Commun. math. Phys. 35, 151—154 (1974)

The Convergence of BPH Renormalization

Eugene R. Speer
Department of Mathematics, Rutgers University, New Brunswick, USA

Received July 30, 1973

Abstract. The convergence of the integrals defining BPH renormalized Feynman amplitudes is derived from the known additive structure of analytic renormalization.

In this paper we derive the convergence of BPH renormalization [1–3] from the known additive structure of analytic renormalization [4], providing an alternate and perhaps simpler route to this important result. We adopt without further remark the notation of [1, 4].

Suppose that $f(\boldsymbol{\lambda})$ is meromorphic in $\mathbb{C}^L$, with at most simple poles on varieties $\Lambda(\chi)=0, \pm 1, \pm 2, \ldots$, where for $\chi \subset \{1, \ldots L\}$, $\Lambda(\chi) = \sum_{l \in \chi} (\lambda_l - 1)$. For $\boldsymbol{\kappa} \in \mathbb{C}^L$, let $\mathscr{V}^\kappa$ be the analytic evaluator of [4; 3.4 (b)], but defined with center $\boldsymbol{\kappa}$: choosing $0 < R_1 < \cdots < R_L \ll 1$ to satisfy $\sum_{i<j} R_i < R_j$, and defining C_i^j as the contour $|\mu_j - \kappa_j| = R_i$,

$$\mathscr{V}^\kappa f(\boldsymbol{\lambda}) = \frac{(2\pi i)^{-L}}{L!} \sum_{s \in S_L} \int_{C^1_{s(1)}} d\mu_1 \ldots \int_{C^L_{s(L)}} d\mu_L \frac{f(\boldsymbol{\mu})}{(\mu_1 - \lambda_1) \ldots (\mu_L - \lambda_L)}$$

whenever $|\lambda_l - \kappa_l| < R_1$. $\mathscr{V}^\kappa f$ is analytic at $\boldsymbol{\kappa}$.

Now let G be a Feynman graph with vertices $V_1, \ldots V_m$ and lines $\{1, \ldots L\}$. If $\hat{\mathscr{X}}$ is a set of vertex parts for G, $U = \{V_1' \ldots V_r'\}$ a generalized vertex, and $Q = \{U_1, \ldots U_s\}$ a partition of U, $\mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1' \ldots V_r')$ is the amplitude defined for $\operatorname{Re}\lambda_l \gg 0$ by $\mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1' \ldots V_r') = \prod_1^s \hat{\mathscr{X}}(U_i) \prod_{\text{conn}} \Delta_l$.

Theorem 1. *If $\boldsymbol{\kappa} \in C^L$ satisfies*

$$\operatorname{Re}\kappa_l \geqq 1, \quad l = 1, \ldots L, \tag{1}$$

then

$$\mathscr{V}^\kappa \mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1', \ldots V_r') = \sum_R \mathscr{T}_{R,\tilde{\mathscr{X}}(Q,\hat{\mathscr{X}})}(V_1', \ldots V_r'), \tag{2}$$

where the $\tilde{\mathscr{X}}$'s are new vertex parts, and the sum is over partitions R of $\{V_1' \ldots V_r'\}$ at least as coarse as Q. Note in particular that if $Q = \{U\}$, $\tilde{\mathscr{X}}(Q, \hat{\mathscr{X}})(V_1' \ldots V_r') = \mathscr{V}^\kappa \hat{\mathscr{X}}(V_1' \ldots V_r')$.

Proof. As in [4, § 4]. The change of center to $\boldsymbol{\kappa}$ and the extension to a generalized graph introduce only a notational difference in the proof.

Condition (1) guarantees that the vertex parts $\tilde{\mathscr{X}}(W)$ have degree less than or equal to the superficial divergence of W.

Theorem 2. *Let $\mathscr{R}$ be the standard BPH renormalization operator* [1]. *Then there are vertex parts $\hat{\mathscr{X}}$ such that, for any $\{V_1', \dots V_r'\}$,*

$$\mathscr{R}\mathscr{T}(V_1', \dots V_r') = \sum_Q \mathscr{V}^\kappa \mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1', \dots V_r'), \tag{3}$$

the sum taken over partitions Q of $\{V_1' \dots V_r'\}$.

