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Strict Convexity of the Pressure: A Note on a Paper of R. B. Griffiths and D. Ruelle

H. Roos

Institute for Theoretical Physics, Groningen, The Netherlands*

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Abstract. Strict convexity of the pressure of a quantum lattice gas is demonstrated in [1] with the help of a trace condition. An interpretation of that condition is given, and, simultaneously, an extension of the result of [1]. In particular, it is shown that the pressure is a continuous function of the lattice gas density.

I. Introduction

It has been demonstrated by Griffiths and Ruelle [1] that the pressure $P(\Phi_i)$ and the time automorphisms $\tau_t(\Phi_i)$, $i=1, 2$, exist are called function of the interaction Φ. One assumption used for the case of a quantum lattice gas is the following:

$$\mathrm{Tr}_Y \Phi(X)=0 \quad \text{for all} \quad Y\subset X, \quad \text{all finite} \quad X\subset \boldsymbol{Z}^\nu . \tag{1}$$

Here, $\boldsymbol{Z}^\nu$ describes the (ν-dimensional) lattice, Tr_Y denotes the partial trace. We are concerned with the interpretation of this condition which is not given in [1].

Definition 1.1. Two interactions Φ_1 and Φ_2 for which the pressures $P(\Phi_i)$ and the time automorphisms $\tau_t(\Phi_i)$, $i=1, 2$, exist are called physically equivalent if $P(\Phi_1)=P(\Phi_2)$ and $\tau_t(\Phi_1)=\tau_t(\Phi_2)$. We then write $\Phi_1 \simeq \Phi_2$.

In view of Theorem 2.2 below, this definition seems to be a sensible one. It will turn out that in every class of equivalent interactions with vanishing trace, there is a unique interaction with vanishing partial traces, i.e. satisfying (1), provided a certain temperedness condition is fulfilled. This allows a generalization of the results of [1]; in particular, we can show the continuity of the pressure as a function of the lattice gas density.

II. Notations and Results

We study a quantum lattice system over $\boldsymbol{Z}^\nu$, with a two-dimensional Hilbert space $\mathscr{H}_x$ attached to every $x\in\boldsymbol{Z}^\nu$, $\mathscr{H}_X=\bigotimes_{x\in X}\mathscr{H}_x$. $X, Y, \Lambda, \ldots$

* On leave of absence from the Institut für Theoretische Physik, Universität Göttingen, Fed. Rep. of Germany.

always denote finite subsets of $\boldsymbol{Z}^\nu$, $N(X)$ is the number of points in X. If $Y \subset X$, $\mathscr{H}_X$ will be identified with $\mathscr{H}_Y \otimes \mathscr{H}_{X\setminus Y}$; and similarly, we identify $A \in \mathfrak{B}(\mathscr{H}_Y)$ and $A \otimes \mathbf{1}_{X\setminus Y} \in \mathfrak{B}(\mathscr{H}_X)$, $\mathfrak{B}(\mathscr{H})$ = set of bounded operators on $\mathscr{H}$. $A \in \bigcup_\Lambda \mathfrak{B}(\mathscr{H}_\Lambda)$ is called strictly local; $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{B}(\mathscr{H}_\Lambda)}$ is the algebra of observables.

The translationally covariant interaction is given by a function $X \mapsto \Phi(X) \in \mathfrak{B}(\mathscr{H}_X)$, $\Phi(X)$ self-adjoint. Let $f(\xi)$ be a real valued function over $\mathbb{R}^+$, $f(\xi) \geqq 0$, then we define the f-norm of Φ by

$$\|\Phi\|_f = \sum_{X \ni 0} \|\Phi(X)\| \, f(N(X)) . \tag{2}$$

The interactions Φ with $\|\Phi\|_f < \infty$ form a Banach space B_f.

For $A \in \mathfrak{B}(\mathscr{H}_X)$, $Y \subset X$, $\mathrm{Tr}_Y A \in \mathfrak{B}(\mathscr{H}_{X\setminus Y})$ denotes the partial trace, and

$$\mathrm{tr}_Y A = 2^{-N(Y)} \, \mathrm{Tr}_Y A . \tag{3}$$

Writing $\mathrm{tr}_{X\setminus Y}$, we generally mean that Y is a proper subset of X. If a term $\mathrm{tr}_\emptyset$ occurs in a summation it is meant to be zero.

Lemma 2.1. *If $Y, Y' \subset X \subset Z$, $A \in \mathfrak{B}(\mathscr{H}_Z)$, then*

$$\mathrm{tr}_{Z\setminus X} \, \mathrm{tr}_{X\setminus Y} A = \mathrm{tr}_{Z\setminus Y} A \tag{4a}$$

$$\mathrm{tr}_{X\setminus Y} \, \mathrm{tr}_{X\setminus Y'} A = \mathrm{tr}_{X\setminus (Y \cap Y')} A \tag{4b}$$

$$\|\mathrm{tr}_Y A\| \leqq \|A\| . \quad \square \tag{4c}$$

The proof is trivial.

The Hamiltonian belonging to the interaction Φ is

$$H_\Lambda(\Phi) = \sum_{X \subset \Lambda} \Phi(X) . \tag{5}$$

The pressure and the time evolution of the system are defined by

$$P(\Phi) = \lim_{\Lambda \to \infty} N(\Lambda)^{-1} \log \mathrm{Tr}_\Lambda e^{-H_\Lambda(\Phi)} , \tag{6}$$

$$\tau_t(\Phi) A = \lim_{\Lambda \to \infty} \tau_t^\Lambda(\Phi) A , \; \tau_t^\Lambda(\Phi) A = e^{itH_\Lambda(\Phi)} A \, e^{-itH_\Lambda(\Phi)} , \; A \in \bigcup_\Lambda \mathfrak{B}(\mathscr{H}_\Lambda) . \tag{7}$$

The limits are known to exist [2, 3] if $\Phi \in B_{f_1}$ (resp. $\Phi \in B_{f_2}$), $f_1(\xi) = 1/\xi$, $f_2 = e^{\alpha\xi}$, $\alpha > 0$. $\Lambda \to \infty$ means the Van Hove limit (resp. $\Lambda \to \infty$ such that it eventually contains every finite subset of $\boldsymbol{Z}^\nu$).

Our main result is the following

Theorem 2.2. *For every interaction Φ which satisfies*

(a) $\Phi \in B_{f_3}$, $f_3(\xi) = e^{\xi^2}$,

(b) $\mathrm{Tr}_X \Phi(X) = 0$ *for all* $X \subset \boldsymbol{Z}^\nu$, *there exists a* $\tilde{\Phi}$ *such that*

(i) $\tilde{\Phi} \in B_{f_2}$, $f_2(\xi) = e^{\alpha\xi}$,

(ii) $\mathrm{Tr}_Y \tilde{\Phi}(X) = 0$ *for all* $Y \subset X$, *all* $X \subset \mathbf{Z}^\nu$,

(iii) $\tilde{\Phi} \simeq \Phi$,

(iv) $\tilde{\Phi}$ *is unique. If* Φ_1 *and* Φ_2 *satisfy* (a) *and* (b), *then* $\Phi_1 \simeq \Phi_2$ *if and only if* $\widetilde{\Phi_1} = \widetilde{\Phi_2}$.

