Commun. math. Phys. 39, 111—119 (1974)

Expectations and Entropy Inequalities for Finite Quantum Systems

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Received March 19, 1974; in revised form July 16, 1974

Abstract. We prove that the relative entropy is decreasing under a trace-preserving expectation in $B(\mathscr{K}^1)$, and we show the connection between this theorem and the strong subadditivity of the entropy. It is also proved that a linear, positive, trace-preserving map Φ of $B(\mathscr{K})$ into itself such that $\|\Phi\| \leqq 1$ decreases the value of any convex trace function.

The main object of this note is to prove that the relative entropy is decreasing under a trace-preserving expectation from $B(\mathscr{K})$ to a von Neumann subalgebra (Theorem 1). We will show the connection between this theorem and the property of strong subadditivity of the entropy functional in quantum statistical mechanics [1]. The theorem is a generalization of a result by Umegaki [2] [for the case $B(\mathscr{H})$] and hence of an inequality in information theory [3]. The proof rests on a result by Lieb [4] on a generalized Wigner- Yanase- Dyson inequality.

The intuitive content of Theorem 1 is that an expectation always decreases the information content of the states, especially it makes it more difficult to distinguish two states from each other. Theorem 2 makes a similar but weaker statement for a larger class of maps: a positive, tracepreserving map of $B(\mathscr{K})$ into itself with norm at most equal to one decreases the value of any convex trace function on $B(\mathscr{K})$.

Let $A, B \in T_+(\mathscr{K})$ (the positive trace class operators in a separable Hilbert space $\mathscr{K}$). The *entropy* of A is defined by

$$S(A) = \mathrm{Tr}\,\hat{S}(A)\,, \qquad \hat{S}(A) = -A \log A\,.$$

If $\{|i\rangle\}$ is a complete orthonormal set of eigenvectors of A or B then we can define the *relative entropy*[2] through

$$S(A\,|\,B) = \Sigma \langle i|\,(A \log A - A \log B + B - A)\,|i\rangle$$

(see [5] for details). In [5] it was shown that if $S(A\,|\,B) < \infty$ we have

$$S(A\,|\,B) = \mathrm{Tr}\,\hat{S}(A\,|\,B)$$

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[1] For $\mathscr{K}$ read $\mathscr{H}$ throughout.

[2] In [5] this was called the *conditional entropy*.

where

$$\hat{S}(A\,|\,B) = \sup_\lambda \hat{S}_\lambda(A\,|\,B) \in T_+(\mathscr{H}) \tag{1}$$
$$\hat{S}_\lambda(A\,|\,B) = \lambda^{-1}[\hat{S}(\lambda A + (1-\lambda)\,B) - \lambda\hat{S}(A) - (1-\lambda)\,\hat{S}(B)],\ \lambda \in (0,1)$$

$\hat{S}_\lambda(A\,|\,B)$ is monotonously increasing when $\lambda \to 0$.

The following properties are elementary: if $\lambda_i > 0$, $\Sigma\lambda_i = 1$, then $\Sigma\,\lambda_i S(A_i) \leqq S(\Sigma\,\lambda_i A_i)$ (concavity)

$$S(UAU^+) = S(A) \qquad \text{for unitary } U\,,$$
$$S(A+B) = S(A) + S(B) \qquad \text{if } AB = 0\,,$$
$$S(A\,|\,B) \geqq 0, = 0 \qquad \text{iff } A = B\,,$$
$$S(UAU^+\,|\,UBU^+) = S(A\,|\,B) \qquad \text{for unitary } U\,,$$
$$S(A_1 + A_2\,|\,B_1 + B_2) = S(A_1\,|\,B_1) + S(A_2\,|\,B_2)$$

if

$$A_1A_2 = B_1B_2 = A_1B_2 = A_2B_1 = 0\,.$$

An *expectation* from a von Neumann algebra $\mathscr{A}$ to a von Neumann subalgebra $\mathscr{B}$ is a linear map Φ of $\mathscr{A}$ onto $\mathscr{B}$ satisfying

1. $\Phi \circ \Phi = \Phi$,
2. $\|\Phi X\| \leqq \|X\|$, all $X \in \mathscr{A}$.

