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Some Aspects of Markov and Euclidean Field Theories

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Abstract. Various aspects of Markov field theory are treated. We give a Fock space description of the scalar free field with Nelson's Markov property formulated in terms of projections. We consider conditions imposed on analytically continued Wightman distributions at Euclidean points so that a Euclidean Markov field theory will result. Euclidean theories in higher dimensional imaginary times are considered. We show how the generalized free field theory can be interpreted as a Markov Euclidean field theory. The spatially cutoff linear perturbation model is solved in arbitrary space-time dimensions and the Wightman distributions are obtained explicitly in the limit as the cutoff is removed. The appendices contain a discussion and derivation of the Segal isomorphism and we give some generalizations of Feynman-Kac formulas in R^n and in the Fock space of Euclidean field theory.

Introduction

Euclidean field theory techniques, particularly Nelson's Markov property and consequent symmetry, have recently played an important role in constructive quantum field theory (see [1–4]). These methods are among the techniques used by Glimm and Spencer [5] to show that the Schwinger distributions for the spatially cutoff $P(\phi)_2$ model converge as the cutoff is removed, provided the coupling constant is sufficiently small. They obtain the Wightman distributions by analytic continuation and show that the corresponding relativistic quantum field theory has a mass gap. For earlier work on the relation of Euclidean and Minkowski field theories see Symanzik [6].

In this article we consider several aspects of Euclidean field theory. In Section I we describe the free scalar Euclidean field and formulate what we call the pre-Markov property of certain projections. This property is due to Nelson [1] and implies the Markov property – we isolate it because it does not require a probabilistic interpretation in its formulation, and because it can be used directly in many applications of the Markov property. Section II contains a brief discussion of the general problem of obtaining Euclidean quantum fields from relativistic quantum fields and of the relationship between Wightman distributions and the pre-Markov property.

We note that for some purposes it is natural to use a Euclidean theory with imaginary time dimension > 1, and that the techniques of Section I carry over immediately to any (odd) number of imaginary time dimensions. We show that a generalized free scalar field does not have the Markov property, but that additional local fields can be introduced to recover the property.

In Section III the generalized free field construction of Section II is motivated by a probabilistic analogy and a complementary and more detailed construction is given.

In Section IV the results of Section I, Appendix C and the theory of hypercontractive semigroups [7, 8] are applied to the cutoff linear perturbation model in arbitrary space-time dimensions, and the cutoff is removed. The limiting Schwinger and Wightman distributions are obtained explicitly. A previous treatment of the cutoff model has been given by Friedrichs [9].

Probably the fastest and most explicit way of introducing probabilistic methods into quantum field theory is via the Segal isomorphism [10], by means of which Fock space is represented as $L^2(M, du)$ where (M, u) is a probability measure space, and Wick polynomials of smeared field operators are represented as unbounded multiplication operators. Appendix A contains a short and explicit derivation of this isomorphism, together with motivational discussion (this derivation is introductory and elementary and of course no substitute for Segal's original paper [10]). In Appendix B we have included derivations of some generalizations of path space formulas for Markov processes in R^n, and in Appendix C we present analogous formulas relating certain Minkowski Fock space inner products to Nelson space inner products (Nelson space being the Fock space associated with the free Euclidean scalar massive field).

We are greatly indebted to Barry Simon for orientation and for showing us the embedding operator techniques which we describe in Section I and which can be used to give operator-theoretic derivations of the formulas in Appendix C. We also thank George Svetlichny and J. A. Swieca for many helpful discussions and observations.

I. Projections Associated with the Free Scalar Euclidean Field

In this section we briefly describe the Euclidean field theory corresponding to a free scalar relativistic field theory [11, 12], then following Nelson introduce a family of projections indexed by closed sets and examine some of their properties. We also consider the case of a more general inner product.

The Wightman two-point distribution for a scalar field with mass m in $d+1$ dimensional space time is

$$(\Omega, \psi(x)\,\psi(y)\,\Omega) = \Delta_+(x-y;m^2) = \int dk\, e^{ik(x-y)}\,\theta(k_0)\,\delta(k^2-m^2) \tag{1.1}$$

$$= \int d\boldsymbol{k}\, \frac{e^{i\omega(\boldsymbol{k})}(x_0-y_0) - \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}{2\omega(k)}, \quad \omega(\boldsymbol{k}) = (\boldsymbol{k}^2+m^2)^{1/2}.$$

The Hilbert space for the free scalar field is the Fock space $\mathscr{F}(F)$ built over the real one-particle Hilbert space F obtained by completing the inner product space whose elements are equivalence classes of elements of $\mathscr{S}(R^{d+1})$, with equivalence defined with respect to the norm given by the inner product $(\;,\;)_M$

$$(f,g)_M = \tfrac{1}{2}\int dx\, dy\, \Delta_+(x-y)\, f\quad(x)\, g(y)\,. \tag{1.2}$$

Since the support of the Fourier transform of Δ_+ is confined to the hyperboloid sheet $k=m^2$, $k_0 \geqq 0$, each equivalence class may be represented by an element $u(\boldsymbol{x}) \in \mathscr{S}(R^d)$; with this representation F is the Sobolev space $\mathscr{H}^{-1/2}(-\Delta+m^2)$ in d dimensions, and we use the symbol $f(\boldsymbol{x})$ to denote an element of F obtaining

$$(f(\boldsymbol{x}), g(\boldsymbol{x}))_M = \frac{1}{2}\int d\boldsymbol{k}\, \frac{\overline{\tilde{f}(\boldsymbol{k})}\, g(\boldsymbol{k})}{2\omega(\boldsymbol{k})} \tag{1.3}$$

where $\tilde{f}$ denotes the Fourier transform of F defined by

$$\tilde{f}(\boldsymbol{k}) = \int d\boldsymbol{x}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\quad f(\boldsymbol{x})\,. \tag{1.4}$$

Here the Fock space $\mathscr{F}(K)$ built over a real Hilbert space K [13, 10] is the complexification of the direct sum

$$R \oplus K \oplus (K \otimes K)_{\mathrm{sym}} \oplus \cdots (\overset{1}{K} \otimes \cdots \otimes \overset{n}{K})_{\mathrm{sym}} \oplus \cdots, \tag{1.5}$$
$$\Omega = 1 \in R\,.$$

If $A: K \to K'$ is a bounded operator with bound $\leqq 1$, the corresponding bounded operator $\Gamma(A): \mathscr{F}(K) \to \mathscr{F}(K')$ is defined as the closure of

$$I \oplus A \oplus (A \otimes A) \oplus \cdots.$$

Γ preserves the properties of being an orthogonal projection, of being unitary, and of being isometric. Let $\mathscr{F}_0(K)$ denote the set of elements of $\mathscr{F}(K)$ which have only a finite number of non-zero summands in (1.4). For each $h \in K$ one defines corresponding operators $a(h): \mathscr{F}_0(K) \to \mathscr{F}_0(K)$,

$\psi(h)=2^{-1/2}(a(h)+a(h)^+)$ using linearity and continuity to extend

$$\begin{aligned} &a(h)^+(\dots(f_1\otimes\dots\otimes f_n)_{\text{sym}}\dots) \\ &\quad=(n+1)^{1/2}(\dots(h\otimes f_1\otimes\dots\otimes f_n)_{\text{sym}}\dots \\ &a(h)(\dots(f_1\otimes\dots\otimes f_n)_{\text{sym}}\dots) \\ &\quad=n^{-1/2}\sum_{i=1}^{n}(h,f_i)(\dots(f_1\otimes\dots\hat{f}_i\dots f_n)_{\text{sym}}\dots\,. \end{aligned} \tag{1.6}$$

In (1.6) $a(h)^+$ in the adjoint of $a(h)$ restricted to $\mathscr{F}_0(K)$. With this construction $\psi(f(\boldsymbol{x}))$, $f(\boldsymbol{x})\in F$ in the time-zero scalar relativistic quantum field.