(v) $\Phi_1 \simeq \Phi_2$ *implies* $P(\beta\Phi_1) = P(\beta\Phi_2)$ *for all* $\beta > 0$, *and* $\varrho(A_{\Phi_1}) = \varrho(A_{\Phi_2})$ *for all translationally invariant states* ϱ, *where* $A_\Phi = \sum_{X \ni 0} \Phi(X) N(X)^{-1}$ *is the observable of the mean energy per site.* □

The equivalence relation $\simeq$ is defined in Definition 1.1. The requirement (a) can be weakened (compare the proof).

For a potential which does not satisfy (b) we define

$$\Phi^T(X) = \Phi(X) - \mathrm{tr}_X \Phi(X) \cdot \mathbf{1}_X . \tag{8}$$

$$C_\Lambda(\Phi) = \sum_{X \subset \Lambda} \mathrm{tr}_X \Phi(X) . \tag{9}$$

Then we have

Proposition 2.3. (i) *If* $\Phi \in B_f$, *then* $\Phi^T \in B_f$,

(ii) $C_\Lambda(\Phi) = \mathrm{tr}_\Lambda H_\Lambda(\Phi)$,

(iii) *if* $\Phi \in B_{1/\xi}$, *then* $\lim_{\Lambda \to \infty} N(\Lambda)^{-1} C_\Lambda(\Phi) = \pi(\Phi)$ *exists and*

$$P(\Phi^T) = P(\Phi) + \pi(\Phi), \quad \tau_t(\Phi^T) = \tau_t(\Phi) . \quad \square \tag{10}$$

We can generalize the result of Griffiths and Ruelle for sufficiently tempered interactions with the help of Theorem 2.2 and Proposition 2.3:

Theorem 2.4. *Let us assume that* $\Phi_1, \Phi_2 \in B_{f_3}$, *then the pressure* $P(\Phi)$ *is strictly convex between* Φ_1 *and* Φ_2 *if and only if* Φ_1^T *and* Φ_2^T *are not physically equivalent.* □

Remark 2.5. According to (ii) and (iii) of Proposition 2.3, $H_\Lambda(\Phi^T) = H_\Lambda(\Phi) - \mathrm{tr}_\Lambda H_\Lambda(\Phi)$, and $\lim_{\Lambda \to \infty} N(\Lambda)^{-1} \mathrm{tr}_\Lambda H_\Lambda(\Phi)$ exists. In the limit $\Lambda \to \infty$, the "energy per site" is thus changed by a finite amount if we go over from Φ to Φ^T, independent of the state of the system. One may consider this as a physically irrelevant renormalization and consider Φ and Φ^T as equivalent in a wider sense. Then Theorem 2.2 implies, loosely speaking, strict convexity of the pressure as a function of the extended equivalence classes.

Let $\mathcal{N}_x$ denote the particle number operator in $\mathcal{H}_x$; $\mathcal{N}(X) = \sum_{x \in X} \mathcal{N}_x$. Define

$$p(\beta, \mu) = P(\beta \Phi_\mu), \qquad \Phi_\mu(X) = \Phi(X) - \mu \delta_{1, N(X)} \mathcal{N}(X) , \qquad \delta_{a,b} = \text{Kronecker symbol} . \tag{11}$$

$[H_A(\Phi_\mu) = H_A(\Phi) - \mu\mathcal{N}(\Lambda)$ gives rise to the statistical operator of the grand canonical ensemble.]

It is known [4] that

$$p^\Phi(\beta,\mu) = \sup(S(\varrho) - \beta\varrho(A_\Phi) + \beta\mu\varrho(\mathcal{N}_0)),$$

where the supremum is to be taken over all invariant states ϱ over $\mathfrak{A}$, $S(\varrho)$ denotes the corresponding entropy. Let the supremum be reached for ϱ_s, then $\varrho_s(\mathcal{N}_0)$ can be considered as the equilibrium density of the system.

Proposition 2.6. *If $\Phi \in B_{f_3}$, then $p^\Phi(\beta,\mu)$ is a continuous function of the equilibrium density $v^\Phi(\beta,\mu) = \varrho_s(\mathcal{N}_0)$.* □

This follows from the strict convexity of $p^\Phi(\beta,\mu)$ with respect to μ which, in turn, is a consequence of Theorem 2.4.

Remark 2.7. The definition of Φ^T and $\tilde{\Phi}^T$ gives non-trivial results for classical interactions too. Our conjecture is that, for classical Φ_i, $\Phi_1 = \Phi_2$ if and only if $\Phi_1^T \simeq \Phi_2^T$. This would prove the strict convexity of $P(\Phi)$ for strongly tempered classical interactions by the same method as for quantum interactions. But since the class of classical interactions considered in [1] is appreciably larger, it does not seem worth proving that conjecture.

Remark 2.8. Looking through the proof of Theorem 2.2 one easily realizes that, for $\Phi_1, \Phi_2 \in B_{f_3}$ and $\mathrm{Tr}_X(\Phi_1(X) - \Phi_2(X)) = 0$ for all X, the equality $\tau_t(\Phi_1) = \tau_t(\Phi_2)$ already implies $P(\Phi_1) = P(\Phi_2)$. From $A_{\Phi_1} = A_{\Phi_2}$ for physically equivalent interactions, it follows that the equilibrium states $\varrho_s^{\Phi_1}$ and $\varrho_s^{\Phi_2}$, as defined by the above variational principle, coincide if $\Phi_1 \simeq \Phi_2$.

III. Proofs

If $f(\xi) \geqq \bar{f}(\xi)$ for sufficiently large ξ, then $\Psi \in B_f$ implies $\Psi \in B_{\bar{f}}$ and $\Psi \in B_{\max(f,\bar{f})}$.

Proof of Theorem 2.2. It suffices to assume

$$\Phi \in B_{f_4},\; f_4(\xi) = e^{\alpha\xi} \sum_{\nu=1}^{\xi-1} \prod_{\mu=1}^{\nu} \left(1 + \binom{\xi}{\mu}\right). \tag{12}$$

With the estimate $\binom{\xi}{\mu} < 2^\xi$ one easily gets

$$f_4(\xi) \leqq e^{\xi^2 \ln 2 + \alpha\xi} \leqq e^{\xi^2} = f_3(\xi) \quad \text{for large } \xi. \tag{13}$$

For the sake of convenience, we take $\alpha = 1$ and assume $\Phi \in B_{f_0}$, $f_0(\xi) = e^{3\xi} f_4(\xi)$. Clearly, $\Phi \in B_{f_0}$ if $\Phi \in B_{f_3}$.

