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A Laplace Transform on the Lorentz Groups

II. The General Case

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Abstract. We extend the results of a previous paper to arbitrary non-integrable but polynomially bounded functions defined over any connected semi-simple Lie group of real-rank one. Our approach is based on the method of bilateral horospheres and is a direct generalisation of that used earlier. All the features of the more restricted transform are retained in this more general formalism.

Introduction

In a recent paper [1] we introduced a "Laplace transform" on the Lorentz groups $SO(n, 1)$, based on Gel'fand's method of horospheres. That paper was restricted to (nonintegrable) quasiregular or Class I representations, which can be regarded as defined by right translations of functions over the two-sheeeted hyperboloids $SO(n, 1)/SO(n)$; however, a theory of horospheres for the regular representation of an arbitrary connected semi-simple Lie group is now available [2], and in this paper we show how the results of [1] can be extended to arbitrary functions defined over the group itself. Although it is not difficult to treat the quite general case, we shall for simplicity restrict our considerations to the groups of real-rank one, which we shall call the Lorentz groups. These comprise the real Lorentz groups, $SO(n, 1)$; the hermitian, $SU(n, 1)$; the symplectic, $\mathrm{Sp}(n, 1)$; and a real form of F_4 which we shall call the octavian Lorentz group [1].

Our approach is that of [1]: we divide the Fourier transformation into two steps, the integral transform $f(g) \to \hat{f}(h)$, which maps a smooth function over the group into one over the manifold of horospheres, and the Fourier transform over the horospheres themselves. The former we regularize by a process similar to the "analytic continuation in the co-ordinates" used before; the latter we replace by a pair of classical Laplace transforms. We show that the combination of the two, which we call the Laplace transform on G, converges for all polynomially

[1] This remark will not be explained.

bounded functions except those with certain discrete asymptotic behaviours, and the kernel of the mapping is the set of all functions which transform under finite-dimensional representations of G.

The inversion formula proceeds via the Laplace transforms, since unlike [1] we are unable to invert the horospheric transform directly. We derive it by considering the inversion formula for the Fourier transform of an integrable $f(g)$; there are then three possibilities if we apply this to our (regularised) Laplace transforms:

(a) the expression converges to $f(g)$,
(b) it converges to something else,
(c) it diverges.

We show that it can diverge if and only if $f \to \hat{f}$ diverges too; and a simple analysis shows that we are faced with case (b) above, in that it converges to $f(g)$ redefined modulo its finite-dimensional components. All these features were first met in Ref. [1].

We refer constantly throughout this paper to the results of [1], but very little acquaintance with that paper is actually necessary to understand this, although some is obviously desirable. The relevant results of Ref. [2] are summarised in Section I, which is devoted to familiarising the reader with the theory of horospheres for the regular representation of a group of real-rank unity.

I. Preliminaries

Let G be a connected semi-simple Lie group of real-rank one (that is, a generalised Lorentz group) and $G = NAK$ an Iwasawa decomposition. The centraliser of A in K we denote M, and the normaliser M'; then M'/M is finite and is called the Weyl group W. We can write almost any element $g \in G$ in the form $g = namv$ where $n \in N$, $a \in A$, $m \in M$, and v belongs to the subgroup V contragredient to N. Let r denote half the sum (with multiplicities) of the positive restricted roots, and if σ is an arbitrary complex number set

$$A^{\sigma} = e^{\sigma r \ln A}$$

as a character on A. Then the irreducible representations of G are given by

$$\begin{aligned} T_g^{(\sigma,j)} : f(v) &= A^{1+\sigma} \mathscr{D}^j(m) : f(v'), \\ vg &= namv' \end{aligned} \tag{1}$$

where $\mathscr{D}^j(m)$ is a unitary irreducible representation of M and $f \in L^2(v, D_j)$ is a smooth function on V with values in the finite-dimensional space D_j in which act the $\mathscr{D}^j$.

The manifold of horospheres (in Gel'fand's sense) of the space G/K is a fibre bundle over V with fibre A; and the set of *bilateral* horospheres H of G introduced in [2] is a bundle over $V \times N$ with fibre $A \times M$. The Fourier transform of $f \in C^{\infty}(G) \cap L^2(G)$ consists of an integral transform $f(g) \to \hat{f}(h)$ $(h \in H)$, followed by a Fourier transform on the fibres of H into unitary irreducible representations of the direct product group $A \times M$; and we showed in [2] that this can be written as

$$\tilde{f}^{(\sigma,j)}(n,v) = \int_G f(g)\, A^{1+\sigma}(vg^{-1}n)\, \mathscr{D}^j[M(vg^{-1}n)]\, dg \tag{2}$$

where we have set $vg^{-1}n = n_1 A M v_1$. Therefore the Fourier transform is a continuous mapping of $f(g)$ into a function over the bundle whose base is the product of two elementary representation spaces of G and whose fibres are the product of the real line (the dual space of A) and the Kreĭn algebra of M.

There is however no particular need to use this parameterisation: given the fibre bundle H, it is often convenient to choose a different cross-section and so obtain a pair of elementary representation spaces of G different from N and V. This is made extensive use of in [3] for the one-sided case of the quasiregular representations of $SL(2, C)$, and its application to the regular representation of that group was indicated in [2] Section V.1. For the general case it is slightly less simple because of the difficulty of ensuring that there is no ambiguity in the parameterisation (e.g. $g = kak'$ is not suitable), but the principle is the same. In this paper we shall make considerable use of that cross-section of the bundle determined in the obvious way by the parameterisation $g = \theta' m a \theta$, where $\theta, \theta' \in K/M$; we shall refer to this as the spherical basis for the space of horospheres. Its advantage is quite simply that it is compact; since we are dealing with nonintegrable functions that means that by use of this cross-section we eliminate the need for a further regularisation process.