It then follows that [6,7]

3. $\Phi I = I$,
4. $\Phi(XY) = (\Phi X)\,Y$, all $Y \in \mathscr{B}$,
5. $\Phi X \geqq 0$ for $X \geqq 0$,
6. $\Phi(X)^+\,\Phi(X) \leqq \Phi(X^+X)$.

In the following we will only consider the case of an expectation from $B(\mathscr{H})$ to a von Neumann subalgebra $\mathscr{A}$. We call Φ *tracepreserving* if $\operatorname{Tr}\Phi X = \operatorname{Tr} X$ for all $X \in T(\mathscr{H})$. If Φ is tracepreserving then the adjoint of Φ on the space of normal states is just the restriction of Φ to the unit sphere of $T(\mathscr{H})$. Furthermore if $X \in T(\mathscr{H})$ then ΦX is the unique element of $\mathscr{A}$ such that

$$\operatorname{Tr}\Phi(X)\,Y = \operatorname{Tr} XY \tag{2}$$

for all $Y \in \mathscr{A}$.

We now state the main theorem.

Theorem 1. *Let Φ be a trace-preserving expectation from $B(\mathscr{H})$ to a von Neumann subalgebra $\mathscr{A}$. If $A, B \in T_+(\mathscr{H})$ then $S(\Phi A\,|\,\Phi B) \leqq S(A\,|\,B)$.*

The proof will be given via a number of lemmas where A, B, Φ, and $\mathscr{A}$ will be as given in the statement of the theorem.

Lemma 1. *Let $X \in T(\mathscr{K})$ and let $K(X)$ be the weakly closed convex hull of the set $\{UXU^{-1}, U \text{ unitary} \in \mathscr{A}'\}$. Then*

$$K(X) \cap \mathscr{A} = \{\Phi X\}$$

$\Phi Y = \Phi X$ for all $Y \in K(X)$.

Furthermore, let $E(\mathscr{A}')$ be the set of nonnegative real functions on the set $U(\mathscr{A}')$ of unitary operators in $\mathscr{A}'$ which are nonzero only on a finite number of points and which satisfy $\Sigma f(U) = 1$. Put $fX = \Sigma f(U)\, UXU^{-1}$.

Then there is a sequence $\{f_n\} \subset E(\mathscr{A}')$ such that $f_n X \to \Phi X$ weakly.

Proof. From the normality of the trace follows that Φ is normal (compare [7] Proposition 6.1.1.), hence ultra-weakly continuous. Furthermore (2) is easily seen to imply that $\Phi(UXU^{-1}) = \Phi X$ for all unitary $U \in \mathscr{A}'$, hence as Φ is ultra-weakly continuous $\Phi Y = \Phi X$ for all $Y \in K(X)$. The first statement of the lemma follows from [8] Theorem 2 and the last from [9] p. 168 (property P').

Lemma 2. *$S(A|B)$ is jointly convex in A and B: if $\lambda_i > 0$, $\Sigma \lambda_i = 1$, $S(\Sigma \lambda_i A_i | \Sigma \lambda_i B_i) \leqq \Sigma \lambda_i S(A_i | B_i)$.*

Proof. From a theorem by Lieb [4] we know that $\mathrm{Tr}(A^{1-p}B^p)$, $p \in (0,1)$, is jointly concave in A, B. Differentiation at $p=0$ together with the fact that the function is affine for $p=0$ gives the statement.

Introduce the auxilary quantity

$$H(A) = S(A) + \mathrm{Tr}\, A \log \mathrm{Tr}\, A\,.$$

Lemma 3. *Let P be a projection in $\mathscr{K}$ and put $A_P = PAP$ etc. Then*

$$H(A_P) \leqq H(A)\,.$$

$$S(A_P | B_P) + S(A_{I-P} | B_{I-P}) \leqq S(A|B)\,.$$

Proof. The first inequality is a direct consequence of Theorem 2 in [10]. Note that $U = 2P - I$ is unitary and that $A' \equiv A_P + A_{I-P} = \frac{1}{2}(A + U^+ A U)$.