Lemma 3.1. *If* $\Psi \in B_{1/\xi}$, *then* $\sum_{Z:Z\supset X} \mathrm{tr}_{Z\setminus X}\Psi(Z)$ *exists, and*

$$\left\| \sum_{Z:Z\supset X} \mathrm{tr}_{Z\setminus X}\Psi(Z) \right\| \leqq N(X)\, \|\Psi\|_{1/\xi}.$$

This follows from $\|\mathrm{tr}_{Z\setminus X}\Psi(Z)\| \leqq \|\Psi(Z)\|$ and

$$\sum_{Z:Z\supset X} \|\Psi(Z)\| = \sum_{x\in X}\sum_{Z\ni x} \|\Psi(Z)\|\, N(Z)^{-1} = N(X)\, \|\Psi\|_{1/\xi}.$$

Due to this Lemma, we can define a sequence of interactions Φ_k, $k=0,1,2,\dots$, by

Definition 3.2. $\Phi_0(X)=\Phi(X)$;

$$\Phi_k(X)=\begin{cases}\Phi_{k-1}(X) & \text{if } N(X)<k\,,\\ \Phi_{k-1}(X)+\sum_{Z:Z\supset X}\mathrm{tr}_{Z\setminus X}\Phi_{k-1}(Z) & \text{if } N(X)=k\,,\\ \Phi_{k-1}(X)-\sum_{Y:Y\subset X,\,N(Y)=k}\mathrm{tr}_{X\setminus Y}\Phi_{k-1}(X) & \text{if } N(X)>k\,.\end{cases}$$

Because of

$$\Phi_k(X)=\Phi_{N(X)}(X) \quad \text{for} \quad k\geqq N(X)\,, \tag{14}$$

the sequence converges in an obvious sense to

$$\tilde{\Phi}(X)=\Phi_{N(X)}(X)\,. \tag{15}$$

We are going to show that $\tilde{\Phi}$ has the properties required in Theorem 2.2.

Remark 3.3. It is clear from the definition that Φ_k and $\tilde{\Phi}$ are translationally covariant. If Φ is of finite range, or if $\Phi(X)=0$ for $N(X)\geqq N_0$, the same holds true for Φ_k and $\tilde{\Phi}$.

(i) Calculation of the norm. For $k<N(X)$, we have

$$\|\Phi_k(X)\| \leqq \|\Phi_{k-1}(X)\| + \sum_{Y:Y\subset X,\,N(Y)=k} \|\Phi_{k-1}(X)\| = \|\Phi_{k-1}(X)\| \left(1+\binom{N(X)}{k}\right),$$

consequently, because also $k-1<N(X)$,

$$\|\Phi_k(X)\| \leqq \|\Phi(X)\|\; p(N(X);k),\; p(\xi;k)=\prod_{\mu=1}^{k}\left(1+\binom{\xi}{\mu}\right). \tag{16}$$

Insertion of (16) into

$$\|\Phi_{N(X)}(X)\| \leqq \|\Phi_{N(X)-1}(X)\| + \sum_{Z:Z\supset X} \|\Phi_{(N(X)-1)}(Z)\|$$

yields

$$\|\Phi_{N(X)}(X)\| \leqq \|\Phi(X)\|\; p(N(X);N(X)-1) + \sum_{Z:Z\supset X} \|\Phi(Z)\|\; p(N(Z);N(X)-1),$$

hence

$$\|\tilde{\Phi}\|_{e^\xi} = \sum_{X \ni 0} \|\Phi_{N(X)}(X)\| \, e^{N(X)} \leqq \sum_{X \ni 0} \|\Phi(X)\| \, p(N(X); N(X)-1) e^{N(X)} \\ + \sum_{Z \ni 0} \|\Phi(Z)\| \sum_{X: X \subset Z, X \ni 0} p(N(Z); N(X)-1) e^{N(X)}, \tag{17}$$

where the second term is obtained by rearranging the terms of the original sum $\sum_{X \ni 0} \sum_{Z: Z \subset X}$.

Note that $p(\xi; \xi - 1) e^{\xi} \leqq f_0(\xi)$ and

$$\sum_{X: X \subset Z, X \ni 0} p(N(Z); N(X)-1) e^{N(X)} \leqq \sum_{\nu=1}^{N(Z)-1} e^{\nu} p(N(Z); \nu - 1) \binom{N(Z)}{\nu} \\ \leqq f_0(N(Z)),$$

therefore, we conclude from (17) that

$$\|\tilde{\Phi}\|_{e^\xi} \leqq 2 \|\Phi\|_{f_0} < \infty .$$

(ii) Vanishing of the partial traces. We have by assumption $\mathrm{tr}_X \Phi_0(X) = \mathrm{tr}_X \Phi(X) = 0$. Now suppose that

$$(T_l)\colon \ \mathrm{Tr}_{X \setminus Y} \Phi_l(X) = 0 \quad \text{for} \quad N(Y) \leqq l, N(Y) < N(X)$$

holds for all $l \leqq k-1$. We then show the validity of (T_k). If $N(X) < k$, then $N(Y) < N(X) \leqq k-1$, and $\mathrm{tr}_{X \setminus Y} \Phi_k(X) = \mathrm{tr}_{X \setminus Y} \Phi_{k-1}(X) = 0$. If $N(X) = k$, then $N(Y) \leqq k-1$, and $\mathrm{tr}_{X \setminus Y} \Phi_k(X) = \mathrm{tr}_{X \setminus Y} \Phi_{k-1}(X) + \sum_{Z \supset X} \mathrm{tr}_{Z \setminus Y} \Phi_{k-1}(Z) = 0$, where we used Lemma 2.1. Finally, if $N(X) > k$, we get, again applying Lemma 2.1,

$$\mathrm{tr}_{X \setminus Y} \Phi_k(X) = \mathrm{tr}_{X \setminus Y} \Phi_{k-1}(X) - \sum_{Y': Y' \subset X, N(Y') = k} \mathrm{tr}_{X \setminus (Y \cap Y')} \Phi_{k-1}(X) . \tag{18}$$

For $N(Y) \leqq k-1$, the r.h.s. vanishes because $N(Y \cap Y') \leqq k-1$. If $N(Y) = k$, then $N(Y \cap Y') \leqq k-1$ unless $Y' = Y$, and all terms in the sum vanish except one which cancels the first term of the r.h.s. of (18). Therefore, (T_k) holds for all k; with $k = N(X)$, we get $\mathrm{tr}_Y \tilde{\Phi}(X) = 0$ for all $Y \subset X$.