The corresponding Euclidean field theory is obtained by replacing $\Delta_+(x,m^2)$ by its analytic continuation to the Euclidean region

$$\begin{aligned} &\Delta_E(x,m^2)=\int\frac{dk_0\,d\boldsymbol{k}\,e^{ikx}}{2\pi\,W(k)}=\int\frac{e^{-\omega(\boldsymbol{k})s-i\boldsymbol{k}\cdot\boldsymbol{x}}}{2\omega(\boldsymbol{k})}d\boldsymbol{k} \\ &x=(s,\boldsymbol{x})\in R^{d+1}\qquad W(k)=k_0^2+\boldsymbol{k}^2+m^2\,. \end{aligned} \tag{1.7}$$

The corresponding Hilbert space is $\mathscr{N}=\mathscr{F}(N)$ where N is the one particle Euclidean space obtained by completing the real inner product space whose elements are in $\mathscr{S}(R^{d+1})$ and for which the inner product is

$$\begin{aligned} (f(x),g(y))_E&=\frac{1}{2}\int dx\,dy\,\Delta_E(x-y,m^2)\,f(x)\,g(y) \\ &=\frac{1}{2}\int dk_0\,d\boldsymbol{k}\frac{\overline{\tilde{f}(k_0,\boldsymbol{k})}\,\tilde{g}(k_0,\boldsymbol{k})}{2\pi\,W(k)}\,. \end{aligned} \tag{1.8}$$

N is the Sobolev space $\mathscr{H}^{-1}(-\Delta+m^2)$ in $d+1$ dimensions. N is a representation space for the Euclidean group on R^{d+1} with the corresponding operators $u(b,R)$ defined by

$$(u(b,R)\,h)\,(x)=h(b+Rx)$$

for $h\in N$, $h\in R^{d+1}$, and Euclidean rotation R. We let $\Gamma(u(b,R))=U(b,R)$ and for $(b,R)=((t,0),I)$ we let u_t and U_t denote the corresponding unitary operators. We have $U(b,R)\,\phi(h(x))\,U^{-1}(b,R)=\phi(u(b,R)\,h(x))$ for the Euclidean field $\phi(h(x))$, $h\in N$.

There is a natural isometric embedding $j_0:F\to N$ given by

$$(j_0f)\,(s,\boldsymbol{x})=\delta(s)\,f(\boldsymbol{x}) \tag{1.9}$$

i.e.

$$(j_0f)\tilde{}\,(k_0,\boldsymbol{k})=\tilde{f}(\boldsymbol{k})\,. \tag{1.10}$$

The adjoint $j_0^+ : N \to F$ is given by

$$(j_0^+ h)^\sim(\boldsymbol{k}) = \int dk_0 \frac{\omega(\boldsymbol{k})}{\pi W(k)} \tilde{g}(k_0, \boldsymbol{k}). \tag{1.11}$$

We have $j_0^+ j_0 = I : F \to F$ and thus $e_0 = j_0 j_0^+$ is an orthogonal projection. Setting $j_t = u_t j_0$ we find

$$\begin{aligned}(j_t^+ j_{t'} f)^\sim(\boldsymbol{k}) &= e^{-\omega(\boldsymbol{k})|t-t'|} \tilde{f}(\boldsymbol{k}) \\ &= (e^{-|t-t'|h_0} f)^\sim(\boldsymbol{k})\end{aligned} \tag{1.12}$$

where h_0 is the single particle relativistic Hamiltonian. We let $J_t = \mathrm{U}_t \Gamma(j_0)$, $E_t = U_t \Gamma(e_0) U_t^{-1}$ and have

$$\begin{aligned}J_t^+ J_{t'} &= e^{-|t-t'|H_0} : \mathscr{F} \to \mathscr{F} \\ J_t J_t^+ &= E_t : \mathscr{N} \to \mathscr{N}\end{aligned} \tag{1.13}$$

where H_0 is the free relativistic Hamiltonian.

From here on we denote the Euclidean field $\phi(h(s, \boldsymbol{x}))$, $h \in N$, and the relativistic field by $\psi(f(\boldsymbol{x}))$, $f(\boldsymbol{x}) \in F$; one has

$$\psi(f(\boldsymbol{x})) = J_0^+ \, \phi(\delta(s) \, f(\boldsymbol{x})) \, J_0 \qquad \forall \, f \in F. \tag{1.14}$$

We let α_t denote the one-parameter family of mappings from the algebra of time-zero field operators into the algebra of Euclidean field operators defined [for bounded functions f and polynomials restricted to $\mathscr{F}_0(F)$] by

$$\alpha_t(g\{\psi(f_\alpha(\boldsymbol{x}))\}) = g(\{\phi(\delta(s-t) \, f_\alpha(\boldsymbol{x}))\}). \tag{1.15}$$

We find

$$\begin{aligned}J_t^+ \, \alpha_t(g\{\psi(f_\alpha(\boldsymbol{x}))\}) \, J_t &= g(\cdot) : \mathscr{F} \to \mathscr{F} \\ E_t \alpha_t(g(\cdot)) \, E_t &= J_t g(\cdot) \, J_t^+ : \mathscr{N} \to \mathscr{N}.\end{aligned} \tag{1.16}$$

Each element h of N has a corresponding tempered distribution $[h]$, defined by $[h](u) = (h, (-\Delta + m^2) u)_E$; by supph, $h \in N$ we mean the support of $[h]$. The support of $j_0(f(x))$ is contained in the hyperplane $s = 0$ for all $f \in F$ – moreover every element of N with support in $s = 0$ is of the form $j_0(f(\boldsymbol{x}))$, $f \in F$. This may be shown using the fact that any tempered distribution T with support in $s = 0$ is of the form

$$T = \sum_{n=0}^{N} a_n(\boldsymbol{x}) \, \delta^{(n)}(s), \qquad a_n(\boldsymbol{x}) \in \mathscr{S}'(R^d), \tag{1.17}$$

and if T is to lie in N then $a_n = 0$, for $n > 0$.

Since the limit of a sequence of distributions each of which has support in a closed set C also has support in C, the set N_C of elements of N with support in C is a closed subspace; we let p_C be the orthogonal projection on N_C and define $P_C = \Gamma(p_C)$. Equivalently, P_C is the projection on the subspace $\mathcal{N}_C \subset \mathcal{N}$ generated by applying polynomials in $\phi(f)$, $\mathrm{supp} f \subset C$, to the vacuum. In particular we have $P_{(s=t)} = E_t$. The basic property [1] of these projections is given in

Theorem 1.1. *If $C_1 \subset R^{d+1}$ and $C_2 \subset R^{d+1}$ are closed then*

$$\begin{aligned} &\text{(i)} \ P_{\overline{C'}} P_C = P_{\partial C} \\ &\text{(ii)} \ P_{C_1} P_{C_2} = P_{\partial C_1 \cup (C_1 \cap C_2)} P_{\partial C_2 \cup (C_1 \cap C_2)} \end{aligned} \tag{1.18}$$

where ∂C_i and $\overline{C_j'}$ denote the boundary of C_i and the closure of its complement, respectively.

Proof. For (i) it suffices to show that $\mathrm{supp}(p_C p_{\overline{C'}} f) \subset \overline{C'}$ for all $f \in N$, implying $P_{\partial C} P_C P_{\overline{C'}} = P_C P_{\overline{C'}}$ from which (1.17i) follows by taking adjoints. To show this let $v \in \mathscr{S}(R^{d+1})$ have support contained in C^{int}. We have [twice using the fact that $\mathrm{supp}((-\Delta + m^2)v) \subset C^{\mathrm{int}}$]

$$\begin{aligned} [P_C P_{\overline{C'}} f](v) &= (P_C P_{\overline{C'}} f, (-\Delta + m^2) v)_E = (P_{\overline{C'}} f, (-\Delta + m^2) v)_E \\ &= (P_{\overline{C'}} f, (-\Delta + m^2) v)_E = [P_{\overline{C'}} f](v) = 0 \,. \end{aligned}$$

The proof of (1.18 ii) is similar.

We refer to (18.i) (which follows from (18.ii)) as the pre-Markov property.

Remark. More generally, if the inner product $(\ , \ )_E$ is changed to the inner product $(\ , \ )_{E_Q}$ obtained by replacing $(k_0^2 + \boldsymbol{k}^2 + m^2)^{-1}$ with $Q(k)(k_0^2 + \boldsymbol{k}^2 + m^2)^{-n}$ in (1.7), $Q(k)$ a polynomial, for all $h \in N$ one can define $[h]_Q(u) = (h, (-\Delta + m^2)^n u)_{E_Q}$, $u \in (R^{d+1})$ and set $\mathrm{supp}_Q(h) = \mathrm{supp}[h]_Q$. Projections P_C^Q can be introduced as before – they satisfy Theorem 1, since for $h \in \mathscr{S}$, $\mathrm{supp}_Q h \subset \mathrm{supp} h$, and for $h \in N$, $\mathrm{supp} h \subset C$ one has that $P_C h = h$. In this case P_C^Q is a subspace of the space generated by applying polynomials in $\phi(h)$, $\mathrm{supp} h \subset C$, to the vacuum.