The inverse formula to (2) for real-rank unity was given in [2] as

$$\begin{aligned} c_G f(g) = {} & \sum_j \int_\Gamma d\sigma \frac{[p_j(\sigma)]^2}{(\dim D_j)} \int A^{1-\sigma}(vg^{-1}n) \\ & \cdot \operatorname{Tr}\{\tilde{f}^{(\sigma,j)}(n,v)\,[\mathscr{D}^j[M(vg^{-1}n)]]^{-1}\}\, dn\, dv \\ & - 2\pi i \sum_{j,s} \frac{\operatorname{Res}[p_j(s)]^2}{(\dim D_j)} \left(\frac{\partial}{\partial s}\right) \int A^{1-s}(vg^{-1}n) \\ & \cdot \operatorname{Tr}\{\tilde{f}^{(s,j)}(n,v)\,[\mathscr{D}^j[M(vg^{-1}n)]]^{-1}\}\, dn dv\,. \end{aligned} \tag{3}$$

Here Γ is a contour along $\operatorname{Re}\sigma = 0$, and the j-summation is over all equivalence classes of unitary irreducible representations of M. The trace is to be taken in the space D_j. The s-sum is over all the poles of $p_j(\sigma)$ in the right half-plane; indeed, this whole result can be written formally as a single contour integral by interpreting Γ as being "to the right of" all singularities of the Plancherel measure (but not of $\tilde{f}^{(\sigma,j)}$ itself). The constant c_G is given by

$$(c_G)^{-1} = \tfrac{1}{2}\pi V(K/M)$$

where V is the volume of K/M in the metric induced by the negative of the Killing form. We have assumed that the Haar measure dg in (2) is normalised according to

$$dg = e^{-2r \ln a}\, dn\, da\, dm\, dv$$

if $g = namv$, and that the total measures of K and M are unity.

We draw attention to the symmetry between (2) and (3). Notice, too, the resemblance to the quasiregular case: there we had an integral over a cross-section of the horospheres, and the measure in the σ-integral was the Plancherel measure; here we have an integral over a cross-section of the fibre bundle, and the measure now is the *square* of the Plancherel measure. The discrete series of representations if present enters as the residues of the integral at its poles; as shown in [2], these are actually simple.

In this paper we shall use (3) in a form adapted to the spherical basis of H. In [1] we labelled the points of the horospheres (the cones) by k, and the cross-sections by $\not{k}$; here we shall use h and $\hbar$ for that purpose.

II. The Horospheric Transform

2.1. Definition

The Fourier transformation of $f \in C^\infty(G) \cap L^1(G)$ has two parts: the mapping $f \to \hat{f}$ which takes f into a space of functions over the manifold H of horospheres $V \times N \times M \times A$, and the actual expansion of $\hat{f}(H)$ into homogeneous functions on A. In this section we shall define a regularisation procedure for the first step.

The mapping $f \to \hat{f}$ is defined for $f \in L^1(g)$ by

$$\hat{f}(n, v; m, a) = \int_G f(g)\, \delta[1 - aA(vg^{-1}n)]\, \delta[mM^{-1}(vg^{-1}n)]\, dg \qquad (4)$$

where $vg^{-1}n = n_1 AMv_1$; and if $f \in C^\infty(G) \cap L^1(G)$ this converges and defines a function $f \in C^\infty(H) \cap L^1(H)$. The regularisation of this transform we define as follows:

$$\hat{f}(n, v; m, a) = \operatorname*{Lim}_{\eta \to 0} \{\int \mathrm{f(g)\, J_\varrho(n, v; g; \eta)\, \delta[mM^{-1}(vg^{-1}n)]\, dg}|_{\varrho = -1}\} \qquad (5)$$

where

$$J_\varrho(n, v; g; \eta) = e^{-i\pi\varrho/2} J_\varrho^0(n, v; g; \eta) + e^{i\pi\varrho/2} J_\varrho^0(n, v; g; -\eta), \tag{6a}$$

$$J_\varrho^0(n, v; g; \eta) = [4\Gamma(\varrho+1)\cos\pi\varrho/2]^{-1}(1 - aA(vg^{-1}n) + i\eta|g|)^\varrho. \tag{6b}$$

Here $|g|$ is a "norm function" on G defined in the following manner. Set $g = ka^+k'$, with $k, k' \in K$ and $a^+ = \exp h$ with h in the positive Weyl chamber of $\mathfrak{a}$; we then take

$$|g|^2 = \cosh 2h. \tag{7}$$

For matrix groups this is the familiar definition $|g|^2 = N^{-1}\Sigma|g_{ij}|^2$ where N is the dimension of the adjoint representation of G.

Equation (5) is to be interpreted as follows. First we integrate over G for positive η and $\operatorname{Re}\varrho$ sufficiently negative to ensure convergence; then we analytically continue in ϱ to $\varrho = -1$; finally we let $\eta \to 0$. The integrals converge for all polynomially bounded f, and define functions which are analytic both in ϱ and either half-plane $\operatorname{Re}\eta \gtrless 0$: hence the continuation procedures are well-defined. If the final limit $\eta \to 0$ in (5) converges, we shall call $\hat{f}$ the horospheric transform of f.

We now prove these statements. Set $\varrho = x + iy$, with x, y real and $x < 0$; suppose $\operatorname{Re}\eta > 0$; let D be a precompact set in G containing the points $|g| = 1$, and D' its complement; and consider

$$\begin{aligned}
&\left|\int_G f(g)\,[1 - aA(vg^{-1}n) + i\eta|g|]^\varrho\, dg\right| \\
&\leqq \operatorname{const}|\operatorname{Re}\eta|^x e^{-\pi\min(0,y)} \int_D |f(g)|\, dg \\
&\quad + \operatorname{const}|\operatorname{Re}\eta|^x e^{-\pi\min(0,y)} \int_{D' \ni kak'} |f(ka^+k')|\, e^{x|h|} \prod_{\alpha>0} (\operatorname{sh}\alpha(h))\, dh\, dk\, dk' \\
&\leqq \operatorname{const}(x, y) + \operatorname{const}(x, y) \int f_s(e^h) \exp|h|\,(x + 2|r|)\, dh
\end{aligned}$$

where $f_s(a^+)$ is the maximum value of $|f(ka^+k')|$ as k, k' vary and we have written $|h|$ for the length of the vector $h \in \mathfrak{a}$. Hence by choosing x sufficiently negative the integral certainly converges for any polynomially bounded $f(g)$, and it is clear from the proof that the convergence is uniform in compact sets in the ϱ and η complex planes, so that it actually defines an analytic function of both these variables. Returning to (5), the extra factor there of $\delta(mM^{-1})$ is easily seen to make no difference to the result, and so the convergence of (5) to an analytic function of ϱ and η is proved.