Proof. This is the standard equivalence of two additive renormalizations; we adapt the proof of [5]. Define $\hat{\mathscr{X}}(V_1'', \dots V_s'')$ inductively by

$$\mathscr{X}(V_1'', \dots V_s'') = \sum_Q{}' \tilde{\mathscr{X}}(Q, \hat{\mathscr{X}})(V_1'', \dots V_s'') + \hat{\mathscr{X}}(V_1'', \dots V_s''), \tag{4}$$

where $\mathscr{X}$ is the vertex part for $\mathscr{R}$ and Σ' is over partitions Q of $\{V_1'', \dots V_s''\}$ into at least two sets. Assume inductively that for $s<r$, $\hat{\mathscr{X}}(V_1'', \dots V_s'') = \mathscr{V}^\kappa \hat{\mathscr{X}}(V_1'', \dots V_s'')$, so that (4) is

$$\mathscr{X}(V_1'', \dots V_s'') = \sum_Q \tilde{\mathscr{X}}(Q, \hat{\mathscr{X}})(V_1'', \dots V_s''). \tag{5}$$

Inserting (5) [and (4) if $s=r$] into $\mathscr{R}\mathscr{T} = \Sigma\, \mathscr{T}_{Q,\mathscr{X}}$, rearranging, and using (2), we have

$$\mathscr{R}\mathscr{T}(V_1', \dots V_r') = \sum_Q{}' \mathscr{V}^\kappa \mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1', \dots V_r') + \hat{\mathscr{X}}(V_1', \dots V_r'). \tag{6}$$

Now apply the BPH M-operator to (6), using $M\mathscr{R}=0$, $M\mathscr{V}^\kappa = \mathscr{V}^\kappa M$, and $M\hat{\mathscr{X}} = \hat{\mathscr{X}}$, to find

$$\hat{\mathscr{X}}(V_1', \dots V_r') = -\mathscr{V}^\kappa M \sum_Q{}' \mathscr{T}_{Q,\hat{\mathscr{X}}}(V_1', \dots V_r').$$

Since $(\mathscr{V}^\kappa)^2 = \mathscr{V}^\kappa$, $\mathscr{V}^\kappa \hat{\mathscr{X}} = \hat{\mathscr{X}}$; this verifies the induction assumption and, when inserted into (6), yields (3).

Corollary 1. *$\mathscr{R}\mathscr{T}(V_1, \dots V_m)$ is holomorphic in*

$$\Omega = \{\lambda \,|\, \mathrm{Re}\,\lambda_l > 1 - 1/L, \quad \textit{for all } l\}.$$

Proof. Any possible pole of $\mathscr{R}\mathscr{T}$ in Ω has the form $\Lambda(\chi)=k$, with $k \geqq 0$, and hence contains a point $\boldsymbol{\kappa}$ satisfying (1). But from (3), $\mathscr{R}\mathscr{T}$ cannot be singular at $\boldsymbol{\kappa}$; this completes the proof.

There remains only to show that this analyticity comes from the convergence of the corresponding integral. The model for the following proof is this: if $f(t)$ is C^∞ on $[0, 1]$, and $\int t^{z-1} f(t)\, dt$ is analytic at $z=0$, then necessarily $f(0)=0$ [7], $f(t)=tg(t)$ with $g(t)$ C^∞ on $[0,1]$, and $\int t^{z-1} f(t)\, dt$ converges for $z \geqq -1$.

Theorem 3. *Let*

$$\mathcal{R}\mathcal{T}(V_1, \dots V_m) = \lim_{\varepsilon \to 0^+} \int_0^\infty \dots \int_0^\infty \left(\prod_1^L \alpha_l^{\lambda_l - 1} e^{-\varepsilon \alpha_l} d\alpha_l \right) f(\boldsymbol{\alpha}, \boldsymbol{p}) \tag{7}$$

be the usual Feynman-parametric representation, known to exist and converge for $\mathrm{Re}\,\lambda_l$ *sufficiently large. Then this integral converges absolutely for* $\boldsymbol{\lambda} \in \Omega$.

Proof. $f(\boldsymbol{\alpha}, \boldsymbol{p})$ in (7) is an entire function of $\boldsymbol{\alpha}$ divided by a product of Symanzik d-functions for various sub and quotient graphs. For any ordering $l_1 < \dots < l_L$ of $\{1, \dots L\}$, let $\chi_i = \{l_1, \dots l_i\}$, and introduce in the region $\alpha_{l_1} \leqq \dots \leqq \alpha_{l_L}$ scaling variables $\{t_{\chi_i}\}$, defined by $\alpha_{l_i} = \prod_{j \geqq i} t_{\chi_j}$. Under this scaling, each d function factors as a product of t_χ's times a function non-zero in $t_\chi \geqq 0$, so that $\mathcal{R}\mathcal{T}$ is the $\varepsilon \to 0^+$ limit of a sum of terms

$$\int_0^\infty dt_{\chi_L} \int_0^1 \dots \int_0^1 \prod_{i<L} dt_{\chi_i} \prod_1^L t_{\chi_i}^{\Lambda(\chi_i) - j(\chi_i)} g_\varepsilon(\boldsymbol{t}, \boldsymbol{p}), \tag{8}$$

with g analytic in the integration region. We choose each $j(\chi_i)$ as small as possible, and will show that then $j(\chi_i) \leqq 0$; from (8), this will complete the proof.

