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On Infinite Direct Products of Continuous Unitary One-Parameter Groups

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Abstract. We discuss the infinite product of unitary operators in an incomplete direct product of Hilbert spaces. Necessary and sufficient conditions are derived under which this infinite product leads to a continuous unitary one-parameter group provided each factor is assumed to have this property. A certain minimal condition guarantees the existence of a renormalized unitary group. An application is made to product representations of the canonical commutation relations in order to determine the admissible test functions.

1. Introduction

Our investigations about infinite products of unitary one-parameter groups are motivated by the following physical situation. Consider a system which consists of infinitely many dynamically independent subsystems. The time evolution of the subsystems may be described by unitary operators $U_r(t)$ acting in separable Hilbert spaces H_r; $r = 1, 2, \ldots$ labelling the subsystems. Independence of the various subsystems is achieved most simply if one takes as representation space H for the total system an incomplete direct product of the Hilbert spaces H_r, $H = \bigotimes_r (H_r, \phi_r)$ [1]. $\{\phi_r\}_{r=1}^{\infty}$, a sequence of unit vectors with $\phi_r \in H_r$, which is called reference vector determines a separable subspace of the nonseparable complete direct product. The (naively suggested) time evolution of the total system should then be given by $U(t) = \bigotimes_r U_r(t)$. However, in general the infinite product of continuous one-parameter groups of unitary operators does not lead to a continuous unitary one-parameter group. $U(t) = \bigotimes_r U_r(t)$ may happen to be not even unitary in $H = \bigotimes_r (H_r, \phi_r)$.

So, we are dealing with a family $\{U_r(\lambda)\}_{r=1}^{\infty}$ of unitary operators $U_r(\lambda)$, $\lambda \in \mathbb{R}$, acting in separable Hilbert spaces H_r such that

$$U_r(\lambda)\, U_r(\lambda') = U_r(\lambda + \lambda'), \tag{1.1}$$

and $U_r(\lambda)$ is weakly (strongly) continuous in the real parameter λ. Let us define a truncated product of the $U_r(\lambda)$ by

$$U^{(n)}(\lambda) = \bigotimes_r U_r^{(n)}(\lambda)\,, \tag{1.2}$$

with $U_r^{(n)}(\lambda) = U_r(\lambda)$ for $r \leqq n$ and $U_r^{(n)}(\lambda) = \mathbb{1}_r$ for $r > n$. Certainly, $U^{(n)}(\lambda)$ represents a continuous unitary one-parameter group in each incomplete direct product space. This is because $U^{(n)}(\lambda)$ can be written as a finite product of commuting operators each of which having the stated property. In the next Section we shall formulate necessary and sufficient conditions under which $U^{(n)}(\lambda)$ or a subsequence $U^{(n_k)}(\lambda)$ converges strongly to a continuous unitary one-parameter group. We shall see that a certain minimal condition implies the existence of real (renormalization) constants $\{\alpha_r\}_{r=1}^{\infty}$ so that $\bigotimes_r U_r(\lambda)\, e^{-i\lambda\alpha_r}$ is unitary. In Ref. [2] this formalism has been used to construct a unitary scattering operator for a model with infrared singularities. Section 3 contains an application of our results to determine the admissible test functions for direct product representations of the canonical commutation relations (CCRs). A necessary condition on those test functions given by Woods [3] turns out to be also sufficient.

2. Relation to Probability Theory and Main Results

Before going into details we recall some consequences which follow from von Neumann's definition of convergence of an infinite product [1]. Let $\{z_r\}_{r=1}^{\infty}$ be complex numbers, $z_r = |z_r|\, e^{i\theta_r}$ with $-\pi < \theta_r \leqq \pi$. Then $\prod_r z_r$ converges to a nonzero value if and only if

(i) $\prod_r |z_r|$ converges irrespective of any order of the factors to a nonzero value *and*

(ii) $\sum_r |\theta_r| < \infty$.