Remark. The proof used the fact that $|g| > 0$ everywhere upon G. This is certainly a very useful property, because it enables us to perform a single integration, with $\operatorname{Re}\varrho < 0$, but it is not essential: the one crucial requirement is that $|g| \to \infty$ polynomially whenever $a^+ \in \mathfrak{a}$ becomes large. If $|g| = 0$ on some (necessarily compact) subset G_0 of G, then to define the integral in (5) we must integrate separately over a compact $G_1 \supset G_0$, with $\operatorname{Re}\varrho > 0$, and its complement, with $\operatorname{Re}\varrho < 0$. There is quite apart from this great latitude in the precise specification of the regularisation of (4), since what has significance is only the result and not the intermediate stages: thus we could have taken

$$\begin{aligned} \hat{f}_1(n, v; m, a) = \int f(g)\, J_\varrho(n, v; g; 0)\, |n_1(vg^{-1}n)|^\sigma\, |v_1\ (vg^{-1}n)|^\tau \\ \cdot\, \delta[mM^{-1}(vg^{-1}n)]\, dg|_{\varrho=-1,\sigma=0=\tau} \end{aligned} \tag{8}$$

and $\hat{f}_1$ is identical with $\hat{f}$ when they are defined at all. We have chosen (5) as our principal definition because of its symmetry and elegant form.

We now show that if $f \in L^1(G)$, then (5) reduces to (4). In this case the integrals are uniformly convergent for all η, and so we can take the limit $\eta \to 0$ before integrating. (6) becomes

$$[4\Gamma(\varrho+1)\cos\pi\varrho/2]^{-1}\,\{e^{-i\pi\varrho/2}(1-aA+i0)^\varrho + e^{i\pi\varrho/2}(1-aA-i0)^\varrho\}\,. \tag{9}$$

Consider the measure dg: since this is Haar measure, $dg = d(vg^{-1}n)$ for fixed v, n; but we have set $vg^{-1}n = n_1 AMv_1$, and so in this parameterisation we find $dg = dn_1 \cdot dv_1 \cdot dM \cdot \exp(-2r\ln A)\, dA$. Now return to (9). This is just $[2\Gamma(1+\varrho)]^{-1}\,|1-aA|^\varrho$, and at $\varrho = -1$ this generalised function is regular and takes the value $\delta(1-aA)$. Hence (5) reduces to (4) as we claimed.

2.2. Convergence and Zeros

We must now investigate the convergence of the limit $\eta \to 0$ in (5). From the foregoing argument it is clear that this exists for all integrable $f(g)$, and so we can confine our considerations to the behaviour of f at large $|g|$. The same is true if we wish to find the kernel of the mapping $f \to \hat{f}$, since the existence of an inversion formula for the Fourier transform implies that no $f \in L^1(G)$ is annihilated by (4). Let us choose a "spherical basis" for the spaces V, N; that is, let us now realise the space H of horospheres as the set of points $\phi \times \phi' \times M' \times A$, where ϕ, ϕ' belong to some fixed coset K/M (compare the Remark at the end of Section II of Ref. [2]), so that there is now only one improper integral to deal with. We consider the following mixed transform $f \to \check{f}$:

$$\check{f}^j(\phi, \phi'; a) = \operatorname{Reg} \int f(g)\, \delta[1 - aA(\phi g^{-1}\phi')]\, \mathscr{D}^j[M(\phi g^{-1}\phi')]\, dg\,. \tag{10}$$

Here $\phi g^{-1}\phi' = nAMv$ with $\phi, \phi' \in K/M$.

Set $g = \theta' m w^\varepsilon \exp h \cdot \theta$, where $\theta, \theta' \in K/M'$, w is a fixed representative in M' of the nontrivial element of W, $h \in \mathfrak{a}^+$, and $\varepsilon = 0, 1$; then the θ, θ' integrals converge, and the regularisation of the divergent integral over h is performed by the methods of the previous section. As $h \to \infty$ we are therefore faced with behaviour of the form

$$\int F(\exp h, m)\,(\operatorname{ch} 2h)^{\varrho/2}\, \mathscr{D}^j(m)\, \Pi \operatorname{sh}\alpha(h)\, dm\, dh\,.$$

where we have ignored all terms of non-leading asymptotic behaviour. The $(\operatorname{ch} 2h)^{\varrho/2}$ is of course the contribution of the regularisation prescription; the last term is the product of $\operatorname{sh}\alpha(h)$ for all the positive roots, with multiplicities.

Let us set $e^h = a^{-1}$. Then we are interested in the behaviour near $a=0$ of the expression

$$\int_0^\infty F(a^{-1}, m)\, a^{-\varrho-1}\, \mathscr{D}^j(m)\, |\Pi \operatorname{sh}\alpha(\ln a)|\, dm\, da\,. \tag{11}$$

The m-integral is of a C^∞ function over a compact set, and hence always converges, so we need consider only the behaviour of the a-integral near zero: which is not worse than

$$\int_0 F(a^{-1})\, a^{-\varrho-1-2r}\, da \tag{12}$$

where $F(a)$ is the transform over M of $F(a, m)$.

Now by hypothesis f is polynomially bounded; so we can write

$$F(a \to \infty) = \sum_i F_{\mu_i}(a) \sum_{j=0}^{\infty} c_j^i a^{-j} \tag{13}$$

where the c_j^i are constants and

$$\lim_{a\to\infty} a^{-\mu_i-\varepsilon} F_{\mu_i}(a) = 0$$

for all $\varepsilon > 0$. Hence (12) behaves as

$$\sum_i c_0^i \int_0 a^{-\mu_i-\varrho-2r-1} f_i(a^{-1})\, da$$

where the $f_i(a)$ can increase with a only slower than any positive power – that is, they must be essentially logarithmic in nature. Standard theory then tells us that the integral has singularities only when $\mu_i + \varrho + 2r$ is a non-negative integer; and because the regularisation (5) is actually equivalent to setting $\varrho \to 0$ above, this implies that the mapping $f \to \hat{f}$ can diverge only when $f(g)$ has components [in the sense of (13)] behaving asymptotically like $|g|^{n-2r}$ where n is a non-negative integer.