(iii) Calculation of $P(\tilde{\Phi})$ and $\tau_t(\tilde{\Phi})$. This is the most laborious part of the proof. $P(\tilde{\Phi})$ and $\tau_t(\tilde{\Phi})$ are well defined because $\tilde{\Phi} \in B_{e^\xi}$. We want to show

$$P(\beta \tilde{\Phi}) = P(\beta \Phi), \ \ \tau_t(\tilde{\Phi}) A = \tau_t(\Phi) A, \ \ A \in \mathfrak{A} \tag{19a, b}$$

by establishing the following Lemmas:

Lemma 3.4. *For a special Van Hove-sequence $\Lambda \to \infty$, to be defined below, we have*

$$|P_\Lambda(\beta \tilde{\Phi}) - P_\Lambda(\beta \Phi)| \leqq N(\Lambda)^{-1} \beta \|H_\Lambda(\tilde{\Phi}) - H_\Lambda(\Phi)\| < \beta \varepsilon \quad \text{if} \quad \Lambda \supset \Lambda_0(\varepsilon) . \quad \square$$

Lemma 3.5. *For the special sequence $\Lambda \to \infty$ of Lemma 3.4, and for any strictly local $A \in \mathfrak{B}(\mathscr{H}_{\Lambda_1})$, we have*

$$\|[H_\Lambda(\tilde{\Phi}), A]^{(m)} - [H_\Lambda(\Phi), A]^{(m)}\| < \varepsilon, \quad m = 1, 2, \dots, N,$$

if $\Lambda \supset \Lambda_0'(\varepsilon, N, A)$. □

The multiple commutator $[B, A]^{(m)}$ is defined by $[B, A]^{(0)} = A$, $[B, A]^{(m)} = [B, [B, A]^{(m-1)}]$. The limits $\lim_{\Lambda \to \infty} P_\Lambda(\Phi)$ (resp. $\lim P_\Lambda(\tilde{\Phi})$, $\lim \tau_t^\Lambda(\Phi)A$, $\lim \tau_t(\tilde{\Phi})A$) are independent of the chosen sequence $\Lambda \to \infty$, hence Lemma 3.4 implies (19a). $\tau_t^\Lambda(\Phi)A$ (resp. $\tau_t^\Lambda(\tilde{\Phi})A$) can, for small t, $|t| < t_0(\Phi)$, be approximated by

$$\sum_{m=0}^{N} \frac{(it)^m}{m!} [H_\Lambda(\Phi), A]^{(m)} \left(\text{resp. } \sum_{m=0}^{N} \frac{(it)^m}{m!} [H_\Lambda(\tilde{\Phi}), A]^m\right),$$

uniformly in Λ (see for instance [2], Section 7.6). Therefore, Lemma 3.5 yields $\tau_t(\tilde{\Phi})A = \tau_t(\Phi)A$ for strictly local A and sufficiently small t, hence (19b).

We start proving

Lemma 3.6.

$$H_\Lambda(\tilde{\Phi}) - H_\Lambda(\Phi) = \sum_{l=1}^{N(\Lambda)} \sum_{X : X \subset \Lambda, N(X) = l} \sum_{Z : X \subset Z \not\subset \Lambda} \operatorname{tr}_{Z \setminus X} \Phi_{l-1}(Z), \tag{20}$$

$$\|H_\Lambda(\tilde{\Phi}) - H_\Lambda(\Phi)\| \leqq \sum_{x \in \Lambda} \sum_{Z : Z \not\subset \Lambda, Z \ni x} \|\Phi(Z)\| f_0(N(Z)). \quad \square \tag{21}$$

Proof. Insertion of Definition 3.2 into

$$H_\Lambda(\Phi_k) = \sum_{X \subset \Lambda, N(X) > k} \Phi_k(X) + \sum_{X \subset \Lambda, N(X) = k} \Phi_k(X) + \sum_{X \subset \Lambda, N(X) < k} \Phi_k(X),$$

and reordering of the terms gives, for all $k = 1, 2, \dots$,

$$H_\Lambda(\Phi_k) = H_\Lambda(\Phi_{k-1}) + \sum_{X : X \subset \Lambda, N(X) = k} \sum_{Z : X \subset Z \not\subset \Lambda} \operatorname{tr}_{Z \setminus X} \Phi_{k-1}(Z). \tag{22}$$

Furthermore, due to (14), we have

$$H_\Lambda(\tilde{\Phi}) = \sum_{X \subset \Lambda} \Phi_{N(X)}(X) = \sum_{X \subset \Lambda} \Phi_{N(\Lambda)}(X) = H_\Lambda(\Phi_{N(\Lambda)}). \tag{23}$$

Iteration of (22) together with (23) yields (20).

By (16), the norm of (20) is bounded by

$$\sum_{l=1}^{N(\Lambda)} \sum_{X : \dots} \sum_{Z : \dots} \|\Phi(Z)\| \, p(N(Z); l-1) \leqq \sum_{x \in \Lambda} \sum_{Z : Z \ni x, Z \not\subset \Lambda} \|\Phi(Z)\| \, r(\Lambda; Z)$$

with

$$r(\Lambda; Z) = \sum_{l=1}^{N(\Lambda)} \sum_{X: X \subset \Lambda \cap Z, N(X)=l} p(N(Z); l-1)$$
$$= \sum_{l=1}^{N(\Lambda)} \binom{N(\Lambda \cap Z)}{l} p(N(Z); l-1).$$

We have to put $\binom{n}{l} = 0$ if $l > n$. The estimate

$$r(\Lambda; Z) \leqq \sum_{l=1}^{N(Z)} \binom{N(Z)}{l} p(N(Z); l-1) \leqq f_0(N(Z))$$

finally proves (21).

