \varphi \in C(k). \quad \text{Q.E.D.} \quad (A.2.38)$$

Finally, we want to prove that the operator G_R^{-1} of Eq. (5.24) is a bounded operator in $H_0(\mu, 1)$. We can write

$$G_R^{-1}(\delta)\delta f(s)=\frac{\cos 2\delta(s)}{\eta(s)}\delta f(s)-\frac{s^{\varkappa}}{\eta(s)|D(s)|^2}\cdot K_R\left\{\frac{|D(s')|^2\sin\Delta(s')}{s'^{\varkappa}\varrho(s')}\delta f(s')\right\}(s). \quad (A.2.39)$$

The factor multiplying δf inside the brackets in the last term has the asymptotic form [cf. Eqs. (5.11), (5.15'), (5.18)]

$$\frac{|D(s')|^2\sin\Delta(s')}{s'^{\varkappa}\varrho(s')} \underset{s'\to\infty}{\simeq} \begin{cases} \text{const}\,\frac{1}{s'}(\ln s')^{-1+2\sigma} & \text{for } \delta \text{ Class 1, } \gamma=0, \text{ and } 2\sigma \geqq 1 \\ \text{const}\,(\ln s')^{-1+2\sigma} & \text{for } \delta \text{ Class 1, } \gamma=0, \text{ and } 2\sigma<1 \\ \text{const}\,(\ln s')^{-1-\gamma} & \text{for } \delta \text{ Class 1 and } \gamma \neq 0 \\ \text{const}\,s'^{-\mu} & \text{for } \delta \text{ Class 2} \end{cases} \quad (A.2.40)$$

and is finite for all s', since the zero at threshold of $\varrho(s')$ is cancelled by a similar zero of $\Delta(s')$. If $\delta f \in H_0(\mu, 1)$ the asymptotic form of the second term in Eq. (A.2.39) is

$$\frac{s^{\varkappa}}{\eta(s)|D(s)|^2} K_R \left\{ \frac{|D(s')|^2 \sin \Delta(s')}{s'^{\varkappa} \varrho(s')} \delta f(s') \right\}(s)$$

$$\underset{s \to \infty}{\sim} \begin{cases} \text{const}\,(\ln s)^{-1} & \text{for } \delta \text{ Class 1} \\ \text{const}\, s^{-\mu} & \text{for } \delta \text{ Class 2}, \end{cases} \tag{A.2.41}$$

which, since all factors are Hölder continuous, implies that

$$G_R^{-1}(\delta)\delta f \in H_0(\mu, 1) \quad \text{if} \quad \delta f \in H_0(\mu, 1) \quad \text{and} \quad \eta(s) \neq 0. \tag{A.2.42}$$

With some extra work, an explicit bound for the norm of G_R^{-1} as an operator in $H_0(\mu, 1)$ can be obtained, but here we only need the fact that a bound exists.

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