Now in general the transform $f \to \hat{f}$ will indeed diverge under these conditions; exceptionally, however, it may not, and we assert that the finite-dimensional representations of G are in fact annihilated. To see this, consider the transform $\hat{f}_1$ given by (8). We require

$$\frac{1}{\Gamma(\varrho)} \int f(g)\, |1 - aA|^\varrho A^{-2}\, |N|^\sigma\, |V|^\tau\, \delta(mM^{-1})\, dA\, dM\, dN\, dV \big|_{\substack{\varrho=-1\\ \sigma=\tau=0}} \tag{14}$$

where we have written A for $A(vg^{-1}n)$ (and similarly for N, V and M), and replaced dg by $d(vg^{-1}n)$ since this is Haar measure. Now $f(g)$ is by hypothesis polynomial in the elements of G, and hence in those of $vg^{-1}n$ also. In particular, it is polynomial in the elements of N and V; so we can perform both the N and V integrals above, and find that they vanish identically. Therefore the transform $\hat{f}_1$ vanishes as we asserted; and since this coincides with $\hat{f}$, it follows that $\hat{f}(n, v; m, a)$ itself is zero.

Therefore the finite-dimensional representations are annihilated by the transform $f \to \hat{f}$. To show that these functions fill up the kernel, suppose that some other, non-polynomial, $F(g)$ lies therein too. By consideration of the asymptotic behaviour on H (see the Proposition of the next section) it follows that functions with different asymptotic behaviour as $|g| \to \infty$ are annihilated separately, if at all, and we already know that no $f \in L^1(G)$ lies in the kernel; so that if we write

$$f(kak') = \sum_i F_{\mu_i}(|g|)\, \Theta_i(k, k') \tag{15}$$

where as $|g| \to \infty\, F_{\mu_i}(|g|)$ has the single asymptotic behaviour $|g|^{\mu_i} f_i(|g|)$, with the f_i satisfying the same conditions as earlier, then it follows that each term in the series satisfying $\mu_i \geqq -2r$ is annihilated, whereas the remainder are not, and hence cannot occur in the expansion (15) since by hypothesis $\hat{F}(h) = 0$. Under the conditions of the problem, there are therefore only a finite number of terms annihilated here; and hence because the kernel of the mapping $f \to \hat{f}$ is invariant under the action of the group (since the operators T_g^R, T_g^L commute with the horospheric transform), it follows that the $F_{\mu_i}(|g|)\, \Theta_i(k, k')$ for $\mu_i \geqq -2r$ lie in a finite-dimensional invariant subspace of $C^\infty(G)$ – that is, they must after all belong to the space of finite-dimensional representations that we considered earlier. Hence the kernel of the transform $f \to \hat{f}$ consists only of the finite-dimensional representations, as we asserted.

We summarize the results of this section in the form of a Lemma:

Lemma 1. *Let $f \to \hat{f}$ be the horospheric transform of a C^∞ polynomially bounded function $f(g)$. Let $|g|$ be a norm function on G, and suppose that when $g = k \cdot \exp h \cdot k'$ becomes large at fixed $k, k' \in K$, then $f(g)$ has the asymptotic behaviour*

$$f(g) \sim |g|^{-2r} \sum_i F_{\mu_i}(|g|) \sum_{j=0} c^i{}_j |g|^{-j}\, \Theta^i{}_j(k, k')$$

where the $c^i{}_j$ are constants and $\lim_{r \to \infty} r^{-\mu_i - \varepsilon} F_{\mu_i}(r) = 0$ for all positive ε. Then the transform converges for all $f(g)$ whose asymptotic behaviour contains no components $F_{\mu_i}(|g|)\, \Theta^i(k, k')$ with μ_i a non-negative integer. The kernel of the mapping is the space of all functions which transform under a finite-dimensional representation of the group G.

Remark. For Class I representations of the real Lorentz groups $SO(n, 1)$ this was proved earlier in Ref. [1], and that proof can readily be extended to all the matrix groups dealt with here. Moreover, if G is such that its Plancherel measure $p_j(\sigma)$ has poles at the integer points, then by considering the Fourier transform of $Q(g)$ regarded as a *generalised* function, and making use of the Parseval identity derivable from the inversion formula (3), it follows at once that the polynomials on G are annihilated; unfortunately however this simple proof does not apply if G has no discrete series, and then we must use the method above.

Finally, the argument following (14) apparently shows that $f(g)$ is annihilated if it has *any* power-law dependence upon the elements of N or V. But the only such dependence possible within $C^\infty(G)$ is polynomial, for which the argument is valid; other powers can enter only in the form of a Fourier integral expansion into the vectors of some pseudo-basis of $L^2(N)$ or $C^\infty(N)$, and then (14) must be understood as the transform of a *generalised* function, which is non-zero.

III. The Laplace Transform

Proposition. *Let $f(g) \in c^\infty(G)$ be a function with a convergent horospheric transform, and $|g|$ a norm function on G. Suppose that as $|g| \to \infty$ $f(g)$ has leading asymptotic behaviour*

$$f(g) = f(k \cdot \exp h \cdot k') \sim |g|^\lambda \,\Theta(k, k')\,;$$

then $\hat{f}(H)$ behaves as

$$\hat{f}(e^{-\beta} H) \sim e^{\lambda\beta} \hat{f}(H)\,,$$
$$\hat{f}(e^{+\beta} H) \sim e^{\beta(\lambda+2r)} \hat{f}(H)$$

where $H \in H$ and $\mathbb{R} \ni \beta \to \infty$. In addition, $\hat{f}(H)$ is infinitely differentiable at all points of H.