Condition (i) is equivalent to $\sum_r |1 - |z_r|| < \infty$ and all $z_r \neq 0$.

The following lemma states a necessary condition for the convergence of $U^{(n_k)}(\lambda)$. Any subsets $\Delta \subset \mathbb{R}$ occurring are assumed to be Borel sets. By $\mu(\Delta)$ we denote the Lebesgue measure of Δ.

Lemma 2.1. *Let n_k, $k = 1, 2, \ldots$, be a subsequence of the positive integers, $n_{k+1} > n_k$ for all k, and let $U^{(n_k)}(\lambda)$ be given by* (1.2). *If $U^{(n_k)}(\lambda)$ (with $k \to \infty$) converges strongly in $H = \bigotimes_r (H_r, \phi_r)$ to a continuous unitary one-parameter group $U(\lambda)$ then there exists a subset $\Delta \subset \mathbb{R}$, $\mu(\Delta) \neq 0$, such that*

$$\sum_r (1 - |\langle \phi_r, U_r(\lambda)\, \phi_r \rangle|) < \infty \quad \textit{for all } \lambda \in \Delta\,. \tag{2.1}$$

Proof. The group relation (1.1) implies $U_r(0) = \mathbb{1}_r$ for all r and, therefore, $U(0) = \mathbb{1}$. Because of the continuity of $U(\lambda)$ we have for all λ close enough to zero, $\lambda \in \Delta$ say,

$$\left|\left\langle \bigotimes_r \phi_r, U(\lambda) \bigotimes_r \phi_r \right\rangle\right| = \lim_{k\to\infty} \prod_{r=1}^{n_k} |\langle \phi_r, U_r(\lambda)\,\phi_r\rangle|$$

$$= \prod_{r=1}^{\infty} |\langle \phi_r, U_r(\lambda)\,\phi_r\rangle| \neq 0 .$$

Note that $|\langle \phi_r, U_r(\lambda)\,\phi_r\rangle| \leqq 1$ for all r and $\lambda \in \mathbb{R}$, thus $\prod_r |\langle \phi_r, U_r(\lambda)\,\phi_r\rangle|$ converges irrespective of any ordering. Equation (2.1) is necessary in order that an infinite product converges to a nonzero value [1].

To prove the following theorem strong use is made of the intimate relationship between continuous one-parameter groups of unitary operators and characteristic functions of random variables.

Theorem 2.2. *The following conditions are equivalent:*

(a) *There are real numbers* $\{\alpha_r\}_{r=1}^{\infty}$ *so that* $U_{\mathrm{ren}}^{(n)}(\lambda) = \bigotimes_{r=1}^{n} U_r(\lambda)\, e^{-i\lambda\alpha_r} \otimes \mathbb{1}$ *converges strongly in* $H = \bigotimes_r (H_r, \phi_r)$ *to a continuous unitary one-parameter group* $U_{\mathrm{ren}}(\lambda) = \bigotimes_r U_r(\lambda)\, e^{-i\lambda\alpha_r}$.

(b) *There exists a subset* $\Delta \subset \mathbb{R}$, $\mu(\Delta) \neq 0$, *and*

$$\sum_r (1 - |\langle \phi_r, U_r(\lambda)\,\phi_r\rangle|) \equiv \sum_r (1 - |f_r(\lambda)|) < \infty \quad \textit{for all } \lambda \in \Delta .$$

Proof. (a)→(b): Is immediate by Lemma 2.1.

