Commun. math. Phys. 36, 133—136 (1974)

On the Euclidean Version of Haag's Theorem in $P(\varphi)_2$ Theories

Robert Schrader
Institut für Theoretische Physik, Freie Universität Berlin, Berlin

Received December 3, 1973

Abstract. We prove that the (physical) measures giving the euclidean Green's functions for the $P(\varphi)_2$ theory for small coupling constants are mutually singular under specified conditions on the ground-state energy density. In particular the free measure and the physical measure are mutually singular when the coupling constant is small.

In a recent article [7] Newman proved that the euclidean Green's functions of the $P(\varphi)_2$ theory for small coupling constants (see e.g. [1]) are the moments of a measure $\mu_{\infty,\lambda}$ on the σ-algebra Σ generated by the cylinder sets on $\mathscr{D}'(\mathbb{R}^2)$. The $\mu_{\infty,\lambda}$ are called the physical measures.

In this note we present a result on the support of these physical measures. Under conditions to be given below, we prove that these measures have disjoint support.

To state the precise result, we recall some notations and definitions. μ_0, the free measure, is defined to be the Gaussian measure on $\mathscr{D}'(\mathbb{R}^2)$ with mean zero and covariance $(-\Delta+m^2)^{-1}$. Δ is the Laplacian on $\mathbb{R}^2$ and $m>0$ is the free mass, which will stay fixed.

We write

$$E(u)=\int_{\omega\in\mathscr{D}'(\mathbb{R}^2)} u(\omega)\,d\mu_0(\omega)$$

for an element $u\in L^1(\mathscr{D}'(\mathbb{R}^2),\Sigma,d\mu_0)$.

Let P be a normalized (see [2]), semibounded real polynomial, let $\varphi(x)$ be the euclidean field in $\mathbb{R}^2$ and let

$$V_{l,\lambda}=\lambda\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}} :P(\varphi(x)):dx$$

$$l\in\mathbb{Z}^+,\quad \lambda\geqq 0$$

be the euclidean action in the box of side-length l centered at the origin corresponding to the $P(\varphi)_2$ theory. We set

$$X_{l,\lambda}=E(\exp-V_{l,\lambda})^{-1}\exp-V_{l,\lambda}\geqq 0$$

such that $E(X_{l,\lambda})=1$. The following asymptotic behaviour for small λ has been proved using the Feynman-Kac-Nelson formula [6], see [2, 7]:

$$E(\exp - V_{l,\lambda}) = \exp\left(l^2(\alpha_\infty(\lambda) + 0(l^{-1}))\right) \tag{1}$$

where $-\alpha_\infty(\lambda)$ is the ground state energy density. $\alpha_\infty(\lambda)$ is non-negative, convex and hence continuous for $\lambda \geqq 0$ [3]. Let now λ_1 be as in Lemma 6 of [7]. Then the physical measures $\mu_{\infty,\lambda}$ exist for $0 \leqq \lambda < \lambda_1$. We define Ω to be the subset of $[0, \lambda_1)^2$ consisting of all (λ', λ) with $0 \leqq \lambda' < \lambda < \lambda_1$ and

$$\alpha_\infty(\delta\lambda' + (1-\delta)\lambda) < \delta\alpha_\infty(\lambda') + (1-\delta)\alpha_\infty(\lambda) \tag{2}$$

for some $\delta \in [0, 1]$. Note that by the convexity of α_∞, (2) is always true, if $<$ is replaced by $\leqq$, so Ω is also the set of all points (λ', λ) such that α_∞ is not a linear function in the closed interval $[\lambda', \lambda]$. Since $\alpha_\infty(\lambda) = 0(\lambda^2)$ for small λ [3], Ω contains all points of the form $(0, \lambda)$ for $0 < \lambda < \lambda_1$. In particular Ω is not empty. Since α_∞ is continuous, Ω is open in the space $[0, \lambda_1)^2$.