Remark. The proof of this is straightforward and we omit it. The asymptotic behaviour as $\beta \to \infty$ of $\hat{f}(e^\beta H)$ differs from that found in Lemma 2 of Ref. [1] because we now have two nilpotent manifolds V, N and not just one. Notice that the behaviour on H is in a sense the worst possible for a given behaviour on G; in particular, if $f \in L^1(G)\,[L^2(G)]$, then $\hat{f} \in L^1(H)\,[L^2(H)]$. Finally, notice that $\hat{f}$ can have no polynomial dependence upon H (because the corresponding $f(g)$ lie in the kernel of $f \to \hat{f}$), and no dependence which is polynomial-cum-logarithmic (because then the transform $f \to \hat{f}$ diverges and so $\hat{f}$ is undefined).

We have now defined a mapping $f \to \hat{f}$ which associates with $f(g)$ a polynomically bounded function on the horospheres. In direct analogy with Ref. [1] we define the Laplace transform on G as the *Laplace* transform of $\hat{f}(n, v; m, a)$ over a and the *Fourier* transform over m: this gives a pair of transforms

$$\begin{aligned} \tilde{f}_{+}^{(\sigma j)}(n, v) &= \int_0^\infty \hat{f}(e^{\beta}, m; n, v)\, \mathscr{D}^j(m)\, e^{-(\sigma+1)r\beta}\, dm\, d\beta \\ \tilde{f}_{-}^{(\sigma, j)}(n, v) &= \int_0^\infty \hat{f}(n, v; m; e^{-\beta})\, \mathscr{D}^j(m)\, e^{-(\sigma-1)r\beta}\, dm\, d\beta . \end{aligned} \tag{6}$$

Because of the Proposition above, both these transforms exist if

$$\mathrm{Re}(r\sigma) > \lambda - r$$

and, indeed, define analytic functions of σ in this domain: the asymmetry in the definition of the $\tilde{f}_{\pm}$ is chosen to ensure this.

We can now obtain an explicit expression for the map $f \to \tilde{f}$ by arguing along lines identical with those of Ref. [1], Section III. We find that

$$\tilde{f}_{+}^{(\sigma, j)}(n, v) = \lim_{\eta \to 0} \{ \int f(g)\, \Phi^{(+)}(g, \eta; \varrho, \sigma)\, \mathscr{D}^j[M(vg^{-1}n)]\, dg|_{\varrho=-1} \} \tag{17}$$

where

$$\begin{aligned} \Phi^{(+)}(g, \eta; \varrho, \sigma) = &-\frac{i}{2} \operatorname{cosec} \pi r(1+\sigma)\, [A(vg^{-1}n)]^{\sigma+1} \\ &\cdot \{ e^{i\pi r(\sigma+1)}(1 + i\eta|g|)^{\varrho - r(\sigma+1)} - e^{-i\pi r(\sigma+1)}(1 - i\eta|g|)^{\varrho - r(\sigma+1)} \} \\ &\cdot \theta(1 - A(vg^{-1}n)) \end{aligned}$$

and we have used the symbol r interchangeably either for half the sum of the positive roots or for the length of this vector, as the context requires; $\tilde{f}_{-}^{(\sigma, j)}$ is defined similarly. These expressions are precisely equivalent to those given by the alternative (and much simpler) kernels

$$\Phi_0^{(\pm)}(g, \eta; \varrho, \sigma) = (1 + i\eta|g|)^{\varrho} [A(vg^{-1}n)]^{1 \pm \sigma}\, \theta[1 - (A(vg^{-1}n))^{\pm 1}] . \tag{18}$$

We shall call (17) the Laplace transform on G. Notice that the integral in (17) extends only over part of the group, by virtue of the θ-functions in the definitions of the kernels Φ_0. It is of course this which provides the analyticity in σ; but although it helps the convergence, we cannot dispense with the regularisation of the horospheric transform introduced in the last section.

Now suppose that $f \in L^1(G)$. Then the Fourier transform exists, and is given by the aid of the integral kernel

$$\Phi(g, \sigma) = \Phi^{+}(g, \sigma) + \Phi^{-}(g, -\sigma) \tag{19}$$

where $\mathrm{Re}\,\sigma = 0$,

$$\Phi^{\pm}(g, \sigma) = A^{(1+\sigma)r}\, \theta(1 - A^{\pm 1})$$

and $A \equiv A(vg^{-1}n)$. These very much simpler kernels are sufficient too if $f \in L^2(G)$, in the usual way.

Remark. If we regard $f(g)$ as a generalised function (that is, as a distribution on the space of smooth functions of compact support on G), then its Fourier transform is of course always defined by the Parseval identity for the group, although its calculation may not be easy. It is clear that if this transform is a regular function, then it coincides with the Fourier transform of $f(g)$ as defined above in (19); and the components of f which give rise to zeros or devergences of $f \to \tilde{f}$ correspond in their turn to the inverse transforms of δ-functions or their derivatives. From an analytic viewpoint, the distinction between the mapping $f \to \tilde{f}$ for a classical and a generalised function lies in the way we choose to perform the regularisation: in the former case it is performed by "analytic continuation in the co-ordinates" as above, and in the latter by analytically varying $f(g)$ itself; and the two procedures may give different results. As a simple example of this we cite Ref. [1], Eq. (14). Finally, it is of interest to note that the sophisticated procedure used above reduces exactly [see (18)] to the elementary device of multiplying $f(g)$ by an analytic regulator function before taking the Laplace transform, and then continuing this function to the identity.

IV. The Inversion Formula

4.1. The Fourier Inversion Formula

We start by deriving the inversion formula for the Fourier transform of a (non-integrable) $f(g)$. This transform is defined as the analytic continuation of

$$\tilde{f}^{(\sigma,j)}(n,v) = \tilde{f}_{+}^{(\sigma,j)}(n,v) + \tilde{f}_{-}^{(-\sigma,j)}(n,v) \tag{20}$$

and reduces to the usual definition if $f \in L^1(G)$. Our method is to take the inversion formula found in Ref. [2],

$$c_G f(1) = \sum_j \int_\Gamma d\sigma \frac{[p_j(\sigma)]^2}{\dim D_j} \int A^{1-\sigma}(vn) \operatorname{Tr}\{\tilde{f}^{(\sigma,j)}(n,v) [\mathscr{D}^j(M(vn))]^{-1}\} \, dn \, dv \tag{21}$$

and apply it to the transform (5); then an analysis of the essential points of the proof of the quasi-regular inversion formula in Ref. [1], Section 2.2, will show us how, if at all, (21) can go wrong.