(b)→(a): Let $f_r(\lambda) = \langle \phi_r, U_r(\lambda)\,\phi_r\rangle$. Since f_r is a continuous function of positive type satisfying $f_r(0) = 1$, $\{f_r\}_{r=1}^{\infty}$ may be regarded as characteristic functions of mutually independent random variables $\{\sigma_r\}_{r=1}^{\infty}$ on a probability space ([4], Chapters II and III). The Condition (b) then implies by Theorem 2.7 in Chapter 3 of [5] that the series $\sum_r (\sigma_r - \alpha_r)$ converges with probability 1 for some sequence of real numbers $\{\alpha_r\}_{r=1}^{\infty}$. Since $f_r(\lambda)\, e^{-i\lambda\alpha_r}$ is the characteristic function of $\sigma_r - \alpha_r$, it follows from Theorem 2.7 in [5] that

$$\sum_r |1 - f_r(\lambda)\, e^{-i\lambda\alpha_r}| < \infty \quad \text{for all } \lambda \in \mathbb{R}, \tag{2.2}$$

the convergence being uniform in every finite interval of λ. According to Lemma 3.2 of [2] (see also Theorem 3.1 of [6]) $U_{\mathrm{ren}}(\lambda) = \bigotimes_r U_r(\lambda)\, e^{-i\lambda\alpha_r}$ is unitary in $H = \bigotimes_r (H_r, \phi_r)$ for all $\lambda \in \mathbb{R}$ and $U_{\mathrm{ren}}(\lambda) =$ strong-$\lim_{n\to\infty} U_{\mathrm{ren}}^{(n)}(\lambda)$. The group relation is obvious by the form of $U_{\mathrm{ren}}(\lambda)$. To prove continuity we remark that for any $\Psi \in H$ and any $\varepsilon > 0$ there exists $N < \infty$ and $\psi_N \in \bigotimes_{r=1}^{N} H_r$ such that $\left\| \Psi - \psi_N \otimes \left(\bigotimes_{r>N} \phi_r \right) \right\| < \varepsilon$ ([7], Lemma 3.1). Thus it is sufficient to prove that

$$\left\langle \bigotimes_r \phi_r, U_{\text{ren}}(\lambda) \bigotimes_r \phi_r \right\rangle = \prod_r f_r(\lambda)\, e^{-i\lambda\alpha_r}$$

is continuous in λ, and this follows from the uniform convergence of $\sum_r |1 - f_r(\lambda)\, e^{-i\lambda\alpha_r}|$.

Remark 2.3. The sequence $\{\alpha_r\}_{r=1}^{\infty}$ is not uniquely determined. Let $\alpha'_r = \alpha_r + \varepsilon_r$ and $\sum_r |\varepsilon_r| < \infty$. Then $\lim_{n\to\infty} \bigotimes_{r=1}^{n} U_r(\lambda)\, e^{-i\lambda\alpha'_r} = e^{-i\lambda\Sigma\varepsilon_r} U_{\text{ren}}(\lambda)$ which is unitary, continuous in λ and fulfills (1.1) along with $U_{\text{ren}}(\lambda)$. If, on the other hand, (2.2) holds also true with $\{\alpha_r\}_{r=1}^{\infty}$ replaced by $\{\alpha'_r\}_{r=1}^{\infty}$, then we have [1] (equivalence of product vectors) that

$$\prod_r \langle U_r(\lambda)\, e^{-i\lambda\alpha'_r} \phi_r, U_r(\lambda)\, e^{-i\lambda\alpha_r} \phi_r \rangle = \prod_r e^{i\lambda(\alpha'_r - \alpha_r)}$$

exists for all $\lambda \in \mathbb{R}$. By Corollary 2.9 of Ref. [8] we get $\sum_r |\alpha'_r - \alpha_r| < \infty$. Therefore, the generator of $U_{\text{ren}}(\lambda)$ is unique up to a finite additive constant.

The next lemma indicates how to determine the sequence of renormalization constants $\{\alpha_r\}_{r=1}^{\infty}$. Let $f_r(\lambda) = |f_r(\lambda)|\, e^{i\beta_r(\lambda)}$. Since $f_r(\lambda)$ is continuous in λ and $f_r(0) = 1$ we choose $\beta_r(0) = 0$ and $\beta_r(\lambda)$ continuous as long as $f_r(\lambda) \neq 0$.