The semigroup property of $Q_t = E_0 U_t E_0$ follows from Theorem 1; one-has

$$Q_{t'} Q_t = Q_{t+t'} \,. \tag{1.19}$$

Proof. From the definitions of the translations U_t and projections E_t we have $U_{t'} E_t = E_{t+t'} U_{t'}$. Thus defining $C_t^{\pm} = \{x \in R^{d+1} \mid \pm(s-t) \geqq 0\}$ one finds (using $E_0 E_{C_t^-} = E_0$, $P_{C_t^+} E_{t+t'} = E_{(t+t')}$, $0 \leqq t' \leqq t' + t$)

$$\begin{aligned} Q_{t'} Q_t &= E_0 U_{t'} E_0 U_t E_0 = E_0 E_{t'} U_{t'} E_t U_t E_0 \\ &= E_0 E_{t'} E_{t+t'} U_{t'} U_t E_0 = E_0 P_{C_{t'}^-} P_{C_{t'}^+} E_{t+t'} U_{t+t'} E_0 = E_0 E_{t+t'} U_{t+t'} E_0 \\ &= Q_{t'+t} \,. \end{aligned} \tag{1.20}$$

Nelson [2] uses the semigroup property of Q_t along with a reflection principle to construct a positive self-adjoint Hamiltonian. In the next section we examine briefly the question of generalizing the analysis of this section to an arbitrary quantum field theory.

II. Euclidean Field Theories via Analytic Continuation and the Markov Property

In this section we mention several difficulties which occur in the construction of a Euclidean field theory beginning with the Schwinger functions of a relativistic theory and mimicing the Wightman reconstruction theorem [11]. We inquire whether a satisfactory resolution of these difficulties is sufficient to guarantee the Pre-Markov property, and find a negative answer.

We begin with a theorem of Ruelle [14, 12] (applied in the present context by Symanzik [6]), which states that the Wightman functions $W(z_1, \ldots, z_n)$ corresponding to vacuum expectation values of n local fields have extensions $\overline{W}(z_1, \ldots, z_n)$ which are holomorphic in regions containing all points $(z_1, \ldots, z_n) \in C^{n(d+1)}$ such that $(\mathrm{Im}\, z_i) = (s_i, 0)$ and $(\mathrm{Re}(z_i - z_j))^2 < 0$ whenever $s_i = s_j, i \neq j$. Specialising to the case $\mathrm{Re}(z_i) = (0, \boldsymbol{x}_i)$ one defines Schwinger functions

$$S(x_1, \ldots, x_n) = \overline{W}(\tilde{x}_1, \ldots, \tilde{x}_n)$$
$$x_i = (s_i, \boldsymbol{x}_i), \qquad \tilde{x}_i = (is_i, \boldsymbol{x}_i), \tag{2.1}$$

in the subset of $R^{n(d+1)}$ in which no two arguments coincide.

Since the Wightman functions are translation invariant hence functions only of the difference variables $(z_i - z_j)$, and finitely covariant under Lorentz transformations, and since an Euclidean rotation of the x_i is equivalent to a (real or complex) Lorentz transformation of the $\tilde{x}_i$'s, the Schwinger functions are invariant under Euclidean translations and finitely covariant under Euclidean rotations.

In order to use the Wightman reconstruction theorem it is necessary first to find tempered distributions corresponding to the Schwinger functions. For free scalar fields the extension can be made uniquely if one demands that the Schwinger distribution is not more singular than the Schwinger function. For generalised free fields with spectral measures $d\varrho(m^2)$ with $\int_0^\infty (1+m^2)^{-1} d\varrho(m^2) = \infty$ and for Wick polynomials $:\varphi^n:(x)$ of free fields (with $d > 1$) the problem is more complicated, and is formally analogous to that of constructing Green's functions from Wightman functions.

For free and generalized free fields these problems can be solved by introducing higher (odd-) dimensional imaginary times, using

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dk_0^{(1)} \ldots dk_0^{(1)} \left(\sum_{i=1}^{n} (k_0^{(2)})^2 + \boldsymbol{k}^2 + m^2 \right)^{-(n+1)/2} = C(n) \frac{1}{\sqrt{\boldsymbol{k}^2 + m^2}} .$$

Given a positive tempered measure $d\varrho(m^2)$ with support in $[\varepsilon, \infty)$, $\varepsilon > 0$ one can always pick $n = n(\varrho)$ so large that the integral

$$\int_0^\infty \left(\sum_{i=1}^n (k_0^{(i)})^2 + \boldsymbol{k}^2 + m^2 \right)^{-(n+1)/2} d\varrho(m^2)$$

converges. Similarly one can always pick $n(d, N)$ such that the Wick polynomial $:\phi^N:(s_1, \ldots, s_n, \boldsymbol{x})$ of a free Euclidean field, $m \neq 0$, (in n imaginary- time dimensions and d space dimensions, n odd) is an operator-valued tempered distribution. The embedding and projection techniques of Section I carry over to any finite (odd) number of imaginary time dimensions and formal perturbation-theoretic integrals converge in higher imaginary-time dimensions.

Beyond existence of the Schwinger distributions one needs the analogue of the Wightman positivity axiom. Symanzik has shown that this follows from plausible assumptions in his framework [6]. In any case, supposing that this program has been carried through and that a Euclidean field theory with fields $\phi_\alpha(x)$ and Hilbert space $\mathscr{N}_E$ constructed, the question arises of formulating, and analyzing the pre-Markov property. As in the free field case we let P_C denote the orthogonal projection on the subspace $\mathscr{N}_C \subset \mathscr{N}_E$ generated by polynomials in fields smeared with functions with support in C, noting that for this to be a valid definition the set of allowed smearing functions must be sufficiently large.

We turn now to a specific example in which these steps can be carried out explicitly.

Consider a scalar generalized free field with Wightman two-point function given by

$$(\Omega, \psi(x)\, \psi(y)\, \Omega) = \int \Delta_+(x - y; m^2)\, d\varrho(m^2)\,. \tag{2.2}$$

The corresponding Euclidean two point function is

$$\int \Delta_E(x - y; m^2)\, d\varrho(m^2) \tag{2.3}$$

and the Hilbert space and Euclidean field can be constructed as in Section I, making everywhere the substitution

$$(k_0^2 + \boldsymbol{k}^2 + m^2)^{-1} \to \int (k_0^2 + \boldsymbol{k}^2 + m^2)^{-1}\, d\varrho(m^2) \tag{2.4}$$

Nelson's semi-group property (1.20) can be checked directly, since the projection $E_0 = P_{(s=0)}$ can be calculated explicitly (e.g. using the natural embedding of the subspace of the relativistic Hilbert space which is generated by time zero fields). One finds that with this construction a necessary condition for the Markov property to hold is that $d\varrho(m^2) \propto \delta(m^2 - m_0^2)$. (This is of course to be expected, since for more general $d\varrho$ the time zero fields do not generate the Hilbert space.)

At first glance it might appear that the same restrictive analysis should apply to the two-point function of an arbitrary Euclidean theory with the Markov property. However we have used a property of free and generalized free fields not to be expected in general: if $\mathcal{N}_E$ is decomposed as $\sum_{j=1}^{i-1} \oplus N_E^{(j)}$, where $N_E^{(i)}$ is the orthogonal complement of $N_E^{(i-1)}$ in the subspace generated by elements of the form $\prod_{j=1}^{\infty} \phi(f_j)^{n_j}\Omega$, $\sum_{j=1}^{\infty} n_j \leqq i$, then in the preceding example $P_C N_E^{(i)} \subset N_E^{(i)}$.