Let us then first rewrite (21) in a form more appropriate to non-integrable functions: that is, let us recast it for the compact realisation of the horospheres. As in Section I, let $\{\hbar\}$ denote the space of "cross-sections" of the horospheres, the space of pairs $\{\theta, \theta'\}$ where $\theta, \theta' \in K/M$; and instead of writing $A(vg^{-1}n)$ we shall write $A(\hbar, g)$. Then (21) becomes, in brief,

$$\begin{aligned} c_G f(g) = \sum_j \int_\Gamma d\sigma \ldots \int A^{1-\sigma}(\hbar, g) \operatorname{Tr}\{\tilde{f}_+^{(\sigma,j)}(\hbar)\,[\mathscr{D}^j(M(\hbar, g))]^{-1}\}\, d\hbar \\ + \sum_j \int_\Gamma d\sigma \ldots \int A^{1+\sigma}(\hbar, g) \operatorname{Tr}\{\tilde{f}_-^{(\sigma,j)}(\hbar)\,[\mathscr{D}^j(M(\hbar, g))]^{-1}\}\, d\hbar \end{aligned} \tag{22}$$

where Γ is "to the right of" all singularities of the integrand. The integral over $\hbar$ is now over a compact set, so we have only one improper integral to consider.

Now it is inconvenient to work with the pair of Laplace transforms in (22); let us therefore consider the inversion formula rather as a linear functional B on the space of the mixed transforms $\check{f}^j(\hbar, a)$ of (10). To do this, we first use the classical Plancherel theorem for generalised functions over the real line, which implies that the contribution of the continuous series can be written in the form of an integral of $\check{f}$ over $a \times \{\hbar\}$, where the measure on a is essentially the Fourier transform of $[p_j(\sigma)]^2$. Now $p_j(\sigma)$ is known to behave as a product of a polynomial in σ and factors of the form tan $(s\sigma + t)$ where s and t are constants; hence as $|\sigma| \to \infty$ in any direction except along the real axis, $p_j(\sigma)$ behaves asymptotically as a polynomial in σ, of degree p say. Therefore its Fourier transform behaves[2] (asymptotically) as $\tilde{p}_j(\alpha) \sim \text{const}\, |\alpha|^{-p-1}$ as $\alpha \to \infty$.

It follows from arguments like those of Section II that the functional B can have singularities only on the space of those functions $\check{f}^j(\hbar, a)$ which have polynomial asymptotic components (we use this phrase also to include any logarithmic or/non-power behaviour). But by Lemma 1 and the Proposition of Section III, such $\check{f}$ can arise only from $f(g)$ which also have polynomial asymptotic behaviour; and by the hypothesis of the convergence of $f \to \hat{f}$ either there are no such components of f or they are annihilated. Hence on the space $\{\check{f}\}$ B has no singularities which arise from asymptotic behaviour.

Now if the transform $L: f \to \hat{f}$ converges, then BL defines a linear functional on $C^\infty(G)$; and by an argument entirely analogous to the one used to deduce the Proposition above, we find that if $B(\hbar, a) \sim |a|^q$ as $a \to \infty$, then, as an integral kernal, $BL(g) \sim |g|^q$ as $|g| \to \infty$[3]. Therefore the singularity subspace of BL too is exactly the singularity subspace of $L: f \to \hat{f}$.

[2] The Fourier transform of a pure polynomial is often written as a series of derivatives of δ-functions; but that formulation is only valid if the function space on which the δ-functions act excludes-polynomial behaviour at infinity. Such representations of the transform as

$$|\tilde{x}|^\lambda(k) = -2 \sin \lambda\pi/2 \cdot \Gamma(\lambda + 1) \cdot |k|^{-\lambda-1}$$

are however always valid.

Suppose then that G has but one class of Cartan subalgebras – that is, that it has no discrete series of representations. Then $p_j(\sigma)$ is a polynomial, and we know that when restricted to the subspace $C^\infty(G) \cap L^1(G)$, BL takes the form $\delta(g)$. But there is only one way in which a generalised function defined by an integral or integral kernel can find its support concentrated upon a lower dimensional manifold, and that is by the confluence of a pole and a compensating zero [cf. $\delta(x) = [\Gamma(\mu)]^{-1} |x|^{\mu-1}$ at $\mu = 0$]. Therefore BL is the product of a complex-number-valued function which has a zero and a distribution-valued-function which has a pole at $g = 1$; and behaves polynomially, as $|g| \to \infty$, as seen from the last paragraph. But as we have just observed, such asymptotic behaviour cannot give rise to a singularity unless $L: f \to \hat{f}$ diverges, and therefore the entire contribution to BLf of the asymptotic region $|g| \to \infty$ is cancelled by the factor of zero.

Hence BL takes the form $\delta(g)$ too upon the space of interest, of (non-polynomial) C^∞ functions which are not integrable; and therefore (22) is the inversion formula that we seek.

Now suppose that G does have a discrete series. In Ref. [2] it was shown that (22) gave the inversion formula then too provided that the contour Γ was interpreted as being "to the right of all singularities", in the sense that we run Γ along the imaginary axis and then subtract off all the contributions from poles of the Plancherel measure [but not from singularities of $\tilde{f}^{(\sigma,j)}(\hbar)$] crossed when we formally move the contour of integration off to infinity in the right half-plane. The proof of that depended largely upon analyticity arguments, and an analysis of the possible singularities arising from the asymptotic behaviour in exactly the same manner as the above leads again to the conclusion that, provided $L: f \to \hat{f}$ converges, then modulo the kernel of L (22) is still the inversion formula.