Lemma 2.4. *Let Condition* (b) *of Theorem* 2.2 *be fulfilled and let* $\beta_r(\lambda)$ *be as above. Then* $\sum_r \left| \frac{1}{\lambda} \beta_r(\lambda) - \alpha_r \right| < \infty$ *for every* $\lambda \neq 0$ *and* $\sum_r |\beta_r(\lambda) - \lambda\alpha_r|$ *converges uniformly in* λ *on every bounded interval.*

Proof. Condition (b) implies that $|f_r(\lambda)| \xrightarrow[r]{} 1$ uniformly in λ on every bounded interval I. Let us assume this interval to be centered at zero. Then $f_r(\lambda) \neq 0$ and $\beta_r(\lambda)$ is continuous in λ for $r > N$ and $\lambda \in I$ with some finite $N = N(I)$. Moreover, $\prod_{r>N} f_r(\lambda)\, e^{-i\lambda\alpha_r} = \prod_{r>N} |f_r(\lambda)|\, e^{i(\beta_r(\lambda) - \lambda\alpha_r)}$ converges uniformly in λ to a nonzero value. This implies uniform convergence of $\prod_{r>N} e^{i|\beta_r(\lambda) - \lambda\alpha_r|} = e^{i\sum_{r>N} |\beta_r(\lambda) - \lambda\alpha_r|}$. Now, $\beta_r(\lambda) - \lambda\alpha_r$ is continuous for $\lambda \in I$ and $r > N$ and $\beta_r(\lambda) - \lambda\alpha_r = 0$ for $\lambda = 0$ and all r. It follows that $\sum_{r>N} |\beta_r(\lambda) - \lambda\alpha_r|$ converges uniformly on I. Since I was arbitrary, $\sum_r \left| \frac{1}{\lambda} \beta_r(\lambda) - \alpha_r \right| < \infty$ for every $\lambda \neq 0$.

According to Remark 2.3 one may especially choose $\alpha_r = \beta_r(1)$.

The minimal Condition (b) of Theorem 2.2 guarantees the existence of real numbers $\{\alpha_r\}_{r=1}^{\infty}$ such that $\bigotimes_r U_r(\lambda)\,\phi_r$ is equivalent to $\bigotimes_r e^{i\lambda\alpha_r}\,\phi_r$. According to Theorem 5.1 of Ref. [3] one would argue that this condition is even sufficient to find a subsequence n_k so that $U^{(n_k)}(\lambda)$ converges strongly to a unitary one-parameter group. The last half (sufficiency part) of Theorem 5.1 in [3], however, is incorrect as can be seen from the following.

Theorem 2.5. *Let $U^{(n_k)}(\lambda)$ be given by* (1.2) *and let $\beta_r(\lambda)$ be defined as in Lemma* 2.4. *$U^{(n_k)}(\lambda)$ converges strongly in $H = \bigotimes_r (H_r, \phi_r)$ to a continuous unitary one-parameter group $U(\lambda)$ if and only if*

(i) $\sum_r (1 - |f_r(\lambda)|) < \infty$ *for* $\lambda \in \Delta$, $\mu(\Delta) \neq 0$, *and*

(ii) $\sum_k \left(\sum_{r=n_{k-1}+1}^{n_k} \beta_r(\lambda_0) \right)$ *converges for some* $\lambda_0 \neq 0$.

Proof. Necessity: Lemma 2.1 shows necessity of Condition (i). By Theorem 2.2 and Lemma 2.4 we have that

$$V^{(n_k)}(\lambda) = \bigotimes_{r=1}^{n_k} U_r(\lambda)\, e^{-i\frac{\lambda}{\lambda_0}\beta_r(\lambda_0)} \otimes \mathbb{1}$$

converges strongly for all λ with any $\lambda_0 \neq 0$. It follows that

$$U^{(n_k)}(\lambda)\, V^{(n_k)}(-\lambda) = e^{i\frac{\lambda}{\lambda_0}\sum_{r=1}^{n_k}\beta_r(\lambda_0)}$$

converges for all λ [note that $U^{(n_k)}(\lambda)$ commutes with all $V^{(n_l)}(\lambda')$]. By Corollary 2.9 of Ref. [8] this implies convergence of $\sum_k \left(\sum_{r=n_{k-1}+1}^{n_k} \beta_r(\lambda_0) \right)$.