Now let us define a special sequence $\Gamma_k \subset \boldsymbol{Z}^\nu$. We choose $a \in \boldsymbol{Z}^\nu$, $a = (a^1, \ldots, a^\nu)$ and $\Lambda(a) = \{x \in \boldsymbol{Z}^\nu; -a^i \leqq x^i < a^i,\ i = 1, \ldots, \nu\}$ in such a way that

$$\sum_{X: 0 \in X \not\subset \Lambda} \|\Phi(X)\| \ f_0(N(X)) < \frac{\varepsilon}{2} \quad \text{for} \quad \Lambda \supset \Lambda(a). \tag{24}$$

Definition 3.7. Let $\Lambda + x$ denote the set Λ translated by x;

$$\Gamma_1 = \Lambda(a), \ \Gamma_k = \bigcup_{x \in \boldsymbol{Z}^\nu: -a^i \leqq x^i \leqq a^i} (\Gamma_{k-1} + x).$$

Γ_k consists of k^ν translates of $\Lambda(a)$, hence

$$N(\Gamma_k) = N(\Lambda(a)) k^\nu. \tag{25}$$

Furthermore, $\Gamma_k \to \infty$ in the sense of Van Hove, and $\Gamma_k \supset \bigcup_{x \in \Gamma_{k-1}} (\Lambda(a) + x)$. This implies, due to (24) and the translation covariance of the interaction, that

$$\sum_{X: x \in X \not\subset \Gamma_k} \|\Phi(X)\| \ f_0(N(X)) < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in \Gamma_{k-1}. \tag{26}$$

Lemma 3.8. *Let us assume $\Lambda \subset \Gamma_k$ and $N(\Lambda \cap \Gamma_{k-1})/N(\Lambda) > 1 - \varepsilon_1$, then*

$$\sum_{x \in \Lambda} \sum_{X: x \in X \not\subset \Gamma_k} \|\Phi(X)\| \ f_0(N(X)) < N(\Lambda)\left(\frac{\varepsilon}{2} + \varepsilon_1 \|\Phi\|_{f_0}\right). \quad \square \tag{27}$$

Proof. We split the sum $\sum_{x \in \Lambda} = \sum_{x \in \Lambda \cap \Gamma_{k-1}} + \sum_{x \in \Lambda \cap (\Gamma_k \setminus \Gamma_{k-1})}$. To the first term, we can apply (26), the second one is bounded by $N(\Lambda \cap (\Gamma_k \setminus \Gamma_{k-1})) \cdot \|\Phi\|_{f_0} \leqq N(\Lambda)\, \varepsilon_1 \|\Phi\|_{f_0}$, hence (27).

Choose k large, such that $N(\Gamma_{k-1})/N(\Gamma_k) = (k - 1/k)^\nu > 1 - \varepsilon/2\|\Phi\|_{f_0}$, and apply Lemma 3.8 with $\Lambda = \Gamma_k$, then

$$\sum_{x \in \Gamma_k} \sum_{X: x \in X \not\subset \Gamma_k} \|\Phi(X)\| \ f_0(N(X)) < N(\Gamma_k)\varepsilon. \tag{28}$$

Proof of Lemma 3.4. We note that

$$|P_\Lambda(\beta\tilde{\Phi}) - P_\Lambda(\beta\Phi)| \leqq N(\Lambda)^{-1} \|H_\Lambda(\beta\Phi) - H_\Lambda(\beta\tilde{\Phi})\| = N(\Lambda)^{-1}\beta\|H_\Lambda(\Phi) - H_\Lambda(\tilde{\Phi})\|.$$

Putting $\Lambda = \Gamma_k$, k sufficiently large, and using (21) and (28), we get

$$|P_{\Gamma_k}(\beta\tilde{\Phi}) - P_{\Gamma_k}(\beta\Phi)| < \beta\varepsilon.$$

Proof of Lemma 3.5. We use the same sort of estimates as in [2], Section 7.6, and the fact that

$$\|[H_\Lambda(\tilde{\Phi}) - H_\Lambda(\Phi), A]\| \leqq N(\Lambda_1)\cdot\varepsilon \quad \text{if} \quad A \in \mathfrak{B}(\mathscr{H}_{\Lambda_1}), \tag{29}$$

$$\begin{aligned}&[H_\Lambda(\Phi), A]^{(m)} - [H_\Lambda(\tilde{\Phi}), A]^{(m)} \\ &\quad = \sum_{r=0}^{m-1} [H_\Lambda(\tilde{\Phi}), [H_\Lambda(\Phi) - H_\Lambda(\tilde{\Phi}), [H_\Lambda(\Phi), A]^{(r)}]]^{(m-r-1)}.\end{aligned} \tag{30}$$

(29) is a consequence of Lemma 3.8, because only those $\Phi(X)$ and $\tilde{\Phi}(X)$ give a contribution for which $X \cap \Lambda_1 \neq \emptyset$. Working out the details is an awful task, and will be done in the appendix.

(iv) The uniqueness of $\tilde{\Phi}$ follows from an argument of Griffiths and Ruelle ([1], Section IV). Suppose there exists a Φ' such that $\mathrm{Tr}_Y \Phi'(X) = 0$ and $\tau_t(\Phi') = \tau_t(\Phi) = \tau_t(\tilde{\Phi})$, then $\Phi' = \tilde{\Phi}$. By the same argument, $\Phi_1 \simeq \Phi_2$ implies $\tilde{\Phi}_1 = \tilde{\Phi}_2$. The inverse is trivial. This completes the proof of Theorem 2.2.

(v) Due to the uniqueness of $\tilde{\Phi}$, we have $\tilde{\Phi}_1 = \tilde{\Phi}_2$ if $\Phi_1 \simeq \Phi_2$, and, according to (19a), $P(\beta\Phi_1) = P(\beta\tilde{\Phi}_1) = P(\beta\tilde{\Phi}_2) = P(\beta\Phi_2)$. In the same way, it follows that, for invariant states ϱ, $\varrho(A_{\Phi_1}) = \varrho(A_{\Phi_2})$, provided we know that $\varrho(A_\Phi) = \varrho(A_{\tilde{\Phi}})$. Define $A_\Phi(\Lambda) = N(\Lambda)^{-1} \sum_{x\in\Lambda}\sum_{X\ni x} \Phi(X)N(X)^{-1}$, and consider

$$\|A_\Phi(\Lambda) - A_{\tilde{\Phi}}(\Lambda)\| \leqq \|A_\Phi(\Lambda) - N(\Lambda)^{-1}H_\Lambda(\Phi)\| + \|A_{\tilde{\Phi}}(\Lambda) - N(\Lambda)^{-1}H_\Lambda(\tilde{\Phi})\| + N(\Lambda)^{-1}\|H_\Lambda(\Phi) - H_\Lambda(\tilde{\Phi})\|.$$

If we choose $\Lambda = \Gamma_k$, k sufficiently large, the third term on the r.h.s. will be small due to (21) and (28). Note that we can replace $f_0(\xi)$ by $1/\xi$ in Lemma 3.8 and in (28). Application of (28) to

$$\|A_\Phi(\Lambda) - N(\Lambda)^{-1}H_\Phi(\Lambda)\| = N(\Lambda)^{-1}\Big\|\sum_{x\in\Lambda}\sum_{X:x\in X\not\subset\Lambda}\Phi(X)N(X)^{-1}\Big\|$$

and to the corresponding expression with $\tilde{\Phi}$ then shows that $\|A_\Phi(\Lambda) - A_{\tilde{\Phi}}(\Lambda)\| < 3\varepsilon$, hence $|\varrho(A_\Phi(\Lambda)) - \varrho(A_{\tilde{\Phi}}(\Lambda))| < 3\varepsilon$ with arbitrarily small ε. Due to the invariance of ϱ, we have $\varrho(A_\Phi(\Lambda)) = \varrho(A_\Phi)$, and therefore $\varrho(A_\Phi) = \varrho(A_{\tilde{\Phi}})$. This completes the proof of Theorem 2.2.