4.2. The Laplace Inversion Formula

The formula (22) held for even a non-integrable $f(g)$ because of the convention regarding Γ as "to the right of" all singularities of $p_j(\sigma)$. When $f(g)$ is allowed to grow at infinity, the singularities of the Laplace transforms move into the right half-plane; fortunately, the inversion formula for the classical Laplace transform also requires a contour now genuinely to the right of all singularities, and so the two are compatible.

Let us therefore substitute (20) in (21), and remember that the $\tilde{f}_{\pm}^{(\sigma,j)}(\hbar)$ are analytic to the right of Γ. Consider the term arising from

[3] There is an error in Eq. (18) of Ref. [1]: the last factor in the denominator should read $(\mathrm{ch}\, r/2)^{\mu+n}$ instead of $(\mathrm{ch}\, r/2)^{\mu+n-2}$. This propagates from a misprint in the last equation in Chapter V, Section 2.3 of Ref. [4]. Since we are only concerned with its value at $r = 0$, none of the conclusions of Ref. [1] are in any way affected.

$\tilde{f}_+$. The integral over $\hbar$, which is of course over the entire cross-section of H, can be written as the sum of an integral over the region $A(\hbar, 1) > 1$ and one over $A(\hbar, 1) < 1$; and in the former case the integrand decreases in the right half-plane, so that we can close the contour Γ to the right and pick up only the contributions from the (double) poles introduced by the Plancherel measure. For the region $A(\hbar, 1) < 1$ this is impossible. We can treat the term in $\tilde{f}_-$ similarly, to obtain the resulting Laplace inversion formula, which we give as a Theorem.

Theorem. *Let $f(g)$ be a c^∞ polynomially-bounded function on a connected semi-simple Lie group G of real-rank unity whose Laplace transforms $\tilde{f}_\pm^{(\sigma,j)}(\hbar)$, as given by* (17), *are convergent. Then we have*

$$\begin{aligned} c_G f'(g) = & \sum_j \int_\Gamma d\sigma \frac{[p_j(\sigma)]^2}{(\dim D_j)} \int_{A(\hbar,g)<1} A^{1-\sigma}(\hbar, g) \\ & \cdot \operatorname{Tr}\{\tilde{f}_+^{(\sigma,j)}(\hbar)\,[\mathcal{D}^j(M(\hbar,g))]^{-1}\}\, d\hbar \\ & + \sum_j \int_\Gamma d\sigma \frac{[p_j(\sigma)]^2}{(\dim D_j)} \int_{A(\hbar,g)>1} A^{1+\sigma}(\hbar, g) \\ & \cdot \operatorname{Tr}\{\tilde{f}_-^{(\sigma,j)}(\hbar)\,[\mathcal{D}^j(M(\hbar,g))]^{-1}\}\, d\hbar \\ & - 2\pi i \sum_{j,s} \frac{\operatorname{Res}[p_j(\sigma)]^2}{(\dim D_j)} \left(\frac{\partial}{\partial s}\right) \int A^{1-s}(\hbar, g) \\ & \cdot \operatorname{Tr}\{\tilde{f}_+^{(s,j)}(\hbar)\,[\mathcal{D}^j(M(\hbar,g))]^{-1}\}\, d\hbar \\ & - 2\pi i \sum_{j,s} \frac{\operatorname{Res}[p_j(\sigma)]^2}{(\dim D_j)} \left(\frac{\partial}{\partial s}\right) \int A^{1+s}(\hbar, g) \\ & \cdot \operatorname{Tr}\{\tilde{f}_-^{(s,j)}(\hbar)\,[\mathcal{D}^j(M(\hbar,g))]^{-1}\}\, d\hbar\,. \end{aligned}$$

Here Γ is a contour of integration parallel to the imaginary axis of σ and to the right of all singularities of the integrand except the poles of the Plancherel measure; and the discrete sums are over all the poles of $p_j(\sigma)$ to the right of Γ.

The left-hand side of this equation, $f'(g)$, is to be understood as $f(g)$ redefined modulo all functions which transform under a finite-dimensional representation of G; the remaining notation is that used throughout this paper.

4.3. Discussion

It is worth commenting on our proof. We have constantly interchanged orders of integration without at the time discussing the validity of this; similarly we have made use of the Parseval identity on the real line. The justification rests on the definition of all the processes involved: the generalised functions were all defined *in the sense of their regularisations*, and that involved analytic continuation in the parameters. The analyticity implicit there at once allows us to interchange orders of integration; similarly the Fourier transform of a *generalised* function K over $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is always defined by the relation

$$\int \overline{f(x)}\, K(x)\, dx = \frac{1}{2\pi} \int \overline{\tilde{f}(k)}\, \tilde{K}(k)\, dk\,;$$

and whether or not $K(x)$ is itself integrable this is sufficient. Notice that we have throughout treated $f(g)$ itself on a different footing: we have insisted that it shall be a function, and not a distribution.

We have been unable to derive an inversion formula directly for the horospheric transform $f \to \hat{f}$, which was possible in Ref. [1], and so we have circumvented the difficulties there caused by the divergence of $f \to \hat{f}$ for certain asymptotic behaviours in the same way as it was resolved there by Lemma 4 of that paper. It is worth noting however that these divergences would anyway not occur in a treatment of the Class I representations by the present method: for they arose essentially from the fact that the invariant measure on G/K contained the factor $\exp(-r \ln A)$, which can be either an integral or a half-integral power of A; whereas on G itself the measure contains $\exp(-2r \ln A)$, which is always integral. Therefore upon G a direct inversion $\hat{f} \to f'$ would always converge if $f \to \hat{f}$ did so.

The inversion formula has many of the features of that given in Ref. [1], Theorem 2: in particular, the integrals over σ possess "good" analyticity properties in the left-hand half-plane, as in the classical one-dimensional Laplace transform. The contribution of the pole terms is more interesting, however. Let us consider first the case of Class I or quasiregular representations, which was treated in Ref. [1], and return to Theorem 2 of that paper: then the discrete terms contain contributions of two types, which we can categorise as discrete series representations or as end-point contributions coming from the fact that the projection integrals defining the $\tilde{f}_\pm$ do not extend over the whole group. But if we combine all the pole terms in that formula, we find that the end-point contributions cancel out, and so we are left with only the discrete series. But we know that finite dimensional components are annihilated, and hence do not contribute, and so this leaves only the unitary discrete series; which are not realised on G/K and hence do not occur. Therefore the role of the discrete terms in Ref. [1] is just to cancel out all the endpoint contributions in the integrals.