One expects that a theory as simple as the generalized free field should have an associated Euclidean theory whith the Markov property; such an interpretation can be given at least in the case

$$d\varrho(m^2) = \sum_{i=1}^{N} |a_i|^2 \, \delta(m^2 - m_i^2) \, .$$

To give it we introduce additional independent free fields $\psi_i(x)$, $(\Box = m_i^2)\, \psi_i(x) = 0$, and express

$$\psi(x) = \sum_{i=1}^{N} |a_i| \, \psi_i(x) \, . \tag{2.5}$$

Now the Euclidean field theory corresponding to the family $\{\psi_i\}$ of independent free fields can be constructed as in Section 1, and if we let P_C denote the projection onto the subspace generated by applying polynomials in the $\phi_i(f_i)$, $\operatorname{supp} f_i \subset C$, to the vacuum, then the Markov property holds, and Nelson's construction of the relativistic Hamiltonian [2] goes through.

The procedure for a general spectral measure $d\varrho(m^2)$ is clear. One introduces a family of local (and relatively local) fields

$$\psi_\chi(h) = 2^{-1/2}(a_\chi(h)^+ + a_\chi(h))$$

indexed by characteristic functions χ of open intervals (including $(-\varepsilon, \infty)$) in R, with

$$\begin{aligned}
&[a_\chi(f), a_{\chi'}(g)] = [a_\chi(f)^+, a_{\chi'}(g)^+] = 0 \\
&[a_\chi(f), a_{\chi'}(g)^+] = 2^{-1} \int d\varrho(m^2)\, \chi(m^2)\, \chi'(m^2) \int dx\, dy\, f(x)\, g(y)\, \Delta_+(x-y; m^2) \\
&\qquad a_\chi(f)\Omega = 0
\end{aligned}$$

and constructs the corresponding Euclidean theory with fields $\phi_\chi(h_\chi)$ and Hilbert space. P_C is then the projection on the subspace generated by applying polynomials in $\phi_\chi(h_\chi)$, $\operatorname{supp} h_\chi \subset C$, to the vacuum. Here the appropriate space of smearing functions h_χ for ϕ_χ is identifiable with the completion of the space generated by $\{a_\chi(u)^+\Omega\,|\,u\in\mathcal{S}(R^{n+d})\}$, n being an imaginary-time dimension appropriate to $d\varrho(m^2)$.

From this example we conclude that in order to obtain the Markov property it is important to pick a sufficiently large set of local relativistic quantum fields with which to construct the Euclidean theory. In the next section this last construction will be related to a standard construction in probability theory and then carried out in more detail.

III. The Generalized Free Field as a Markov Euclidean Field

Before considering the generalized free field we first consider two Brownian motion diffusion processes with different diffusion constants which take place in $\Omega_1=R^m$ and $\Omega_2=R^n$ respectively. Let $p_1(x_1,y_1,t)$ $(x_1,y_1\in R^m)$ and $p_2(x_2,y_2,t)$ $(x_2,y_2\in R^n)$ be the transition density functions for the processes. The processes give rise to the semi-groups $T_1(t)$ and $T_2(t)$ in $L^2(R^m)$ and $L^2(R^n)$ defined by

$$(T_i(t)\,f_i)\,(x_i)=\int p_i(x_i,y_i,t)\,f_i(y_i)\,du_i(y_i)\,.$$

One way to form a new process is to consider the product process defined on $\Omega_1 x\Omega_2$ where the transition density function for the product process is taken to be the product $p_1(x_1,y_1,t)\;p_2(x_2,y_2,t)$. The semigroup associated with the product process is the direct product of the semigroups $T_1(t)$ and $T_2(t)$ on the Hilbert space $L^2(R^m)\otimes L^2(R^n)$.

If we consider the direct sum Hilbert space $L^2=L^2(R^m)\oplus L^2(R^n)$ then we can define a new semigroup $T_1(t)\oplus T_2(t)=T(t)$ on L^2. The semigroup $T(t)$ is the semigroup associated with what we will call the disjoint union of the processes. Let $\Omega=\Omega_1\vee\Omega_2$ be the disjoint union of Ω_1 and Ω_2. We define the transition density function $P(x,y,t)$ where $x,y\in\Omega$ to be

$$P(x,y,t)=\begin{cases}0 \quad \text{if} \quad x \text{ and } y \text{ are in different components}\\ p_1(x,y,t) \quad \text{if} \quad x,y\in\Omega_1\\ p_2(x,y,t) \quad \text{if} \quad x,y\in\Omega_2\,.\end{cases}$$

Then $T(t)$ is given by

$$(T(t)\,f)\,(x)=\int P(x,y,t)\,f(y)\,du(y)$$

where $u(\Gamma)=u_1(\Gamma\cap\Omega_1)+u_2(\Gamma\cap\Omega_2)$ for $\Gamma\subset\Omega$.

In Minkowski space the two point function of the generalized free field is given by

$$W^2_{MG}(x,y)=(2\pi)^{-d}\int\theta(p^0)\,\delta(p^2-m^2)\,e^{-ip\cdot(x-y)}\,d^dp\,d\varrho(m^2) \qquad (3.1)$$

which determines the scalar product in F_G, the direct integral of single particle spaces $F_\lambda(\lambda=m^2)$ and the Fockification $\mathcal{F}_G=\mathcal{F}(F_G)$ is the

Hilbert space associated with the generalized free field. In (3.1) $d\varrho(m^2)$ is a positive measure polynomially-bounded at infinity with support in $m^2 \in [0, \infty)$. Before discussing the case of an arbitrary generalized free field consider the special case where the measure has the form $d\varrho(m^2) = \delta(m^2 - m_1^2) + \delta(m^2 - m_2^2)$. Here we are in analogy with the direct sum of Markov processes. Then $F_G = F_1 + F_2$ and in Euclidean space $N_G = N_1 + N_2$. We can embed F_G in N_G by embedding F_i in N_i i.e. define $j'_a : F_G \to N_G$ by $j'_a : (f_1, f_2) \to (f_1(\mathbf{x})\,\delta(s-a), f_2(\mathbf{x})\,\delta(s-a))$. The norms in F_G and N_G are given by $|f|^2_{F_G} = |f_1|^2_{F_1} + |f_2|^2_{F_2}$ where $|f_i|^2_{F_i} = (f_i, f_i)$ and $|n|^2_{N_G} = |n_1|^2_{N_1} + |n_2|^2_{N_2}$ where $|n_1|^2_{N_i} = (n_i, n_i)$. We now introduce field operators in the Fockifications $\mathscr{F}(F_G)$ and $\mathscr{F}(N_G)$. In $\mathscr{F}(F_G)$ we define $a_i(f), a_j(f)^+$ and $\psi_i(f) = (a_i(f) + a_i(f)^+)/(2)^{1/2}$ $i = 1, 2$ where operators with different indices commute. In $\mathscr{F}(N_G)$ we define the operators $b_i(f), b_i(f)^+$ and $\phi_i(f) = (b_i(f) + b_i(f)^+)/(2)^{1/2}$ $i = 1, 2$ where operators with different indices commute. Then we have

$$W_{GM}(x, y) \equiv (\Omega, \psi_1(x)\,\psi_1(y)\,\Omega) + (\Omega, \psi_2(x)\,\psi_2(y)\,\Omega)$$

and

$$W_{GE}(x, y) = (\omega, \phi_1(x)\,\phi_1(y)\,\omega) + (\omega, \phi_2(x)\,\phi_2(y)\,\omega)\,.$$

As in Section I we define for any closed set S the orthogonal projection $p_S = p_{1S} \oplus p_{2S} : N_G \to N_G$ where p_{iS} are the projections associated with the mass m_i. Note that $p_{iS} N_i$ is the closed subspace of N_i generated by elements of N_i with support in S.

Then Theorem 1.1 is valid for p_S and its Fockification P_S. Now considering the case of arbitrary $d\varrho(m^2)$ we define $N_G = \int N_\lambda\, d\varrho(\lambda)$, $p_G = \int p_\lambda\, d\varrho(\lambda)$ with $\mathscr{N}_G$ and P_G their Fockifications, respectively. We have

Theorem 2.1. *Eqs.* (1.1) *and* (1.2) *are valid with p and P replaced by* p_G *and* P_G.