Hence, by Lemma 2:

$$S(A'|B') \leqq \tfrac{1}{2} S(A|B) + \tfrac{1}{2} S(U^+AU | U^+BU) = S(A|B)\,.$$

The second statement follows from the fact that

$$S(A'|B') = S(A_P|B_P) + S(A_{I-P}|B_{I-P})\,.$$

Lemma 4. *Let $\{P_n\}$ be a sequence of projections such that $P_m \leqq P_n$ for $m \leqq n$, $\dim P_n$ is finite for all n, and $P_n \to I$ strongly when $n \to \infty$. Put $A_n = P_n A P_n$. Then the sequences $H(A_n)$ and $S(A_n|B_n)$ are monotonously increasing and*

$$S(A_n) \to S(A)\,, \qquad S(A_n|B_n) \to S(A|B)\,.$$

Proof. The monotonicity follows from Lemma 3 and the convergence of $S(A_n)$ from the appendix of [1]. In order to prove the last of the statements we first observe that the convergence $A_n \to A$ is uniform. In fact

$$\mathrm{Tr}\, P_n A^2 \to \mathrm{Tr}\, A^2$$

$$0 \leqq \mathrm{Tr}[P_n(A^2 - A_n^2)] = \mathrm{Tr}[P_n A(I - P_n) A] \leqq \mathrm{Tr}[A^2(I - P_n)] \to 0,$$

hence

$$\mathrm{Tr}(A - A_n)^2 = \mathrm{Tr}(A^2 - A_n^2) = \mathrm{Tr}[A^2(I - P_n)] + \mathrm{Tr}[P_n(A^2 - A_n^2)] \to 0.$$

But $\|A - A_n\|^2 \leqq \mathrm{Tr}(A - A_n)^2$, consequently $\|A_n - A\| \to 0$. As the function $x \log x$ is continuous on $(0, \infty)$ we obtain

$$\|\hat{S}(A_n) - \hat{S}(A)\| \to 0.$$

Hence, for every finite-dimensional projection P

$$\mathrm{Tr}[P\hat{S}_\lambda(A_n | B_n)] \to \mathrm{Tr}[P\hat{S}_\lambda(A | B)].$$

From

$$S(A | B) = \sup\nolimits_P \mathrm{Tr}[P\hat{S}(A | B)]$$

$$\mathrm{Tr}[P\hat{S}(A | B)] = \sup\nolimits_\lambda \mathrm{Tr}[P\hat{S}_\lambda(A | B)]$$

it follows that $S(A | B)$ is lower semicontinuous under the convergence $(A_n, B_n) \to (A, B)$:

$$S(A | B) \leqq \liminf S(A_n | B_n).$$

But from Lemma 3 we know that $S(A_n | B_n) \leqq S(A | B)$, hence $\lim S(A_n | B_n) = S(A | B)$.

Proposition 1. *Assume that* $\{f_k\} \subset E(\mathscr{A}')$ *satisfies* $f_k A \to \Phi A$, $f_k B \to \Phi B$ *weakly. Then*

$$\lim S(f_k A) = S(\Phi A) \geqq S(A)$$

$$S(\Phi A | \Phi B) \leqq \liminf S(f_k A | f_k B) \leqq S(A | B).$$

Proof. First we note that $S(A) \leqq S(\Phi A)$ [11, 12] and that $\Phi f_k A = \Phi A$, hence

$$S(f_k A) \leqq S(\Phi f_k A) = S(\Phi A), \quad \text{all } k,$$

The same inequalities obviously hold for $H(A)$. There is a sequence of projections $\{P_n\}$ in $\mathscr{A}$ satisfying the conditions of Lemma 4 (this follows from the fact that Φ is tracepreserving: use the spectral measure of ΦA where $A \in T_+(\mathscr{K})$ has the support projection I). From the definition of Φ follows that