Proof of Proposition 2.3. (i) and (ii) are simple consequences of Lemma 2.1. Notice that

$$H_\Lambda(\Phi^T) = H_\Lambda(\Phi) - C_\Lambda(\Phi) \cdot \mathbf{1}_\Lambda ; \tag{31}$$

due to (i), $P(\Phi)$ and $P(\Phi^T)$ exist, hence

$$\begin{aligned} P(\Phi) - P(\Phi^T) &= \lim_{\Lambda\to\infty} N(\Lambda)^{-1}(\log \mathrm{Tr}_\Lambda e^{-H_\Lambda(\Phi)} - \log \mathrm{Tr}_\Lambda e^{-H_\Lambda(\Phi^T)}) \\ &= \lim_{\Lambda\to\infty} N(\Lambda)^{-1} C_\Lambda(\Phi) \equiv \pi(\Phi) \end{aligned}$$

exists. This proves the first part of (10), the second one is a trivial consequence of (31) and the definition of τ_t.

Proof of Theorem 2.4. Let us suppose $\Phi_1, \Phi_2 \in B_{f_0}$, $0 \leqq \alpha \leqq 1$, then $\Phi = \alpha\Phi_1 + (1-\alpha)\Phi_2 \in B_{f_0}$ and

$$\Phi^T = \alpha\Phi_1^T + (1-\alpha)\Phi_2^T , \tag{32}$$

$$C_\Lambda(\Phi) = \alpha C_\Lambda(\Phi_1) + (1-\alpha) C_\Lambda(\Phi_2) , \tag{33}$$

$$\widetilde{\Phi^T} = \alpha\widetilde{\Phi_1^T} + (1-\alpha)\widetilde{\Phi_2^T} , \tag{34}$$

because all operations involved are linear. Thus we have

$$\begin{aligned} P(\Phi) &= P(\alpha\Phi_1^T + (1-\alpha)\Phi_2^T) - \alpha\pi(\Phi_1) - (1-\alpha)\pi(\Phi_2) \\ &= P(\alpha\widetilde{\Phi_1^T} + (1-\alpha)\widetilde{\Phi_2^T}) - \alpha\pi(\Phi_1) - (1-\alpha)\pi(\Phi_2) . \end{aligned} \tag{35}$$

If $\Phi_1^T \not\simeq \Phi_2^T$, then we know from Theorem 2.2 that $\widetilde{\Phi_1^T} \neq \widetilde{\Phi_2^T}$, hence, according to [1],

$$P(\alpha\widetilde{\Phi_1^T} + (1-\alpha)\widetilde{\Phi_2^T}) > \alpha P(\widetilde{\Phi_1^T}) + (1-\alpha) P(\widetilde{\Phi_2^T}) .$$

Insertion into (35) gives immediately

$$P(\Phi) > \alpha P(\Phi_1) + (1-\alpha) P(\Phi_2) .$$

On the other hand, if $\Phi_1^T \simeq \Phi_2^T$, then we have $\widetilde{\Phi_1^T} = \widetilde{\Phi_2^T} = \widetilde{\Phi^T}$ and

$$P(\Phi) = P(\Phi^T) - \pi(\Phi) = \alpha P(\Phi_1) + (1-\alpha) P(\Phi_2) .$$

Proof of Proposition 2.6. We have to show the strict convexity of $P(\beta\Phi_\mu)$ with respect to μ. Φ_μ is given by $\Phi_\mu(X) = \Phi(X) - \delta_{1,N(X)}\mathcal{N}(X)$. It follows by a straightforward computation that

$$\begin{aligned} \Phi_\mu^T(X) &= \Phi^T(X) - \mu\delta_{1,N(X)}(\mathcal{N}(X) - \tfrac{1}{2}\mathbf{1}_X) , \\ \widetilde{\Phi_\mu^T}(X) &= \widetilde{\Phi^T}(X) - \mu\delta_{1,N(X)}(\mathcal{N}(X) - \tfrac{1}{2}\mathbf{1}_X) . \end{aligned}$$

Hence $\mu_1 \neq \mu_2$ implies $\widetilde{\Phi_{\mu_1}^T} \neq \widetilde{\Phi_{\mu_2}^T}$, and we can apply the previous theorem.

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Appendix: Proof of Lemma 3.5

Let us consider Eq. (30) with $A \in \mathfrak{B}(\mathscr{H}_{\Lambda_1})$. It suffices to show that each term on the r.h.s. is bounded in norm by $C \cdot \varepsilon$ if $\Lambda = \Gamma_k \supset \Gamma_{k_0}$, where k_0 is chosen large enough such that

$$\Lambda_1 \subset \Gamma_{k_0 - N - 1} . \tag{A 1}$$

The constant C may also depend on N and Λ_1. We insert $H_\Lambda(\Phi) = \sum_{X \subset \Lambda} \Phi(X)$, resp. $H_\Lambda(\tilde{\Phi}) = \sum_{Y \subset \Lambda} \tilde{\Phi}(Y)$, resp. the expression of Eq. (20) into (30). To shorten the notation, we write (for fixed Λ):

$$\sum_{(X,p)} = \sum_p \sum_{X: X \subset \Lambda, N(X) = p} = \sum_{X \subset \Lambda},$$

furthermore

$$\sum_{p_\iota} = \sum_{p_1, \dots, p_r = 1}^{N(\Lambda)}, \qquad \sum_{(X_\iota, p_\iota)} = \sum_{(X_1, p_1)} \dots \sum_{(X_r, p_r)} .$$

We put $s = m - r - 1$, the indices i and j run from 1 to r and from 1 to s, respectively. Then we get

$$\begin{aligned}
&[H_\Lambda(\tilde{\Phi}), [H_\Lambda(\Phi) - H_\Lambda(\tilde{\Phi}), [H_\Lambda(\Phi), A]^{(r)}]]^{(s)} \\
&= \sum_{p_\iota} \sum_l \sum_{q_j} \sum_{(X_i, p_\iota)} \sum_{(X,l)} \sum_{Z: X \subset Z \not\subset \Lambda} \sum_{(Y_j, q_j)} [\tilde{\Phi}(Y_s), [\dots [\tilde{\Phi}(Y_1), [-\mathrm{tr}_{Z \setminus X} \Phi_{l-1}(Z), \\
&\quad \cdot [\Phi(X_r), [\dots [\Phi(X_1), A] \dots] .
\end{aligned} \tag{A2}$$

Let us define $S_1 = \Lambda_1, S_{i+1} = S_i \cup X_i, i = 1, \dots, r-1, S = S_r \cup X_r$, $T_1 = S \cup X$, $T_{j+1} = T_j \cup Y_j, j = 1, \dots, s-1$. We may restrict the summations to those X_i, X and Y_j for which $X_i \cap S_i \neq \emptyset$, $X \cap S \neq \emptyset$, $Y_j \cap T_j \neq \emptyset$. [Notice that $\mathrm{tr}_{Z \setminus X} \Phi_{l-1}(Z) \in \mathfrak{B}(\mathscr{H}_X)$.] These restricted summations will be denoted by $\sum_{(X_i, p_\iota, S_i)} = \sum_{(X_1, p_1, S_1)} \dots \sum_{(X_r, p_r, S_r)}$, etc.