Sufficiency: If (i) is satisfied then there exists a sequence $\{\alpha_r\}_{r=1}^{\infty}$ such that $U^{(n_k)}(\lambda) \prod_{r=1}^{n_k} e^{-i\lambda\alpha_r}$ converges strongly for all λ to a continuous unitary one-parameter group. According to Lemma 2.4 and the preceding remark we have $\sum_r |\beta_r(\lambda) - \lambda\alpha_r| < \infty$ for all λ, especially for $\lambda = \lambda_0$ so so that Condition (ii) implies convergence of $\sum_k \left(\sum_{r=n_{k-1}+1}^{n_k} \alpha_r \right)$. It follows that $\prod_{r=1}^{n_k} e^{-i\lambda\alpha_r}$ converges to a continuous unitary group $e^{-i\lambda\alpha}$ and we have the desired result.

We turn now to the question whether one can always find a sequence of unit vectors $\{\phi_r\}_{r=1}^{\infty}$ satisfying Condition (b) of Theorem 2.2.

Theorem 2.6. *Given a family of continuous unitary one-parameter groups* $\{U_r(\lambda)\}_{r=1}^{\infty}$ *acting in Hilbert spaces* H_r *there exists a sequence of unit vectors* $\{\phi_r \in H_r\}_{r=1}^{\infty}$ *such that*

$$\sum_r (1 - |\langle \phi_r, U_r(\lambda)\, \phi_r \rangle|) \equiv \sum_r (1 - |f_r(\lambda)|) < \infty \tag{2.3}$$

for all λ *of some subset of* $\mathbb{R}$ *with nonzero Lebesgue measure.*

Proof. Let $U_r(\lambda) = e^{i\lambda A_r} = \int e^{i\lambda x}\, dE_x^{(r)}$ and let x_r be a growth point of $E_x^{(r)}$, i.e., $E_{x_r+\varepsilon}^{(r)} - E_{x_r}^{(r)} > 0$ for every $\varepsilon > 0$. Choose a sequence of positive real numbers $\{\varepsilon_r\}_{r=1}^{\infty}$ such that $\sum_r \varepsilon_r < \infty$. Let $\phi_r \in (E_{x_r+\varepsilon_r}^{(r)} - E_{x_r}^{(r)})\, H_r$ and $\|\phi_r\| = 1$ for all r. Then, for $|\lambda| \leqq 1$,

$$\begin{aligned} 1 - |f_r(\lambda)| &= 1 - |e^{-i\lambda x_r} f_r(\lambda)| \leqq |1 - e^{-i\lambda x_r} f_r(\lambda)| \\ &\leqq \left| \int (1 - e^{i\lambda(x - x_r)})\, d\langle \phi_r, E_x^{(r)} \phi_r \rangle \right| \\ &\leqq \int_0^{\varepsilon_r} |1 - e^{i\lambda y}|\, d\langle \phi_r, E_{x_r+y}^{(r)} \phi_r \rangle \\ &\leqq \operatorname*{Max}_{|\lambda| \leqq 1,\, y \leqq \varepsilon_r} 2 \left| \sin \frac{\lambda y}{2} \right| \leqq \varepsilon_r . \end{aligned}$$

Hence $\sum_r (1 - |f_r(\lambda)|) < \infty$ for $|\lambda| \leqq 1$.

It is clear that (2.3) does not determine the sequence $\{\phi_r\}_{r=1}^{\infty}$ uniquely. As can be seen from the construction above it does not even fix the weak equivalence class.