We are now prepared to state the main result of this note

Theorem. *If $(\lambda', \lambda) \in \Omega$ then the measures $\mu_{\infty,\lambda'}$ and $\mu_{\infty,\lambda}$ are mutually disjoint. In particular the free measure and the physical measure $\mu_{\infty,\lambda}$ are mutually disjoint for small λ.*

It was known that $\mu_{\infty,\lambda}$ is not absolutely continuous with respect to μ_0, so the theorem sharpens this observation. The theorem may be viewed as a strong euclidean version of Haag's theorem on the non-existence of the interaction picture [5, 4]. In agreement with this view is the fact that the theorem results from a volume effect when the thermodynamic limit is taken.

For a proof of the theorem we introduce the following sets in Σ, which depend on λ, λ':

$$\begin{aligned} A_l &= \{\omega \in \mathscr{D}'(\mathbb{R}^2) \mid X_{l,\lambda}(\omega) \leqq X_{l,\lambda'}(\omega)\} \\ B_l &= \bigcap_{l' \geqq l} A_{l'} \qquad l', l \in \mathbb{Z}^+ . \end{aligned} \tag{3}$$

Obviously

$$B_l \subset B_{l'}, \quad B_l \subset A_{l'}, \quad \text{for} \quad l' \geqq l .$$

We set

$$B = \bigcup_l B_l = \sup_l B_l = \varliminf_l A_l . \tag{4}$$

We claim

$$\mu_{\infty,\lambda}(B) = 0 \tag{5}$$

$$\mu_{\infty,\lambda'}(B) = 1 \tag{6}$$

for $(\lambda', \lambda) \in \Omega$. Obviously these relations prove the theorem. For a proof of Rel. (5), (6) we recall the definition of the physical measures:

First for an open U in $\mathbb{R}^2$, let $\Sigma(U)$ be the σ-algebra generated by the fields $\varphi(f)$ with $\operatorname{supp} f \subset U$ and for V closed $\Sigma(V)$ is $\cap \Sigma(U)$ with the intersection taken over all open U containing V. For an element

$u \in L^2(\mathscr{D}'(\mathbb{R}^2), \Sigma, d\mu_0)$ we denote by $E_l u$ the projection onto $L^2(\mathscr{D}'(\mathbb{R}^2), \Sigma_l, d\mu_0)$ where $\Sigma_l = \Sigma(V)$ with

$$V = \left\{ x \in \mathbb{R}^2 \,\middle|\, |x_i| \leqq \frac{l}{2}, \quad i = 1, 2 \right\}.$$

In [7], $\mu_{\infty,\lambda}$ is constructed as

$$\begin{aligned} \int E_l u \, d\mu_{\infty,\lambda} &= \lim_{l' \to \infty} \int E_l u \, d\mu_{l',\lambda} \\ &= \lim_{l' \to \infty} E(X_{l',\lambda} E_l u) \end{aligned}$$

for any l.

To prove Rel. (5) and (6) we need another estimate which is an easy consequence of the proof of Lemma 1 in [7]:

$$E(|E_l(X_{l,\lambda} - X_{l',\lambda})|) \leqq O(e^{-\varepsilon l}) \tag{7}$$

for all $\lambda < \lambda_1$, uniformly for all $l' \geqq l$ and some $\varepsilon = \varepsilon(\lambda)$.

To prove Rel. (5), we note that

$$\begin{aligned} \mu_{\infty,\lambda}(B) &= \sup_l \mu_{\infty,\lambda}(B_l) \\ &\leqq \lim_l \sup \mu_{\infty,\lambda}(A_l) \end{aligned}$$

since $\mu_{\infty,\lambda}$ is a measure. Now $A_l \in \Sigma_l$ so we have

$$\mu_{\infty,\lambda}(A_l) = \lim_{l' \to \infty} E(X_{l',\lambda} A_l).$$

Here and in what follows we have identified sets with their characteristic functions. We write

$$\begin{aligned} E(X_{l',\lambda} A_l) &= E(E_l(X_{l',\lambda} - X_{l,\lambda}) A_l) \\ &\quad + E(X_{l,\lambda} A_l). \end{aligned} \tag{8}$$

The first term on the right hand side is controlled by estimate (7).