We estimate the norm of (A 2) by taking the norms of the terms of the r.h.s., using $\|\mathrm{tr}_{Z \setminus X} \Phi_{l-1}(Z)\| \leqq \|\Phi(Z)\| \, 2^{(l-1)N(Z)}$. This gives

$$\begin{aligned}
&\|[H_\Lambda(\tilde{\Phi}), [\dots [\dots]^{(r)}]]^{(s)}\| \\
&\leqq 2^m \|A\| \sum_{p_i} \sum_{(X_i, p_\iota, S_i)} \prod_{i=1}^r \|\Phi(X_i)\| \sum_l \sum_{(X,l,S)} \sum_{Z: X \subset Z \not\subset \Lambda} \|\Phi(Z)\| \, 2^{(l-1)N(Z)} \\
&\quad \cdot \sum_{q_j} \sum_{(Y_j, q_j, T_j)} \prod_{j=1}^s \|\tilde{\Phi}(Y_j)\| .
\end{aligned} \tag{A 3}$$

We evaluate the sums starting with the q_j- and Y_j-summations. We use the same arguments as in Section 7.6 of [2], with

$$N(S_i) \leqq N(\Lambda_1) + \Sigma p_i, \quad N(S) \leqq N(\Lambda_1) + \Sigma p_i,$$
$$N(T_j) \leqq N(\Lambda_1) + \Sigma p_i + l + \Sigma q_i,$$
$$\prod_{j=1}^{s} N(T_i) \leqq (N(\Lambda_1) + \Sigma p_i + l + \Sigma q_j)^s \leqq s!\, e^{N(\Lambda_1)+l}\, \Pi\, e^{p_i}\, \Pi\, e^{q_j},$$

with the result

$$\sum_{q_j} \sum_{(Y_j, q_j, T_j)} \prod_{j=1}^{s} \|\tilde{\Phi}(Y_j)\| \leqq s!\, e^{N(\Lambda_1)} e^l\, \Pi\, e^{p_i} (\|\tilde{\Phi}\|_{e^\xi})^s. \tag{A 4}$$

For $s = 0$, (A 4) is to be replaced by $1 \leqq e^{N(\Lambda_1)} e^l\, \Pi\, e^{p_i}$.

The next step is to consider

$$\begin{aligned} \sigma(X_1, \ldots, X_r) &\equiv \sum_l \sum_{(X,l,S)} \sum_{Z: X \subset Z \not\subset \Lambda} \|\Phi(Z)\|\, 2^{(l-1)N(Z)} e^l \\ &\leqq \sum_{x \in S} \sum_{Z: Z \not\subset \Lambda, Z \ni x} \|\Phi(Z)\| \sum_{l=1}^{N(\Lambda)} \sum_{X: X \subset Z, N(X) = l} 2^{(l-1)N(Z)} e^l \qquad \text{(A 5)} \\ &\leqq \sum_{x \in S} \sum_{Z: Z \not\subset \Lambda, Z \ni x} \|\Phi(Z)\|\, f_0(N(Z)). \end{aligned}$$

Now let us take $\Lambda = \Gamma_k$, with a fixed $k \geqq k_0$ [k_0 as defined in (A 1)], and try to apply Lemma 3.8 to the r.h.s. of (A 6). This is possible for those $X_1, \ldots, X_r$, for which $S = \Lambda_1 \cup X_1 \cup \cdots \cup X_r$ fulfills

$$\text{(S):} \quad N(S \cap \Gamma_{k-1})/N(S) > 1 - \varepsilon/2\|\Phi\|_{f_0}.$$

We define

$$\chi(\Lambda'; X_1, \ldots, X_r) = \begin{cases} 1 & \text{if} \quad S = \Lambda' \cup X_1, \ldots, X_r \text{ satisfies (S)}, \\ 0 & \text{otherwise}. \end{cases} \tag{A 6}$$

Then we have by application of Lemma 3.8

$$\sigma(X_1, \ldots, X_r)\, \chi(\Lambda_1; X_1, \ldots, X_r) < N(S) \cdot \varepsilon < e^{N(\Lambda_1)} \prod_i e^{p_i} \cdot \varepsilon \tag{A 7}$$

furthermore,

$$\begin{aligned} \sigma(X_1, \ldots, X_r)\,(1 - \chi(\Lambda_1; X_1, \ldots, X_r)) &\leqq N(S)\, \|\Phi\|_{f_0} (1 - \chi(\Lambda_1; X_1, \ldots, X_r)) \\ &\leqq e^{N(\Lambda_1)} \prod_i e^{p_i} \|\Phi\|_{f_0} (1 - \chi(\Lambda_1; X_1, \ldots, X_r)). \qquad \text{(A 8)} \end{aligned}$$

Combining Eqs. (A4) through (A 8) with (A 3), we get

$$\|[H_\Lambda(\tilde{\Phi}), [\ldots [H_\Lambda(\Phi), A]^{(r)}]]^{(s)}\| \leqq \sigma_1 + \sigma_2, \tag{A 9}$$

$$\sigma_1 = C_{s,m} \sum_{p_i} \sum_{(X_i, p_i, S_i)} \prod_{i=1}^{r} \|\Phi(X_i)\| \, e^{2p_i} \cdot \varepsilon \,, \tag{A 10}$$

$$\sigma_2 = C_{s,m} \|\Phi\|_{f_0} \sum_{p_i} \sum_{(X_i, p_i, S_i)} \prod_{i=1}^{r} \|\Phi(X_i)\| \, e^{2p_i}(1 - \chi(\Lambda_1; X_1, \ldots, X_r)) \,, \tag{A 11}$$

$$C_{s,m} = s! \, e^{2N(\Lambda_1)} (\|\tilde{\Phi}\|_{e^{\xi}})^s \cdot 2^m A \,. \tag{A 12}$$

For $r=0$ we have to put $\prod_{i=1}^{r} \cdots = 1$, furthermore, $\chi(\Lambda_1; X_1, \ldots, X_r) = 1$ because $\Lambda_1 \subset \Gamma_{k-N-1}$ and (S) is fulfilled, hence $\sigma_2 = 0$.