IV. Linear Perturbation Model and the Infinite Volume Limit

We consider the spatially cutoff Hamiltonian $H_g = H_0 + \lambda \int g'(x)\psi(x)d^d x$ where $g(x)/\lambda = g'(x) = \chi_l(x)$ the characteristic function of $\prod_{i=1}^{d} [-l_i, l_i]$. Let $V = \psi(g)$. We are interested in the limit $l \to \infty$. Even though the interaction V is not a lower-bounded polynomial it still satisfies the hypothesis for hypercontractive semigroups, such as $V \in L^p$ for some $p > 2$ and $e^{-tV} \in L^1$ for all $t > 0$ (see [7, 8]). This is easily seen by the explicit computation in Q space or Segal space associated with Fock space. We have

$$\begin{aligned} |V|^{2n}_{2n} &= (\Omega, V^{2n}\Omega) = (\Omega, \psi(g)^{2n}\Omega) = (2n)!\,|g|^{2n}_M / 2^n n! \\ |e^{-V}|_1 &= (\Omega, e^{-V}\Omega) = e^{|g|^2_M/2}\,. \end{aligned} \tag{4.1}$$

Similarly using the Segal isomorphism for the Euclidean Fock space we find

$$|\phi(f)|_{E\,2n}^{2n} = (\omega, \phi(f)^{2n}\omega) = (2n)!\,|f|_E^2/2^n n!$$
$$|e^{-\phi(f)}|_{E\,1} = (\omega, e^{-\phi(f)}\omega) = e^{|f|_E^2/2}. \tag{4.2}$$

Thus $H_g = H_0 + \lambda V$ is essentially self-adjoint on Fock space and is lower-bounded. In fact using the field-theoretic Feynman-Kac formula (see Appendix C) and letting $E_l = \inf sp(H_l)$ we have

$$\begin{aligned} -E_l &= \lim_{T\to\infty} T^{-1}\log(\Omega, e^{-TH_g}\Omega) \\ &= \lim_{T\to\infty} T^{-1}\log(\omega, e^{-\phi(u)}\omega) \\ &= \lim_{T\to\infty} \frac{\lambda^2\pi^2}{T}\int \chi_h(x,s)\,[(-\Delta+m^2)^{-1}\chi_h]\,(x,s)\,d^dx\,ds \end{aligned}$$

where $v(x,s) = \lambda u(s)\,\chi_l(x)$ and u is the characteristic function of $[O, T]$. Thus we find

$$-E_l = \lambda^2\pi^2/(2\pi)^{d/2}\int_0^\infty \frac{e^{-m^2t}dt}{(2t)^{d/2}}\,[\int \chi_l(x)\,e^{-|x-y|^2/4t}\,\chi_l(y)\,d^dx\,d^dy]$$

and

$$|E_l| \leqq \frac{\lambda^2\pi^2 l^d}{2^d m^2}.$$

Now we consider the infinite volume limit. We take the limit in the Schwinger distributions in Nelson space to obtain Euclidean invariant distributions. From Appendix C we consider the l, $T\to\infty$ limit of $S_h^n(x_1, s_1, \ldots, x_n, s_n)$ where $h = \chi_T(s)\,g(x)$ and χ_T is the characteristic function of $[-T, T]$. Thus

$$S_h^n(f_1, s_1, \ldots, f_n, s_n) = \frac{(\omega, \phi(f_1, s_1)\ldots\phi(f_n, s_n)\,e^{-\lambda S_h\phi(x,s)d^dx\,ds}\omega)_E}{(\omega, e^{-\lambda S_h\phi(x,s)d^dx\,ds}\omega)_E} \tag{4.3}$$

and by (4.2) we see that (4.3) is well-defined since the exponential is in all $L^p(Q_E)$, $(1 \leqq p < \infty)$ and so is $\phi(f_1, s_1)\ldots\phi(f_n, s_n)\,\omega$.

We smear (4.2) in the s variables and write with $F_i \in C_0^\infty(R^{d+1})$

$$S_h^n(F_1, \ldots, F_n) = (\omega, \phi(F_1)\ldots\phi(F_n)\,e^{-S_h}\omega)_E/(\omega, e^{-S_h}\omega)_E. \tag{4.4}$$

Eq. (4.4) can be evaluated explicitly to give

$$\begin{aligned} S_h^n(F_1, \ldots, F_n) &= \left[\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n}(\omega, e^{\Sigma_i\lambda_i\phi(F_i)}e^{-\lambda S_h}\omega)_E/(\omega, e^{-\lambda S_h}\omega)_E\right]_{\lambda_i=0} \\ &= \left[\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n}e^{1/2\,|\Sigma_i\lambda_iF_i-\lambda h|_E^2}/e^{1/2\,|h|_E^2}\right]_{\lambda_i=0}. \end{aligned} \tag{4.5}$$

Thus $S_h^n(F_1, \ldots, F_n)$ is a polynomial in λ with coefficients given by finite products of $(F_i, F_j)_E$ and $(F_i, h)_E$. Taking the $h\to 1$ limit and using the fact that

$$(F,h)_E = \int \frac{\bar{\tilde{F}}(k)\,\tilde{h}(k)}{k\cdot k + m^2}\, d^{d+1}k = \int \bar{F}(x,s)\,[(-\Delta + m^2)^{-1} h]\,(x,s)\, d^d x\, ds$$

and

$$[(-\Delta+m^2)^{-1}h]\,(s,x) = (2\pi)^{-(d+1)/2} \int_0^\infty e^{-m^2 t}(2t)^{-(d+1)/2}$$
$$\cdot\left[\int e^{-|x-y|^2/4t} e^{-|s-s'|^2/4t} h(y,s')\, d^d y\, ds'\right]$$

and that the above when $h \to 1$ becomes $[(-\Delta+m^2)^{-1}1]\,(s,x) = 1/m^2$ we find

$$\lim_{h\to 1}(h,F)_E = \left(\frac{1}{m}\right)^2 \int F(x,s)\, d^d x\, ds \equiv \left[\left(\frac{1}{m}\right)^2\right](F)$$

and the limit Schwinger distributions are

$$S^n(F_1,\ldots,F_n) = \left[\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n} e^{\frac{1}{2}\{\Sigma_{i,j}\lambda_i\lambda_j(F_i,F_j)_E - \lambda\Sigma_i\lambda_i\left[2\left(\frac{1}{m}\right)^2\right](F_i)\}}\right]_{\lambda_i=0}.$$

Thus

$$S^1(x_1,s_1) = \left(\frac{1}{m}\right)^2 \lambda$$

$$S^2(x_1,s_1,x_2,s_2) = W_E^2 + \left(\frac{1}{m}\right)^4 \lambda^2$$

and for a general n we have

$$S^n(x_1,s_1\ldots x_n s_n) = \left(\omega, \left[\phi(x_1,s_1) + \lambda\left(\frac{1}{m}\right)^2\right]\ldots\left[\phi(x_n,s_n)+\lambda\left(\frac{1}{m}\right)^2\right]\omega\right).$$

By an analytic continuation we obtain the Poincare invariant Wightman distributions

$$W^n(x_1,t_1\ldots x_n,t_n) = \left(\Omega, \left[\psi(x_1,t_1) + \lambda\left(\frac{1}{m}\right)^2\right]\ldots\left[\psi(x_n,t_n)+\lambda\left(\frac{1}{m}\right)^2\right]\Omega\right).$$

Appendix A

The Segal Isomorphism

We give a short derivation and discussion of the Segal isomorphism. Given a countable set of elements f_i, $i=1,2,\ldots$ we consider the real inner product space S with basis elements

$$a(f_{i_1})^+ a(f_{i_2})^+ \ldots a(f_{i_n})^+ \Omega$$
$$i = 1,2\ldots; \quad n = 0,1,2,\ldots \tag{A.1}$$

and inner product (,) defined on basis elements by

$$(a(f_{i_1})^+ \dots a(f_{i_n})^+ \Omega,\ a(f_{j_1})^+ \dots a(f_{j_n})^+ \Omega) \\ = \delta_{nm} \sum_{\text{perm}\,\pi} \delta_{i_1 j_{\pi(1)}} \dots \delta_{i_m j_{\pi(m)}} \tag{A.2}$$

and extended to S by linearity. In (A.1) elements in which $(a(f_i)^+)$ occur in different order are not distinguished. We also use an equivalent labelling of basis elements, setting

$$a(f_{i_1})^+ \dots a(f_{i_n})^+ \Omega = \prod_{i=1}^{\infty} (a(f_i)^+)^{n_i} \Omega \tag{A.3}$$

where each n_i is the number of times $(a(f_i)^+)$ occurs on the left side of (A.3); with this labelling (A.2) becomes

$$\left(\prod_{i=1}^{\infty} (a(f_i)^+)^{n_i} \Omega, \prod_{j=1}^{\infty} (a(f_j)^+)^{n_j} \Omega \right) = \prod_{i=1}^{\infty} (n_i)!\, \delta_{n_i, n_j}. \tag{A.4}$$

With the norm and topology given by the inner product, the completion $\bar{S}$ of S is a separable real Hilbert space.