Now return to our results in this paper. The analysis of the discrete terms proceeds in exactly the same way, and as before the end-point contributions of the summations cancel against those of the integrals. The finite-dimensional representations are annihilated, and so do not occur; but the unitary discrete series are indeed present, and the summations give their contribution exactly as in Ref. [2]. Although we apparently now have double poles in the integral, we have shown in Ref. [2] that the leading term in the Laurent series about these points actually vanishes, and so the poles are actually simple. Notice incidentally that the unitary discrete series enters as a series of derivative terms.

Finally, the integral over $\hbar$ is in principal over *any* cross-section of the fibre bundle; but as in [1], unless this is compact we shall need some further regularisation procedure.

V. Some General Remarks

We have discussed earlier [1] the three alternative approaches to the definition of the Fourier transform of a nonintegrable function $K(g)$ on a nonabelian group. These are:

(i) We can regard $K(g)$ as a *distribution* on some space of test functions and define $\tilde{K}$ by the Parseval identity. Then $\tilde{K}$ is always defined, but may be difficult to calculate.

(ii) We can project $f(g)$ over nonunitary representations of a *semigroup* contained in G. This approach was noticed earlier by other workers but has been most highly developed quite recently by Toller [5] for the group $SL(2, C)$. For that group, the method rests upon noticing that the two sets of elements

$$\mathrm{G}_{\pm} = \{g = v \cdot \exp(\pm K_3 \zeta) \cdot v' : v, v' \in SU(1,1), \zeta > 0\}$$

each define a semigroup; if we then choose a basis for the group in which $SU(1, 1)$ is diagonalised, then the matrix elements of $\exp(\pm K_3 \zeta)$ are just the classical second-kind functions and so readily lend themselves to defining transforms with good asymptotic behaviour.

Unfortunately there are two severe difficulties with this approach. The first is that $SU(1, 1)$ is noncompact, so that if $f(g)$ is nonintegrable the transform diverges unless some further regularisation is introduced; and the second (and more serious) is that the two semigroups do not fill up G. Consequently Toller has been able to define a satisfactory transform – inversion pair only when $f(g)$ is an integrable function which vanishes outside $G_+ \cup G_-$. This is a major deficiency in an otherwise elegant approach.

(iii) We can use special-function projection techniques. The most sophisticated work here is that of Cronström [6] for $SL(2, R)$, who projects with the group analogue of Legendre's second-kind functions $Q_l(\operatorname{ch}\beta)$ (thus always achieving convergence) and gives an inversion formula using the unusual functions $R^l_{mn}(\operatorname{ch}\beta)$, which for the spinless case are

$$R_l(\operatorname{ch}\beta)=\frac{1}{\pi\sqrt{2}}\int_0^\beta \frac{e^{i\sigma\phi}}{(\operatorname{ch}\beta-\operatorname{ch}\phi)^{\frac{1}{2}}}\,d\phi \qquad (l=-\tfrac{1}{2}+i\sigma).$$

These functions are of course just those which arise in the application of our inversion formula in [1], Theorem 2, to the case of spinless functions over $SL(2,R)$: for in such a case we can choose a spherical coordinate system for the hyperboloid $O(2,1)/O(2)$ and obtain terms like

$$\begin{aligned}\int_{u\cdot k>1} \tilde{f}(k_-,s)\,(u\cdot k)^{-\frac{1}{2}+is}\,dk &= \tilde{f}_-(s)\int_{(\;)>1}(\operatorname{ch}\zeta-\operatorname{sh}\zeta\cos\phi)^{-\frac{1}{2}+is}\,d\phi\\ &=\pi\tilde{f}_-(s)\,R_l(\operatorname{ch}\zeta)\end{aligned}$$

(we have used the notation of [1]) and this is identical to Cronström's function. The remainder of his inversion formula differs from ours by not containing any discrete summations; this is a consequence of his projection formula, which unlike ours uses the second-kind functions.

All these approaches, except perhaps the first, have their disadvantages. In particular, the third loses contact with representation theory in the projection formula and the second-kind functions used there; and, moreover, it cannot be generalised even to $SL(2,C)$ because these functions for an arbitrary group (and, in particular, that one) are not locally integrable at the identity. The only way to avoid this difficulty seems to be to split $K(g)$ into two parts, vanishing in a neighbourhood of the identity and infinity respectively, and treat these parts separately – a process possible but clearly unsatisfactory.

Our approach to the problem has disadvantages in its turn: the two-stage regularisation process for the transform, and the existence of zeros and divergences of the map $f\to\hat{f}$. It has however the considerable advantages of being sufficiently general to apply to all groups (though we have given the details here only for the Lorentz groups) and of retaining the manifest basis-independence of the general theory in which it is so firmly rooted.

References

1. Macfadyen, N. W.: Commun. math. Phys. **28**, 87 (1972)
2. Macfadyen, N. W.: The horospheric approach to harmonic analysis on a semi-simple Lie group. Cambridge preprint DAMTP 73/16, to be published
3. Vilenkin, N. J., Smorodinskii, Ya. A.: Sov. Phys. JETP **19**, 1209 (1964)

4. Gel'fand, I. M., Graev, M. I., Vilenkin, N. J.: Generalised functions, Vol. V. New York: Academic Press 1966
5. Bassetto, A., Toller, M.: Ann. Inst. H. Poincaré **18**, 1 (1973)
Ferrara, S., Mattioli, G., Rossi, G., Toller, M.: Nucl. Phys. B **53**, 366 (1973)
6. Cronström, C.: Generalised 0(2, 1) expansions of multiparticle amplitudes–II: The 0(2, 1) Laplace transform. I.C.T.P. preprint IC/72/41

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