We can estimate σ_1 by the same method as in (A 4):

$$\sigma_1 \leqq C_{s,m} r! \, e^{N(\Lambda_1)} (\|\Phi\|_{e^{3\xi}})^r \cdot \varepsilon \,. \tag{A 13}$$

This equation also holds for $r=0$.

For $r \geqq 1$, let us write

$$\sigma_2 = C_{s,m} \|\Phi\|_{f_0} \, \Sigma(r; \Lambda_1; \Gamma_k) \tag{A 14}$$

with

$$\Sigma(r; \Lambda'; \Gamma_k) = \sum_{p_i} \sum_{(X_i, p_i, S_i)} \prod_{1}^{r} \|\Phi(x_i)\| \, e^{2p_i}(1 - \chi(\Lambda'; X_1, \ldots, X_r)) \,, \tag{A 15}$$

$S_1 = \Lambda'$, $S_i = \Lambda' \cup X_1 \cup \cdots \cup X_{i-1}$, $i = 2, \ldots, r$, $\Lambda' \subset \Gamma_k$, $X_i \subset \Gamma_k$. We shall show by induction that

$$\begin{aligned} &\Sigma(r; \Lambda'; \Gamma_k) \\ &\leqq e^{N(\Lambda') + r - 1} N(\Lambda')^r \prod_{\varrho=0}^{r} (\varrho!) (\|\Phi\|_{f_0})^{r-1} \varepsilon \quad \text{if} \quad \Lambda' \subset \Gamma_{k-r-1} \,. \end{aligned} \tag{A 16}$$

Because of $\Lambda_1 \subset \Gamma_{k-N-1}$, Eqs. (A 9)–(A 16) finally yield

$$\|[H_\Lambda(\tilde{\Phi}), [H_\Lambda(\Phi) - H_\Lambda(\tilde{\Phi}), [H_\Lambda(\Phi), A]^{(r)}]]^{(s)}\| \leqq C \cdot \varepsilon \,,$$

$$\begin{aligned} C = {} & N! \, e^{3N(\Lambda_1)} (2\|\Phi\|_{e^{3\xi}} \|\tilde{\Phi}\|_{e^{\xi}})^N \, \|A\| \\ & + \prod_{\varrho=0}^{N} (\varrho!) e^{3N(\Lambda_1)} (2eN(\Lambda_1) \, \|\tilde{\Phi}_{e^{\xi}}\| \, \|\Phi\|_{f_0})^N \, \|A\| \,, \end{aligned}$$

which is the desired estimate.

It remains to prove (A 16). Take $r=1$ and $\Lambda' \subset \Gamma_{k-2}$, then $\chi(\Lambda'; X_1) = 1$ if $X_1 \subset \Gamma_{k-1}$, i.e. $1 - \chi(\Lambda'; X_1)$ is certainly zero unless $X_1 \not\subset \Gamma_{k-1}$, thus

$$\begin{aligned} \Sigma(1; \Lambda'; \Gamma_k) &\leqq \sum_{p_1} \sum_{X_1 : N(X_1) = p_1, X_1 \cap \Lambda' \neq \emptyset, X_1 \not\subset \Gamma_{k-1}} \|\Phi(X_1)\| \, e^{2p_1} \\ &\leqq \sum_{x \in \Lambda'} \sum_{X_1 : x \in X_1 \not\subset \Gamma_{k-1}} \|\Phi(X_1)\| \, e^{2N(X_1)} \,. \end{aligned}$$

We can apply Lemma 3.8 since $\Lambda' \subset \Gamma_{k-2}$ and $e^{2\xi} \leqq f_0(\xi)$, getting

$$\Sigma(1;\Lambda';\Gamma_k) < N(\Lambda')\frac{\varepsilon}{2} < e^{N(\Lambda')} N(\Lambda')\varepsilon\,,$$

i.e. (A 16) holds for $r=1$. Let us suppose its validity for $r-1$ and assume $\Lambda' \subset \Gamma_{k-r-1}$. Notice that $\chi(\Lambda'; X_1, \dots, X_r) = \chi(\Lambda' \cup X_1; X_2, \dots, X_r)$, therefore,

$$\Sigma(r;\Lambda';\Gamma_k) = \sum_{p_1} \sum_{(X_1,p_1,S_1)} \|\Phi(X_1)\|\, e^{2p_1}\, \Sigma(r-1;\Lambda'\cup X_1;\Gamma_k)\,. \quad \text{(A 17)}$$

We split the X_1-summation into two parts: one part with $X_1 \subset \Gamma_{k-r}$ so that we can use (A 16) in estimating $\Sigma(r-1;\Lambda'\cup X_1;\Gamma_k)$, and a second one with $X_1 \not\subset \Gamma_{k-r}$ to which we again apply Lemma 3.8 (with Γ_k replaced by Γ_{k-r}) using

$$\Sigma(r-1;\Lambda'\cup X_1;\Gamma_k) \leqq (r-1)!\, e^{N(\Lambda')+N(X_1)} (\|\Phi\|_{e^{3\xi}})^{r-1}\,.$$

(In the first part, the factor $N(\Lambda'\cup X_1)^{r-1}$ appearing in the bound of $\Sigma(r-1;\Lambda'\cup X_1;\Gamma_k)$ is to be replaced by $(N(\Lambda')+N(X_1))^{r-1} \leqq N(\Lambda')^{r-1}(1+N(X_1))^{r-1} \leqq N(\Lambda')^{r-1}(r-1)!\, e^{1+N(X_1)}$.) This gives

$$\begin{aligned}\Sigma(r;\Lambda';\Gamma_k) &\leqq N(\Lambda')\,\|\Phi\|_{f_0}\cdot e^{N(\Lambda')+r-1} N(\Lambda')^{r-1}(r-1)! \prod_0^{r-1} (\varrho!)\,(\|\Phi\|_{f_0})^{r-2}\cdot\varepsilon \\ &\quad + N(\Lambda')\frac{\varepsilon}{2}\cdot(r-1)!\, e^{N(\Lambda')}(\|\Phi\|_{f_0})^{r-1} \\ &\leqq N(\Lambda')^r e^{N(\Lambda')+r-1}(r-1)! \prod_0^{r-1} (\varrho!)\,(\|\Phi\|_{f_0})^{r-1}\left(1+\tfrac{1}{2}\right)\varepsilon\,,\end{aligned}$$

which is the bound of (A 16) if we replace $1+\frac{1}{2}$ by $r > 1+\frac{1}{2}$. This completes the proof of Lemma 3.5.

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H. Roos
Institut für Theoretische Physik
der Universität
D-3400 Göttingen, Bunsenstr. 9
Fed. Rep. of Germany