By the definition of A_l we have

$$X_{l,\lambda} \leqq (X_{l,\lambda'})^\delta (X_{l,\lambda})^{1-\delta} \quad \text{on} \quad A_l \tag{9}$$

for any $\delta \in [0, 1]$. In particular, if we choose δ such that (2) holds we obtain

$$\begin{aligned} E(X_{l,\lambda} A_l) &\leqq E((X_{l,\lambda'})^\delta (X_{l,\lambda})^{1-\delta}) \\ &\leqq O(e^{-cl^2}) \end{aligned} \tag{10}$$

for some $c > 0$, using estimate (1). Estimates (7)–(10) show that $\mu_{\infty,\lambda}(B) = 0$.

The proof of Rel. (6) is completely analogous:

We first note that for the complement CB of B we have by Rel. (3):

$$CB \subset CB_l \subset \bigcup_{l' \geqq l} CA_{l'}$$

so

$$\begin{aligned} \mu_{\infty,\lambda'}(CB) &\leqq \sum_{l' \geqq l} \mu_{\infty,\lambda'}(CA_{l'}) \\ &= \sum_{l' \geqq l} \lim_{l'' \to \infty} \mu_{l'',\lambda'}(CA_{l'}) \end{aligned}$$

for any l.

Now

$$\mu_{l'',\lambda'}(CA_{l'}) = E(E_{l'}(X_{l'',\lambda'})\, CA_{l'}) .$$

Hence

$$\mu_{l'',\lambda'}(CA_{l'}) \leqq E(|E_{l'}(X_{l'',\lambda'} - X_{l',\lambda'})|) + E(X_{l',\lambda'}\, CA_{l'}) . \tag{11}$$

Again the first term is estimated by $O(e^{-\varepsilon' l'})$ uniformly for all $l'' \geqq l'$. For the second term we observe that now

$$X_{l',\lambda'} \leqq (X_{l',\lambda'})^{\delta}\, (X_{l',\lambda})^{1-\delta} \quad \text{on} \quad CA_{l'}$$

so again by (1)

$$E(X_{l',\lambda'}\, CA_{l'}) \leqq O(e^{-c(l')^2})$$

if δ is chosen so that (2) is satisfied. Summing over $l' \geqq l$ and taking lim sup over l proves Rel. (6), concluding the proof of the theorem.

References

1. Glimm, J., Jaffe, A., Spencer, T.: The particle structure of the weakly coupled $P(\varphi)_2$ model and other applications of high temperature expansions. In: Constructive Quantum Field Theory, ed. Velo, G., Wightman, A. S., Lecture Notes in Physics, Vol. **25**, Berlin-Heidelberg-New York: Springer 1973
2. Guerra, F., Rosen, L., Simon, B.: Commun. math. Phys. **27**, 10—22 (1972)
3. Guerra, F., Rosen, L., Simon, B.: Commun. math. Phys. **29**, 233—247 (1973)
4. Guerra, F., Rosen, L., Simon, B.: The $P(\varphi)_2$ Euclidean quantum field theory as classical statistical mechanics. Ann. Math. (to appear)
5. Haag, R.: Mat. Fys. Medd. **29**, No. 12 (1955)
6. Nelson, E.: Quantum fields and Markoff fields, Proceedings of Summer Institute on Partial Differential equations. Providence: Am. Math. Soc. 1973
7. Newman, C.: The construction of stationary two-dimensional Markoff fields with an application to quantum field theory. NYU, preprint 1973

Communicated by A. S. Wightman

R. Schrader
Institut für Theoretische Physik der FU
D-1000 Berlin 33
Arnimallee 3