Next we consider three specific realisations of this scheme.

(i) *Fock Space Realisation.* Take the $\{f_i\}$ to be an orthonormal basis for a real separable Hilbert space K, and consider S as embedded in the Fock space $\mathscr{F}(K)$, identifying each basis element $(a(f_{i_1})^+) \dots (a(f_{i_n})^+)\Omega$ with the corresponding vector

$$\left(\prod_{i=1}^{\infty} (n_i!) \right)^{1/2} (O \oplus O \oplus \dots \oplus \text{Sym}(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_n}) \oplus O \oplus \dots), \tag{A.5}$$

where

$$\text{Sym}(f_{i_1} \otimes \dots \otimes f_{i_n}) = (n!)^{-1} \sum_{\text{perm}\,\pi} (f_{i_{\pi(1)}} \otimes \dots \otimes f_{i_{\pi(n)}}). \tag{A.6}$$

In this realisation the closure and complexification of S is $\mathscr{F}(K)$.

(ii) *Infinite Tensor Product Realisation.* In this realisation (introduced primarily for motivation), we consider S as contained in $\prod_{i=1,2} \otimes h_i$, the von Neuman (complete) tensor product of Hilbert spaces [15], where for each factor h_i we take $L^2(R, dx)$. Here and below we abuse notation slightly, letting (R, dx) denote the real line with Lebesgue measure, and letting $(R, \pi^{-1/2} e^{-x^2} dx)$ denote the measure space (R, u) where the u-measurable sets are R itself and the Lebesgue measurable sets, and where $u(A)$ is the Lebesgue integral $\pi^{-1/2} \int_A dx\, e^{-xg}$. The basis element $\prod_{i=1}^{\infty} ((a(f_i)^+)^{n_i} \Omega)$ is identified with the vector $\prod_{i=1,2\dots} \otimes ((n_i!)^{1/2} \Psi_{n_i})$, where

Ψ_{n_i} is the n_ith harmonic oscillator eigenfunction,

$$\Psi_{n_i}(x) = \pi^{-1/4} 2^{-n_i/2} (n_i!)^{-1/2} H_{n_i}(x) e^{-x^2/2} ; \tag{A.7}$$

here $H_n(x)$ denotes the nth Hermite polynomial. The closure and complexification of S in this realization is the $((\Psi_0, \Psi_0, \ldots)$-adic) incomplete tensor product $\Pi_{\otimes}^{(\Psi_0, \Psi_0, \ldots)} h_i$. In this realisation the subspace S_N generated by basis elements of the form $\prod_{i=1}^{N} ((a(f_i))^+)^{n_i} \Omega, n_i = 1, 2, \ldots$, has a natural identification with the space of functions $L^2(R^N, d^N x)$. Thus one may identify $\prod_{i=1}^{\infty} ((a(f_i))^+)^{n_i} \Omega, n_i > N = 0$, with $\Psi_{(n_1, \ldots, n_N)} \in L^2(R^N, d^N x)$ where

$$\Psi_{(n_1, n_2, \ldots, n_N)}(x_1, \ldots, x_N) = \prod_{i=1}^{N} \Psi_{n_i}(x_i) . \tag{A.8}$$

It is not possible to extend this identification to all of S – as a function of countably many real variables $\prod_{i=1}^{\infty} (\pi^{-1/4} e^{-x_i^2/2})$ vanishes identically. In our third realisation this difficulty will be overcome and elements of S will be (equivalence classes of) functions of countably many real variables. In order to explain this we first recall several properties of the one-dimensional harmonic oscillator.

On the dense domain $D \subset L^2(R, dx)$ consisting of finite linear combinations of Hermite functions

$$\Psi_n = (n!\, 2)^{-1/2} (x - d/dx) \Psi_0 , \qquad \Psi_0 = (\pi)^{-1/4} e^{-x^2/2} , \qquad |\Psi_n| = 1$$

we define the lowering operator A acting on Ψ_m by $A\Psi_m = (m)^{1/2} \Psi_{m-1}$ and A is extended to D by linearity. The closure of $A \upharpoonright D$ is an operator which we also denote by A. The adjoint A^+ (the raising operator) of A also has the name domain as the closure of A and on D is given by linearity from $A^+ \Psi_m = (m+1)^{1/2} \Psi_{m+1}$. Furthermore

$$\begin{aligned} A \upharpoonright D &= (x + d/dx)/(2)^{1/2} , \\ A^+ \upharpoonright D &= (x - d/dx)/(2)^{1/2} , \\ \Psi_n &= (n!)^{-1/2} A^+ \Psi_0 . \end{aligned} \tag{A.9}$$

As the Hamiltonian H of the one-dimensional harmonic oscillator (with the ground state energy subtracted) we take the closure of $H \upharpoonright D$ and on D, H is defined by linearity from $H\Psi_n = n\Psi_n$, $n = 0, 1, \ldots$ which can also be written as $H\Psi_n = -2^{-1}(d^2/dx^2 - x^2 + 1) \Psi_n$. We note also that $\underset{\sim}{x} = (A + A^+)/(2)^{1/2}$.

Now consider the unitary mapping $U: L^2(R,dx) \to L^2(R, \pi^{-1/2} e^{-x^2} dx)$ defined by $(Uf)(x) = \pi^{1/4} e^{x^2/2} f(x)$. Defining $H' = UHU^{-1}$, $a = UAU^{-1}$, $a^+ = UA^+U^{-1}$, $\Psi'_n = U\Psi_n$, one obtains

$$\begin{aligned} \Psi'_n(x) &= 2^{1/2}(n!)^{1/2} H_n(x) \\ H' \restriction D &= -(1/2)(d_x^2 - 2x\, d_x) \\ a \restriction D &= 2^{-1/2}(d_x), \qquad a^+ \restriction D = 2^{-1/2} \quad (2x - d_x)\,. \end{aligned} \tag{A.10}$$

We observe that $x = 2^{-1/2}(a + a^+)$ and defining $:x^n: = 2^{-n/2} \sum_{m=0}^{n} (a^+)^m a^{n-m}$ find (e.g. noting $\Psi'_n = (n!)^{-1/2} 2^{-n/2} :x^n: \Psi'_0$, and arguing inductively that $:x^n:$ is a polynomial) that

$$:x^n: = 2^{-n} H_n(x)\,. \tag{A.11}$$

Since $\Psi'_0(x) = 1$ (U is introduced solely to achieve this) the product $\prod_{i=1}^{\infty} \Psi'_0(x)$ is a non-zero function of countably many real variables for each sequence $(n_1, n_2, \ldots)$ of integers with finite sum; if $L^2(R,dx)$ is replaced by $L^2(R, \pi^{-1/2} e^{-x^2} dx)$ and Ψ_n by Ψ'_n in the second realisation, the difficulty encountered under (A.8) disappears. This leads to our third realisation.

(iii) L^2 *(Gaussian Measure Space)*. In this realisation we identify the basis element $\prod_{i=1}^{\infty} ((a(f_i)^{n_i})\Omega$ with the function $\prod_{i=1}^{\infty} ((n_i)!)^{1/2} \Psi'_{n_i}((x_i)) \in L^2(M,u)$ where the probability measure space (M,u) is the product of probability measure spaces $\prod_{i=1}^{\infty} (X_i, u_i)$ taking $(R, \pi^{-1/2} e^{-x^2} dx)$ for each factor (X_i, u_i). The collection of u-measurable sets of M is the σ-algebra generated by elementary sets, i.e. sets of the form $B = B_1 x B_2 x \ldots B_n x R x R x \ldots$ with each B_i u_i-measurable, and for such a set B, $u(B)$ is $\prod_{i=1}^{n} u_i(B_i)$. These requirements guarantee existence and uniqueness of the countably additive measure u [16].