$$\Phi(P_n A P_n) = P_n(\Phi A) P_n.$$

As f_k is built up of elements of $\mathscr{A}'$ we see that

$$f_k P_n A P_n = P_n(f_k A) P_n .$$

In the finitedimensional space $\mathscr{K}_n = P_n \mathscr{K}$ the convergence $f_k A_n \to \Phi A_n$ is uniform and obviously, when $k \to \infty$:

$$H(f_k A_n) \to H(\Phi A_n)$$
$$S(f_k A_n | f_k B_n) \to S(\Phi A_n | \Phi B_n) .$$

From Lemma 4 we obtain that $H(A) = \sup H(A_n)$, hence $H(A)$ is lower semicontinuous i.e. $H(\Phi A) \leqq \liminf H(f_k A)$. But $H(f_k A) \leqq H(\Phi A)$ for all k, hence $H(\Phi A) = \lim H(f_k A)$ and $S(\Phi A) = \lim S(f_k A)$.

In the same way it follows that $S(A|B) = \sup S(A_n | B_n)$ and $S(\Phi A | \Phi B) \leqq \liminf S(f_k A | f_k B)$. As $S(\Phi A | \Phi B) \leqq S(f_k A | f_k B)$ for all k we cannot conclude that $S(\Phi A | \Phi B) = \lim S(f_k A | f_k B)$. From Lemma 2 and the unitary invariance we have

$$S(f_k A | f_k B) \leqq \Sigma f_k(U) S(U A U^+ | U B U^+) = S(A|B)$$

hence

$$S(\Phi A | \Phi B) \leqq S(A|B) .$$

Remark. The only difficulty remaining in proving Theorem 1 lies in the fact that we do not know if there is a sequence $\{f_k\} \subset E(\mathscr{A}')$ which implements Φ on both A and B.

Proof of Theorem 1. Choose a sequence of projections $P_n \in \mathscr{A}$ satisfying the conditions of Lemma 4, and let $f_k \in E(\mathscr{A}')$ be such that $f_k A \to \Phi A$ weakly. Hence

$$f_k A_n \to \Phi A_n$$

in norm. For a given k there exists $g_j \in E(\mathscr{A}')$ such that (remember that $\Phi f_k B = \Phi B$) $g_j f_k B \to \Phi B$ weakly when $j \to \infty$, hence

$$g_j f_k B_n \to \Phi B_n$$

in norm. If $\|(f_k - \Phi) A_n\| \leqq \varepsilon(k)$, choose $g_{j,k}$ such that

$$\|(g_{j,k} f_k - \Phi) B_n\| \leqq \varepsilon(k) .$$

Obviously

$$\|(g_{j,k} f_k - \Phi) A_n\| = \|g_{j,k}(f_k - \Phi) A_n\| \leqq \|(f_k - \Phi) A_n\| \leqq \varepsilon(k) .$$

Hence $h_k = g_{j,k} \cdot f_k$ satisfies

$$h_k A_n \to \Phi A_n , \qquad h_k B_n \to \Phi B_n$$

in norm. As in the proof of Proposition 1 it follows that

$$S(\Phi A_n | \Phi B_n) \leqq S(A_n | B_n)$$

and from Lemma 4

$$S(\Phi A | \Phi B) \leqq S(A|B) .$$

Corollary. *Let $\{P_k\}$ be a set of mutually orthogonal projections in $\mathscr{K}$ satisfying $\Sigma P_k = I$. The map $\Phi: A \to \Sigma P_k A P_k$ is a trace-preserving expectation which describes the interaction of a finite quantum system with a classical apparatus measuring an observable with eigenspaces P_k. Consequently*

$$S(\Phi A|\Phi B) = \Sigma S(P_k A P_k | P_k B P_k) \leqq S(A|B).$$

This generalizes an inequality proved in [5].