3. Application to Product Representations of the CCRs

We want to apply our results to product representations of the CCRs in order to determine the admissible test functions. We consider representations which are infinite direct products of Fock and/or Schrödinger representations of the CCRs. So, let $\{V_r\}_{r=1}^{\infty}$ be a sequence of test function spaces which we choose to be complex Hilbert spaces with finite or enumerably infinite dimension. For $g_r \in V_r$ let $W_r(g_r)$ be the unitary Weyl operators of the Fock or, respectively, Schrödinger representation [9]. Acting in H_r the $W_r(\lambda g_r)$ are weakly (strongly) continuous in the real parameter λ and satisfy the multiplication law

$$W_r(g_r)\, W_r(h_r) = W_r(g_r + h_r)\, e^{i(g_r, h_r)}, \tag{3.1}$$

where (g_r, h_r) is given by

$$(g_r, h_r) = \frac{1}{2i} \{\langle h_r, g_r \rangle - \langle g_r, h_r \rangle\}, \tag{3.2}$$

and $\langle g_r, h_r \rangle$ denotes the scalar product in V_r. It follows that for fixed g_r and real parameters λ, λ' we have

$$W_r(\lambda g_r)\, W_r(\lambda' g_r) = W_r((\lambda + \lambda')\, g_r) \tag{3.3}$$

for all r. Consider the infinite Cartesian product $V^* = \bigtimes_r V_r$ of the V_r consisting of all sequences $g = \{g_r\}_{r=1}^{\infty}$ with $g_r \in V_r$. Let V be that subspace of V^* which consists of all sequences with only finitely many non-vanishing components, $V = \{g = \{g_r\}_{r=1}^{\infty} : g_r = 0$ for almost all $r\}$. We extend the bilinear form (3.2) to elements of V by defining

$$(g, h) = \sum_r (g_r, h_r)\,. \tag{3.4}$$

Let $H = \bigotimes_r (H_r, \phi_r)$ be an incomplete tensor product of the H_r with $\|\phi_r\| = 1$ for all r. For $g \in V$, $g = \{g_r\}_{r=1}^{\infty}$ we define in H a representation of the CCRs by

$$W(g) = \bigotimes_r W_r(g_r)\,. \tag{3.5}$$

Let τ be the weakest topology on V such that the map $g \to \bigotimes_r W_r(\lambda g_r)\, \phi_r$ is norm-continuous for all $\lambda \in \mathbb{R}$. Since the representation $W(g)$ is irreducible, τ coincides with the weakest vector topology on V such that $g \to W(g)$ is continuous in the strong operator topology [8, 10]. Let $g^{(n)} = \{g_r^{(n)}\}_{r=1}^{\infty} \in V$ be a Cauchy sequence in τ and let $\Psi = \bigotimes_r \psi_r \in H$. Then

$$\Big\|\bigotimes_r W_r(\lambda g_r^{(n)})\, \psi_r - \bigotimes_r W_r(\lambda g_r^{(m)})\, \psi_r\Big\|^2$$
$$= 2\Big(1 - \operatorname{Re} \prod_r \langle \psi_r, W_r(-\lambda g_r^{(n)})\, W_r(\lambda g_r^{(m)})\, \psi_r \rangle\Big) \to 0$$

for $m, n \to \infty$ and all λ. This implies

$$|\langle \psi_r, W_r(-\lambda g_r^{(n)})\, W_r(\lambda g_r^{(m)})\, \psi_r \rangle| = |\langle \psi_r, W_r(\lambda(g_r^{(m)} - g_r^{(n)}))\, \psi_r \rangle| \to 1$$

for all r and $\lambda \in \mathbb{R}$. Let us choose $\psi_s = \Omega_s$, the Fock (Schrödinger) vacuum for some arbitrary s, and $\psi_r = \phi_r$ for $r \neq s$. Then $\bigotimes_r \psi_r \in H$ and [12]

$$|\langle \psi_s, W_s(\lambda(g_s^{(m)} - g_s^{(n)}))\, \psi_s \rangle| = \exp(-\tfrac{1}{2}\lambda^2 \|g_s^{(m)} - g_s^{(n)}\|^2)\,.$$