[The scaling transformation is the local form of a global desingularization of the integration space (see e.g. [6]) and $j(\chi_i)$ is related to the degree of the pole of f on a certain analytic variety. From this it follows that $j(\chi_i)$ actually depends only on χ_i, not on the original ordering.]

Suppose that $j(\chi) > 0$ for some χ, and choose χ_0 to be a *minimal* subset for which $j(\chi_0) > 0$. Changing variables to $\alpha_l = u\beta_l$, $l \in \chi_0$, with $\sum_{\chi_0} \beta_l = 1$, (7) becomes the $\varepsilon \to 0$ limit of

$$\int_{\substack{0 \\ \sum_{\chi_0} \beta_l = 1}}^\infty \dots \int_0^\infty u^{\Lambda(\chi_0) - j(\chi_0)} du \prod_{l \in \chi_0} \beta_l^{\lambda_l - 1} d\beta_l \prod_{l \notin \chi_0} \alpha_l^{\lambda_l - 1} d\alpha_l h_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}, u, \boldsymbol{p}). \tag{9}$$

The residue of (9) on the pole $\Lambda(\chi_0) = j(\chi_0) - 1$, which vanishes by Corollary 1, is ([7])

$$0 = \int_{\substack{0 \\ \sum_{\chi_0} \beta_l = 1}}^\infty \dots \int_0^\infty \left(\prod_{l \notin \chi_0} \alpha_l^{\lambda_l - 1} d\alpha_l \right) \left(\prod_{l \in \chi_0} \beta_l^{\lambda_l - 1} d\beta_l \right) \Bigg|_{\Lambda(\chi_0) = j(\chi_0) - 1} h_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0, \boldsymbol{p}). \tag{10}$$

By choice of χ_0, (10) converges absolutely if $\mathrm{Re}\,\lambda_l > 1 - 1/L$, $l \in \chi_0$, and $\mathrm{Re}\,\lambda_l > k_l$ for some k_l, $l \notin \chi_0$ (change back to t variables). We now claim that

$$h_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0, \boldsymbol{p}) = 0\,; \tag{11}$$

this establishes the theorem by contradiction, since then $h_\varepsilon = u\boldsymbol{h}_\varepsilon$, with $\boldsymbol{h}_\varepsilon$ analytic, so that $j(\chi_0)$ was not as small as possible.

To prove (11), choose $l_0 \in \chi_0$, and change variables in (10) to $y_l = \ln \alpha_l$, $l \notin \chi_0$, $y_l = \ln(\beta_l/\beta_{l_0})$, $l \in \chi_0 - \{l_0\}$. Then (10) becomes

$$0 = \int_{\mathbb{R}^{L-1}} \cdots \int \prod_{l \neq l_0} e^{(\lambda_l - 1) y_l} \, dy_l [\beta_{l_0}^{j(\chi) + |\chi| - 1} h_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0, \boldsymbol{p})] \, \varrho(y) \qquad (12)$$

with $\varrho(y)$ the Jacobean of the variable change. Taking $\lambda_l = 1 + i\omega_l$, $l \in \chi_0 - \{l_0\}$, and $\lambda_l = 1 + k_l + i\omega_l$, $l \notin \chi_0$, (12) states that the Fourier transform of the continuous L_1 function

$$\left(\prod_{l \notin \chi_0} \alpha_l^{k_l} \right) \beta_{l_0}^{j(\chi) + |\chi| - 1} \, h_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0, \boldsymbol{p}) \varrho$$

vanishes. Since ϱ is strictly positive, $h(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0, \boldsymbol{p}) = 0$, q.e.d.

References

1. Hepp, K.: Commun. math. Phys. **2**, 301 (1966)
2. Bogoliubov, N. N., Parsiuk, O. S.: Acta. Math. **97**, 227 (1957)
3. Parasiuk, O. S.: Ukr. math. J. **12**, 287 (1960)
4. Speer, E. R.: Commun. math. Phys. **23**, 23—36 (1971); **25**, 336 (1972)
5. Hepp, K.: Commun. math. Phys. **14**, 67 (1969)
6. Boyling, J. B.: Nuovo Cimento **53**, 351 (1968)
7. Gelfand, I. M., Shilov, G. E.: Generalized functions, Vol. I. New York: Academic Press 1964

Eugene R. Speer
Department of Mathematics
Rutgers University
New Brunswick, N. J. 08903, USA