We will now show the connection between Theorem 1 and the property of strong subadditivity.

Let ϱ, $\hat{\varrho}$ be two states on a quasilocal algebra over some configuration space (e.g. Z^ν) such that the local algebra of a bounded region is of the type $B(\mathscr{K})$, $\mathscr{K}$ separable. We denote the Hilbert space corresponding to the bounded region Λ by $\mathscr{K}_\Lambda$. The state ϱ restricted to $B(\mathscr{K}_\Lambda)$ is then represented by a density operator ϱ_Λ in $\mathscr{K}_\Lambda$ [13].

Proposition 2. *For $\Lambda \subset \Lambda'$ we have*

$$S(\varrho_\Lambda|\hat{\varrho}_\Lambda) \leqq S(\varrho_{\Lambda'}|\hat{\varrho}_{\Lambda'}).$$

Proof. Let $\mathscr{K}_{\Lambda'} \equiv \mathscr{K}_{12} = \mathscr{K}_1 \otimes \mathscr{K}_2$ where $\mathscr{K}_1 = \mathscr{K}_\Lambda$, $\mathscr{K}_2 = \mathscr{K}_{\Lambda' - \Lambda}$. Then $\varrho_\Lambda \equiv \varrho_1 = \mathrm{Tr}_2 \varrho_{12}$ where $\varrho_{12} \equiv \varrho_{\Lambda'}$ and Tr_2 denotes the partial trace over $\mathscr{K}_2$. Put

$$\mathscr{K}_{12}^n = \mathscr{K}_1 \otimes P_n \mathscr{K}_2$$

where $\{P_n\}$ is a sequence of projections in $\mathscr{K}_2$ satisfying the conditions of Lemma 4. Then we have the uniform convergence

$$\begin{aligned} A_n &= I \otimes P_n \varrho_{12} I \otimes P_n \to \varrho_{12} \\ B_n &= I \otimes P_n \hat{\varrho}_{12} I \otimes P_n \to \hat{\varrho}_{12} \\ A_{1n} &= \mathrm{Tr}_2 A_n \to \varrho_1 \quad \text{etc.} \end{aligned}$$

Define an expectation

$$\Phi: B(\mathscr{K}_{12}^n) \to B(\mathscr{K}_1) \otimes \{\lambda I_2^n\}$$

($I_2^n =$ identity in $\mathscr{K}_2^n$) through

$$\Phi A = \mathrm{Tr}_2 A \otimes C_{2n}$$

where $C_{2n} = (\dim \mathscr{K}_2^n)^{-1} I_2^n$. Then

$$S(\Phi A_n|\Phi B_n) = S(A_{1n} \otimes C_{2n}|B_{1n} \otimes C_{2n}) = S(A_{1n}|B_{1n}) \leqq S(A_n|B_n).$$

From Lemma 4 it follows that

$$S(A_n|B_n) \to S(\varrho_{12}|\hat{\varrho}_{12}).$$

Let $\{Q_m\}$ be a set of projections in $\mathscr{K}_1$ satisfying the conditions of Lemma 4. Then

$$S(A_1|B_1) = \sup S(Q_m A_1 Q_m | Q_m B_1 Q_m)$$

and a reasoning similar to that of Proposition 1 gives that

$$S(\varrho_1|\hat{\varrho}_1) \leqq \liminf S(A_{1n}|B_{1n})$$

hence that

$$S(\varrho_1|\hat{\varrho}_1) \leqq S(\varrho_{12}|\hat{\varrho}_{12}).$$

Remark. The inequality proved above is nothing but a slight generalization of the property of strong subadditivity of the quantum-mechanical entropy [1]. This is easily seen by taking three disjoint regions $\Lambda_1, \Lambda_2, \Lambda_3$ and putting

$$\Lambda' = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3, \qquad \Lambda = \Lambda_1 \cup \Lambda_2$$

$$\varrho_{\Lambda'} = \varrho_{123}, \qquad \hat{\varrho}_{\Lambda'} = \varrho_1 \otimes \varrho_{23}, \qquad \varrho_\Lambda = \varrho_{12}, \qquad \hat{\varrho}_\Lambda = \varrho_1 \otimes \varrho_2.$$