Thus $g_s^{(n)}$ converges in the strong topology on V_s. Since s was arbitrary it follows that there is a canonical map σ from $\bar{V}(\tau)$ (the sequence completion of V in the topology τ) into V^* such that if $\lim g^{(n)} = \hat{g}(\tau)$ with $g^{(n)} \in V$, then $\sigma(\hat{g}) = g = \{g_r\}_{r=1}^{\infty} \in V^*$.

Theorem 3.1. *Let* V, $W(g)$, $\Phi = \bigotimes_r \phi_r$ *and* $\bar{V}(\tau)$ *be as above. Let* $g = \{g_r\}_{r=1}^{\infty} \in V^*$. *Then* $g \in \sigma(\bar{V}(\tau))$ *if and only if*

$$\sum_r (1 - |\langle \phi_r, W_r(\lambda g_r)\, \phi_r \rangle|) \equiv \sum_r (1 - |f_r(\lambda)|) < \infty \tag{3.6}$$

for all λ *of some subset* $\Delta \subset \mathbb{R}$ *with* $\mu(\Delta) \neq 0$.

Proof. Necessity of Condition (3.6) has been shown by Woods [3]. To prove sufficiency we note that according to Theorem 2.2 there is a sequence $\{\alpha_r\}_{r=1}^{\infty}$ of real numbers α_r such that $W(\lambda \hat{g}) = \bigotimes_r W_r(\lambda g_r)\, e^{-i\lambda\alpha_r}$ represents a continuous unitary one-parameter group in H. Moreover, $W(\lambda \hat{g}) = \text{s-}\lim_{n\to\infty} \bigotimes_{r=1}^{n} W_r(\lambda g_r)\, e^{-i\lambda\alpha_r} \otimes \mathbb{1}$ for all λ. If $\sum_r |\alpha_r| < \infty$ then according to Remark 2.3 we may choose $\alpha_r = 0$ for all r. Defining $g^{(n)} = \{g_r^{(n)}\}_{r=1}^{\infty}$ by $g_r^{(n)} = g_r$ for $r \leqq n$, $g_r^{(n)} = 0$ for $r > n$, we have $g^{(n)} \in V$ for all n, $g^{(n)} \to \hat{g}(\tau)$ and $\sigma(\hat{g}) = g$. If $\sum_r |\alpha_r|$ diverges we denote the partial sums by $\gamma^{(n)}$, $\gamma^{(n)} = \sum_{r=1}^{n} \alpha_r$. Because $\sum_r |\alpha_r|$ diverges there is a subsequence $\{\alpha_{r_l}\}_{l=1}^{\infty}$ ($r_{l+1} > r_l$ for all l) of only positive or negative terms such that $\sum_l \alpha_{r_l}$ diverges. Hence for every n there are finite integers $L(n)$, $K(n)$, $K(n) \geqq L(n) > n$, such that $\left|\sum_{l=L(n)}^{K(n)} \alpha_{r_l}\right| \geqq |\gamma^{(n)}|$. It follows that we can find real numbers $x^{(n)}$ with $|x^{(n)}| \leqq 1$ and $x^{(n)} \sum_{l=L(n)}^{K(n)} \alpha_{r_l} = \sum_{l=L(n)}^{K(n)} x^{(n)} \alpha_{r_l} = -\gamma^{(n)}$. Define $g^{(n)} = \{g_r^{(n)}\} \in V$ by $g_r^{(n)} = g_r$ for $r \leqq n$, $g_{r_l}^{(n)} = x^{(n)} g_{r_l}$ for $l = L(n), \ldots, K(n)$, and $g_r^{(n)} = 0$ elsewhere. To compute