In this realisation the closure of S is clearly a closed subspace of $L^2(M,u)$; we now show that it coincides with $L^2(M,u)$. Since linear combinations of the Ψ'_n are dense in $L^2(R, \pi^{-1/2} e^{-x^2} dx)$, it suffices to show that every characteristic function in $L^2(M,u)$ can be approximated by finite linear combinations of products of characteristic functions of elementary sets. Thus it is enough to show that every u-measurable set A is elementary approximatable, i.e., that for any $\varepsilon > 0$ there exists corresponding finite union T of elementary sets with $u(A \Delta T) < \varepsilon$ where

$A \Delta T$ is the symmetric difference between A and T. This can be shown by applying a corollary of the monotone class theorem [17] to the ring generated by elementary sets. The corollary states that in a finite measure space (M, u) any u-measurable set A can be approximated arbitrarily closely in measure by elements of any ring which generates the algebra of u-measurable sets. Thus in this realisation we can identify $\bar{S}$ with $L^2(M, u)$.

Anticipated by the notation, we introduce linear operators $a(f_i)^+$, $a(f_i)$, $:\prod_{i=1}^{N} (\phi(f_i))^{n_i}:$, acting from S to S, and defined by linearity from

$$
\begin{aligned}
a(f_k)^+\left(\prod_{i=1}^{\infty} a(f_i)^{+n_i}\Omega\right) &= \prod_{i=1}^{\infty} a(f_i)^{+(n_i+\delta_{ik})}\Omega \\
a(f_k)\left(\prod_{i=1}^{\infty} (a(f_i))^{+n_i}\Omega\right) &= n_k \prod_{i=1}^{\infty} a(f_i)^{+(n_i-\delta_{ik})}\Omega \qquad \text{(A.12)} \\
:\prod_{i=1}^{N} (\phi(f_i)^{n_i}): &= \prod_{i=1}^{N}\left(\sum_{m_i=0}^{n_i}\binom{n_i}{m_i}(a(f_i))^{+(n_i-m_i)}(a(f_i))^{m_i}\right).
\end{aligned}
$$

The $a(f_i)$ and $a(f_i)^+$ satisfy the commutation relations (on S)

$$
\begin{aligned}
&[a(f_i), a(f_j)] = [a(f_i)^+, a(f_j)^+] = 0\,. \\
&[a(f_i), a(f_j)^+] = \delta_{ij}\,.
\end{aligned} \qquad \text{(A.13)}
$$

In the first realisation the $a(f_i)$, $a(f_i)^+$ give the Fock representation of the canonical commutation relations (see e.g. [18]); in the third realisation thay are the lowering and raising operators a_i, a_i^+. The Wick polynomials $:\prod_{i=1}^{N} \phi(f_i)^{n_i}:$ are the usual Wick polynomials in the first realisation, and in the third realisation they are (by (A.11)) simply Hermite polynomials.

The natural correspondances between these realisations can clearly be implemented by unitary operators- the correspondence between case (i) and (iii) is the Segal isomorphism.

Nelson's interpretation of the free scalar Euclidean field is obtained by using the creation-annihilation operator expression (1.6) to identify $\mathcal{N}$ with L^2 (Gaussian Measure Space), defined above.

Appendix B

Generalizations of the Feynman-Kac formula

In this appendix we derive some formula for transformed Markov processes. We will consider the processes associated with the transition density functions, measures and path space measures given by

1) $p_1(x,y,t)=(4\pi Dt)^{-n/2}e^{-|x-y|^2/4Dt}$ $\quad x,y\in R^n \quad dv_1=\prod_{j=1}^{n} dx_j$
$d\mu_{1x}=$ path space measure

2) $p_2(x,y,t)=\prod_{j=1}^{n}(1-e^{-2\omega t})^{-1/2}\exp\left[\frac{-\omega(y_j-e^{-\omega t}x_j)^2}{1-e^{-2\omega t}}+\omega y_j^2\right]$

$dv_2=\prod\left(\frac{\omega}{\pi}\right)^{1/2}e^{-\omega x_j^2}dv_1, \quad d\mu_{2x}=$ path space measure.

The semigroups associated with processes 1 add 2 will be denoted by e^{-H_1t} and e^{-H_2t} where the Fourier transform of e^{-H_1t} is multiplication by e^{-Dp^2t} on $L^2(R^n,dv_1)$ and e^{-H_2t} is multiplication by $e^{-(n_1+\dots+n_n)t}$ on the Hermite polynomial $H_{n_1}(x_1)H_{n_2}(x_2)\dots H_{n_n}(x_n)$. For both processes 1 and 2 the non-continuous functions have measure zero. We have

Theorem A.1. *Let* $V:R^n\to R$ *be continuous,* $V\geqq C>-\infty$. $\overline{H_i+V}$ *is self-adjoint. Then for* $f,g\in L^1(R^n,dv_i)\cap L^2(R^n,dv_i)$, $t>0$

$$\int f(x_0)\,I_{ix_0}\,dv_i(x_0)=(f,e^{-t(\overline{H_i+V})}g)_i \tag{B.1}$$

where

$$I_{ix_0}=\int e^{-\int_0^t V(x(\tau))d\tau}g(x(t))\,d\mu_{ix_0}. \tag{B.2}$$

Proof. We have $I_{ix_0}=\lim_{n\to\infty}I_{ix_0}=\lim_{n\to\infty}\int e^{-t/n\sum_{m=1}^{n}V(x((mt)/n))}g(x(t))\,d\mu_{ix_0}$. This follows using the dominated convergence theorem since V is continuous and thus $e^{-\int_0^t V(x(\tau))d\tau}=\lim_{n\to\infty}e^{-t/n\sum_{m=1}^{n}V(x((mt)/n))}$ μ_i a.e. and $\left|e^{-\int_0^t V(x(\tau))d\tau}\right|_{\mu_i,\infty}\leqq e^{|C|t}$ and $|g(x(t))|_{1\mu_i}<\infty$. However I_{ix_0} is a cylinder function so that

$$\begin{aligned}I_{ix_0}^n&=\int p_i(x_0,x_1,t/n)\,e^{-t/nV(x_1)}dv_i(x_1)\dots\int p_i(x_{n-1},x_n,t/n)\\&\quad e^{-t/nV(x_n)}g(x_n)\,dv_i(x_n)\\&=[(e^{-t/nH_i}e^{-t/nV})^n g](x_0).\end{aligned}$$

By the Trotter product formula

$$\lim_{n\to\infty}|(e^{-t(\overline{H_i+V})}-[e^{-t/nH_i}e^{-t/nV}]^n)g|_{L^2(dv_i)}=0$$

so there exists a subsequence n_k such that

$$\lim_{k\to\infty}[e^{-t/n_kH_i}e^{-t/n_kV}]^{n_k}g(x_0)=(e^{-t(\overline{H_i+V})}g)(x_0)\quad \text{a.e.}\quad v_i.$$

However since $\lim_{n\to\infty} I^n_{ix_0} = I_{ix_0}$ for each x_0 we have that the sequence itself converges a.e. $\nu_{i'}$ such as

$$\lim_{n\to\infty} [e^{-t/n H_i} e^{-t/n V}]^n g(x_0) = e^{-t(\overline{H_i+V})} g(x) \quad \text{a.e.} \quad \nu_i$$

so that $\int f(x_0) I_{ix_0} d\nu_i(x_0) = (f, e^{-t(H_i+V)} g)_i$.