Then, if the terms are finite,

$$S(\varrho_{123}|\varrho_1 \otimes \varrho_{23}) = S(\varrho_1) + S(\varrho_{23}) - S(\varrho_{123})$$

$$S(\varrho_{12}|\varrho_1 \otimes \varrho_2) = S(\varrho_1) + S(\varrho_2) - S(\varrho_{12}).$$

Hence, from Proposition 2, we get the strong subadditivity:

$$S(\varrho_{123}) + S(\varrho_2) - S(\varrho_{12}) - S(\varrho_{23}) \leqq 0.$$

Conversely the joint convexity of $S(A|B)$ follows from the strong subadditivity. In fact the strong subadditivity implies equation (4) of [14] which by our formula (1) implies the convexity of $S(A|B)$.

There are obviously many positive trace-preserving mappings of $B(\mathscr{K})$ into itself which decrease the relative entropy but which are not expectations (take e.g. any convex combination of unitary transformations). Therefore it is interesting to consider more general classes of transformations which have some suitable averaging property.

Let $f(x)$ be a bounded real-valued function defined in an interval I of the real line, and let A be a selfadjoint operator in $\mathscr{K}$ with spectrum in I. Then we define $f(A)$ as usual through the spectral resolution of A. It is well known that if $f(x)$ is operator convex [15] such that $f(0)=0$ and if Φ is a completely positive map such that $\|\Phi\| \leqq 1$, then

$$f(\Phi A) \leqq \Phi f(A)$$

(Jensen's inequality) [16, 17]. This class of maps includes the expectations [18]. If Φ is trace-preserving then

$$\operatorname{Tr} f(\Phi A) \leqq \operatorname{Tr} f(A)$$

which implies e.g. the increase of the entropy.

Now let $f(x)$ be a convex but not necessarily operator convex function on $(0, \infty)$ and let $f(0)=0$. If $A \in T_+(\mathscr{H})$ we introduce

$$F(A)=\operatorname{Tr} f(A)=\Sigma_i f(a_i)$$

where $\{a_i\}$ are the eigenvalues of A counted in decreasing order of magnitude including degeneracies. We get a more general class of averaging maps by finding all Φ such that $F(\Phi A) \leqq F(A)$. Define

$$\sigma_k(A)=\sum_1^k a_i .$$

From [19] Lemma 4.1 follows that

$$\sigma_k(A)=\sup\{\operatorname{Tr} PA, \dim P=k\} .$$

Hence if $A \leqq B$ then $\sigma_k(A) \leqq \sigma_k(B)$.

Lemma 5. *$F(\Phi A) \leqq F(A)$ for all $A \in T_+(\mathscr{H})$ and all convex $f(x)$ iff $\sigma_k(\Phi A) \leqq \sigma_k(A)$ for all k and $\sigma_\infty(\Phi A)=\sigma_\infty(A)$.*

Proof. The statement follows from [19] Lemma 3.4 and [20] Theorem 108.

The following theorem gives a characterization of the positive maps satisfying the conditions of Lemma 5.

Theorem 2. *Let $\Phi: B(\mathscr{H}) \to B(\mathscr{H})$ be a positive map. Then*

$$\|\Phi\| \leqq 1 , \quad \operatorname{Tr}\Phi A=\operatorname{Tr} A , \quad \text{all } A \in T_+(\mathscr{H}) .$$
$$\Leftrightarrow f(\Phi A) \leqq f(A) \quad \textit{all convex } f, \textit{ all } A \in T_+(\mathscr{H}) .$$