$$\left\| \left(\bigotimes_r W_r(\lambda g_r^{(n)}) - W(\lambda \hat{g}) \right) \Phi \right\|^2$$

$$= 2 \left(1 - \operatorname{Re} \prod_r \langle \phi_r, W_r(\lambda(g_r - g_r^{(n)}))\, e^{-i\lambda\alpha_r}\, \phi_r \rangle \right)$$

let $I(n) = \{r \in \mathbb{N} : r > n,\ r \neq r_l \text{ for } l = L(n), \ldots, K(n)\}$. Then the infinite product becomes

$$\prod_r \langle \phi_r, W_r(\lambda(g_r - g_r^{(n)}))\, e^{-i\lambda\alpha_r}\, \phi_r \rangle$$

$$= \prod_{l=L(n)}^{K(n)} |f_{r_l}(\lambda(1 - x^{(n)}))| \prod_{r \in I(n)} f_r(\lambda)\, e^{-i\lambda\alpha_r}$$

$$\cdot \exp\left(i \left(-\lambda \gamma^{(n)} + \sum_{l=L(n)}^{K(n)} \{\beta_{r_l}(\lambda(1 - x^{(n)})) - \lambda \alpha_{r_l}\} \right) \right)$$

where $\beta_r(\lambda)$ is defined as in Lemma 2.4. Now, by Condition (3.6) $\prod_r f_r(\lambda)\, e^{-i\lambda\alpha_r}$ converges uniformly in every bounded interval of λ (see Theorem 2.2). This implies that

$$\prod_{r\in I(n)} f_r(\lambda)\, e^{-i\lambda\alpha_r} \xrightarrow[n]{} 1$$

and

$$\prod_{l=L(n)}^{K(n)} |f_{r_l}(\lambda(1-x^{(n)}))| \xrightarrow[n]{} 1$$

for all λ [note that $K(n) \geqq L(n) > n$ and $|x^{(n)}| \leqq 1$ for all n]. Considering the third factor we remark that

$$-\lambda\gamma^{(n)} + \sum_{l=L(n)}^{K(n)} \left(\beta_{r_l}(\lambda(1-x^{(n)})) - \lambda\alpha_{r_l}\right)$$
$$= \sum_{l=L(n)}^{K(n)} \left(\beta_{r_l}(\lambda(1-x^{(n)})) - \lambda(1-x^{(n)})\,\alpha_{r_l}\right)$$

and this tends to zero by Lemma 2.4. Hence

$$\left\|\left(\bigotimes_r W_r(\lambda g_r^{(n)}) - W(\lambda\hat{g})\right)\Phi\right\| \xrightarrow[n]{} 0$$

for all λ, i.e., $g^{(n)} \to \hat{g}(\tau)$ and, clearly, $\sigma(\hat{g}) = g$.

As a final remark we note that if $\sigma(\hat{g}_1) = \sigma(\hat{g}_2)$ then $W(\lambda\hat{g}_2) = e^{i\lambda\alpha} \cdot W(\lambda\hat{g}_1)$ for some $\alpha \in \mathbb{R}$. This is because $W(\lambda\hat{g}_2)\, W(-\lambda\hat{g}_1)$ commutes with all $W(f)$ where $f \in V$, for only the weak limit $\sigma(\hat{g}_2 - \hat{g}_1) = 0$ enters in the bilinear form (3.4). Irreducibility and unitarity imply $W(\lambda\hat{g}_2) \cdot W(-\lambda\hat{g}_1) = e^{i\chi(\lambda)}$ or $W(\lambda\hat{g}_2) = e^{i\chi(\lambda)}\, W(\lambda\hat{g}_1)$. Now, the l.h.s. and the second factor of the r.h.s. are continuous unitary groups and, therefore, $\chi(\lambda) = \lambda\alpha$ for some $\alpha \in \mathbb{R}$.

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