Our next theorem gives a formula for the mean value of certain products. We have

Theorem B.2. a) *With* $V, f, g,$ *as in Theorem* B.1 *and* $F_j \in L^\infty(R^n)$ $j=1, \ldots, n$ $t_0>0$, $t_1>0, \ldots, t_n>0$ *then*

$$\int f(u) I_{iu} du(u) \tag{B.3}$$

$$= (f, e^{-t_0(\overline{H_i+V})} F_1 e^{-t_1(\overline{H_i+V})} F_2 \ldots F_n e^{-t_n(\overline{H_i+V})} g)_i \tag{B.4}$$

where

$$I_{iu} = \int F_1(x(t_0)) F_2(x(t_0+t_1)) \ldots F_n(x(t_0+t_1 \ldots + t_{n-1})) \cdot e^{-\int_0^{t_0+\ldots+t_n} V(x(\tau))d\tau} g(x(t_0+\cdots+t_n)) d\mu_{iu}\,. \tag{B.5}$$

b) *For* $i=2, \Omega \mp 1$ *let* $E_2 = \inf sp(\overline{H_2+V})$. *Let* $H_{2R} = \overline{H_2+V} - E_2$ *and assume* Ω_2 *is an eigenvector of* $\overline{H_2+V}$ *with* $\Omega_2>0$ *a.e. and* E_2 *has multiplicity* 1. *Then for* $0<t_1<t_2<\cdots<t_n<T$ *we have*

$$(\Omega_{2'} e^{-t_1 H_{2R}} F_1 e^{-(t_2-t_1)H_{2R}} F_2 \ldots F_n e^{t_n H_{2R}} \Omega_2)_2 \tag{B.6}$$

$$= \lim_{T\to\infty} \left[\frac{(\Omega, e^{-(t_1+T)(\overline{H_2+V})} F_1 e^{-(t_2-t_1)(\overline{H_2+V})} F_2 \ldots F_n e^{-(T-t_n)(\overline{H_2+V})} \Omega)_2}{(\Omega, e^{-2TH_2}\Omega)_2}\right] \tag{B.7}$$

$$= \lim_{T\to\infty} \left[\frac{\int d\nu_2(u) F_1(x(t_1+T)) F_2(x(T+t_2)) \ldots e^{-\int_0^{2T} V(x(\tau))d\tau} d\mu_{2u}}{\int d\nu_2(u) \int e^{-\int_0^{2T} V(x(\tau))d\tau} d\mu_{2u}}\right]. \tag{B.8}$$

Proof of a). Write

$$I_{iu} = \int c^{-\int_0^{t_0} V(x(\tau))d\tau} F_1(x(t_0)) e^{-\int_{t_0}^{t_0+t_1} V(x(\tau))d\tau} F_2(x(t_0+t_1)) \ldots e^{-\int_{t_0+\ldots+t_{n-1}}^{t_0+\ldots+t_n} V(x(\tau))d\tau} g(x(t_0+\cdots+t_n)) d\mu_{iu}$$

and write each integral as a limit of Riemann sums to give a cylinder function which when evaluated yields

$$I^{n_0 \ldots n_n}_{iu} = [(e^{-t_0/n_0 H_i} e^{-t_0/n_0 V})^{n_0} F_1 \ldots (e^{-t_n/n_n H_i} e^{-t_n/n_n V})^{n_n} g](u)\,.$$

Taking the $n_0 \ldots n_n \to \infty$ limit using the Trotter product formula gives

$$I_{iu} = [e^{-t_0(\overline{H_i+V})} F_1 \ldots e^{-t_n(\overline{H_i+V})} g](u) \quad \text{a.e.} \quad \nu_i .$$

Finally taking the inner product with f given us the result.

Proof of b). Eq. (B.7) follows from (B.8) since $\lim_{u\to\infty} e^{-uH_{2R}}\Omega = \Omega_2(\Omega_2, \Omega)_2$ $\Omega_2(\Omega_2, \Omega)_2$ as verified from the spectral theorem and $\Omega_2 > 0$ a.e. implies $(\Omega, \Omega_2) > 0$ so that $(\Omega, e^{-2TH_2}\Omega) \neq 0$ for large T. In (B.3) and (B.4) change to the primed variables defined by

$$t_0 = t_1^1 + T, \quad t_1 = t_2^1 - t_1^1, \ldots, t_n = T - t_n^1$$

so that

$$t_0 = t_1 = T + t_2^1, \ldots, t_0 + t_1 + \cdots + t_n = 2T$$

then drop the primes to get

$$\begin{aligned}(\Omega, e^{-(t_1+T)H_2} F_1 e^{-(t_2-t_1)H_2} \ldots F_n e^{-(T-t_n)H_2}\Omega)_2 \\ = \int d\nu_2(u) \int F_1(t_1+T) F_2(t_2+T) \ldots F_n(T+t_n) e^{-\int_0^{2T} V(x(\tau))\,d\tau} d\mu_{2u} .\end{aligned} \tag{B.9}$$

Now divide (B.9) by $(\Omega, e^{-2TH_2}\Omega)_2$ and multiply by $1 = e^{2TE_2}/e^{2TE_2}$ to get (recalling $H_{2R} = H_2 - E_2$) the equality of (B.7) and (B.8).

Appendix C

Generalizations of Feynman-Kac Formulas in Relativistic Quantum Field Theory

In this appendix we give two formulas relating Fock space inner products and Nelson space inner products (see [4, 5]). These relations are analogous to those of appendix B. One has (see [4]).

Generalized Nelson symmetry: For $a_0 < a_1 < \cdots < a_n$, $t > 0$

$$(\Omega, e^{-t(H_0+Y)}\Omega) = (\Omega, e^{-(a_1-a_0)(H_0+V_t^{(1)}})\ldots e^{-(a_n-a_{n=1})(H_0+V_t^{(n)}}\Omega) \tag{C.1}$$

where

$$Y = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} :P_i(\psi(x)): dx \quad \text{and} \quad V_t^{(i)} = \int_{-t/2}^{t/2} :P_i(\psi(x)): dx$$

and P_i is linear or a lower-bounded polynomial in two dimensional space-time. As a special case of (C.1) for $a_1 = l/2$, $a = -l/2$ and $a_2 = \cdots = a_n = 0$ we obtain the Nelson symmetry formula

$$\begin{aligned}(\Omega, e^{-t(H_0+V_l)}\Omega) &= (\Omega, e^{-l(H_0+V_t)}\Omega) \\ &= (\omega, e^{-\int_0^t \int_{l/2}^{l/2} :P(\phi(x,s))\cdot dx\,ds}\omega)\end{aligned} \tag{C.2}$$

which is analogous to (B.1). Also one has analogous to Theorem B.2 the following:

a) For $0 < t_0 < t_1 \ldots < t_n$ we have

$$\begin{aligned}&\left(\Omega, e^{-t_0 H_l} F_1(\psi(f_1))\, e^{-t_1 H_l} F_2(\psi(f_2)) \ldots F_n(\psi(f_n))\, e^{-t_n H_l} \Omega\right)\\ &\quad = \left(\omega, F_1(\phi(f_1, t_0))\, F_2(\phi(f_2, t_0 + t_1)) \ldots F_n(\phi(f_n, t_0 + \cdots + t_{n-1}))\right.\\ &\qquad \left. \cdot\, e^{-\int_0^{t_0 + \ldots + t_n} V_s\, ds} \omega\right)\end{aligned}$$

where

$$H_l = H_0 + \int_{-l/2}^{l/2} :P(\psi(x)): dx \quad \text{and} \quad V_s = \int_{-l/2}^{l/2} :P(\phi(x, s)): dx\,.$$

b) For $0 < t_1 < t_2 < \cdots t_n < T$

$$\begin{aligned}&\left(\Omega_l, e^{-t_1 H_R} F_1(\psi(f_1))\, e^{-(t_2 - t_1) H_R} F_2(\psi(f_2)) \ldots F_n(\psi(f_n))\, e^{t_n H_R} \Omega_l\right)\\ &= \lim_{T \to \infty} \left[\frac{\left(\Omega, e^{-(t_1 + T) H_l} F_1(\psi(f_1))\, e^{-(t_2 - t_1) H_l} F_2(\psi(f_2)) \ldots e^{-(T - t_n) H_l} \Omega\right)}{(\Omega, e^{-2 T H_l} \Omega)}\right]\\ &= \lim_{T \to \infty} \left[\frac{\left(\omega, F_1(\phi(f_1, t_1)) \ldots F_n(\phi(f_n, t_n))\, e^{-\int_{-l/2}^{l/2} \int_{-T}^{T} P\,.(\phi(x,s)): dx\, ds} \omega\right)}{\left(\omega, e^{-\int_{-l/2}^{l/2} \int_{-T}^{T} :P(\phi(x,s)): dx\, ds} \omega\right)}\right]\end{aligned} \tag{C.3}$$

where $H_l = H_0 + \int_{-l/2}^{l/2} :P(\psi(x)): dx$, $E_l = \inf sp(H_l)$, $H_R = H_l - E_l$ and Ω_l is the eigenvector for H_l with eigenvalue E_l. Also

$$\lim_{u \to \infty} e^{-u(H_l - E_l)} \Omega = (\Omega_l, \Omega)\, \Omega_l\,. \tag{C.4}$$

We remark that the second line of (C.3) follows from the first line using (C.4). Eq. (C.4) follows if E_l has multiplicity one and it is not necessary that E_l be an isolated eigenvalue.

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