Proof. $\Rightarrow$ Note first that $\sigma_\infty(A)=\operatorname{Tr} A$, hence $\sigma_\infty(\Phi A)=\sigma_\infty(A)$. Let P_k be the projection on the subspace of $\mathscr{H}$ spanned by the eigenvectors corresponding to the k largest eigenvalues of A. Put

$$A_k=P_k(A-a_kI)+a_kI=\hat{A}_k+a_kI .$$

Obviously $A \leqq A_k$ and $\sigma_k(A_k)=\sigma_k(A)$. Furthermore $\hat{A}_k \geqq 0$ and $\operatorname{Tr}\hat{A}_k = \sigma_k(\hat{A}_k)=\sigma_k(A)-ka_k$

$$\Phi A_k=\Phi\hat{A}_k+a_k\Phi I$$

where $\Phi\hat{A}_k \geqq 0$ and $\Phi I \leqq I$.

$$\operatorname{Tr}\Phi\hat{A}_k=\sigma_\infty(\Phi\hat{A}_k) \geqq \sigma_k(\Phi\hat{A}_k)=\sigma_k(\Phi A_k)-a_k\sigma_k(\Phi I) \geqq \sigma_k(\Phi A_k)-ka_k .$$

But $\operatorname{Tr}\Phi\hat{A}_k=\operatorname{Tr}\hat{A}_k$, hence

$$\sigma_k(\Phi A_k)-ka_k \leqq \sigma_k(A)-ka_k .$$

From $\Phi A \leqq \Phi A_k$ follows that

$$\sigma_k(\Phi A) \leqq \sigma_k(\Phi A_k) \leqq \sigma_k(A)$$

and the statement follows from Lemma 5.

$\Leftarrow$: The statement is obvious from the fact that $\sigma_1(A) = \|A\|$, $\sigma_\infty(A) = \operatorname{Tr} A$ and Lemma 5.

Remark. This class of maps correspond precisely to the stochastic matrices for probability distributions on a discrete set. If we put $\Phi I = I$ we obtain the analogy of doubly stochastic matrices.

References

1. Lieb, E. H., Ruskai, M. B.: J. Math. Phys. **14**, 1938—1941 (1973)
2. Umegaki, H.: Kodai Math. Sem. Rep. **14**, 59—85 (1962)
3. Kullback, S.: Information theory and statistics. New York: Wiley 1959
4. Lieb, E. H.: Adv. Math. **11**, 267—288 (1973)
5. Lindblad, G.: Commun. math. Phys. **33**, 305—322 (1973)
6. Tomiyama, J.: Proc. Japan Acad. **33**, 608—612 (1957)
7. Arveson, W. B.: Am. J. Math. **89**, 578—672 (1967)
8. Kovács, I., Szücs, J.: Acta Math. Sci., Szeged, **27**, 233—246 (1966)
9. Schwartz, J. T.: *W**-algebras. New York: Gordon and Breach 1967
10. Lindblad, G.: Commun. math. Phys. **28**, 245—249 (1972)
11. Nakamura, M., Umegaki, H.: Proc. Japan Acad. **37**, 149—154 (1961)
12. Segal, I. E.: J. Math. Mech. **9**, 623—629 (1960)
13. Ruelle, D.: Statistical mechanics. New York: Benjamin 1969
14. Baumann, F.: Helv. Phys. Acta **44**, 95—100 (1971)
15. Bendat, J., Sherman, S.: Trans. Am. Math. Soc. **79**, 58—71 (1955)
16. Stinespring, W. F.: Proc. Am. Math. Soc. **6**, 211—216 (1955)
17. Davis, C.: Proc. Am. Math. Soc. **8**, 41—44 (1957)
18. Nakamura, M., Takesaki, M., Umegaki, H.: Kodai Math. Sem. Rep. **12**, 82—90 (1960)
19. Gohberg, I. C., Krein, M. G.: Introduction to the theory of linear non-selfadjoint operators. Providence: Am. Math. Soc. 1969
20. Hardy, G. H., Littlewood, J. E., Polya, G.: Inequalities. Cambridge: University Press 1934

Communicated by H. Araki

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