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On the Semiboundedness of the $(\phi^4)_2$ Hamiltonian *

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Abstract. An elementary alternate proof of the semiboundedness of the locally correct Hamiltonian $H_0 + \int :\phi^4(x): g(x)\, dx$ of the $(\phi^4)_2$ quantum field theory model. The interaction operator is expressed as the sum of a positive operator and operators which are "tiny" relative to N^ε for any $\varepsilon > 0$, where N is the number operator.

The semiboundedness of the space cut-off $(\phi^4)_2$ Hamiltonian was first proved by Nelson [1]. Alternative proofs and generalizations of this result have been given by various authors (see [2] and the references therein). In this note we give an elementary alternate proof in which the interaction operator $V = \int :\phi^4(x): g(x)\, dx (g \geqq 0,\ g \in L^1 \cap L^2)$ is expressed as the sum of a positive operator and operators which are "tiny" relative to N^ε for any $\varepsilon > 0$ (here N is the number operator). The proof is based on the formal identity $:\phi^4: = (:\phi^2: - 2c)^2 - 6c^2$ where c is the infinite constant $\int w(k)^{-1}\, dk$.

In our notation

$$a(k)\, a^+(p) - a^+(p)\, a(k) = \delta(k-p)$$

$$N = \int a^+(k)\, a(k)\, dk$$

$$H_0 = \int a^+(k)\, a(k)\, w(k)\, dk$$

$$\phi(x) = \int [a(k) + a^+(-k)] \exp(ikx)\, w(k)^{-1/2}\, dk$$

where $w(k) = (k^2 + m^2)^{1/2}$ and m is the mass of the free field ϕ.

Let $b > 0$ and define

$$f_n(k) = \begin{cases} w(k)^{-1/2} & |k| \leqq n^b \\ 0 & |k| > n^b . \end{cases}$$

Let $a_n = a_n(x) = \int a(k) \exp(ikx)\, f_n(k)\, dk$ and let a_n^+ be the adjoint of a_n and let

$$c_n = a_n a_n^+ - a_n^+ a_n = \| f_n \|^2 \quad \text{for} \quad n = 0, 1, \ldots .$$

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Let $Y = Y(x)$ and C be the symmetric operators defined on "n-particle states" ψ_n by

$$Y\psi_n = [a_{n+2}^{+\,2} + 2a_n^+ a_n + a_n^2]\,\psi_n$$

$$C\psi_n = c_n \psi_n\,.$$

We now apply the positive operator $\int dx g(x)(Y-2C)^2$ to the n-particle state ψ_n and Wick-order the terms in the resulting expression, using the commutation relations $[a_n, a_n^+] = c_n$:

$$\begin{aligned}&[\int g(x)\,(Y-2C)^2\,dx]\,\psi_n\\ &\quad = [\int g(x)\,\{a_{n+4}^{+\,2}\,a_{n+2}^{+\,2} + 2a_{n+2}^{+\,2}\,a_n^+\,a_n\\ &\quad + 2a_{n+2}^{+\,3}\,a_{n+2} + a_{n+2}^{+\,2}\,a_{n+2}^{\;2} + 5a_n^{+\,2}\,a_n^2\\ &\quad + 2a_n^+\,a_n^3 + 2a_{n-2}^{\;+}\,a_{n-2}\,a_n^2 + a_{n-2}^{\;2}\,a_n^2\\ &\quad + 2(c_{n+2} - c_n)\,a_{n+2}^{\;+2} + 4(c_{n+2} - c_n)\,a_{n+2}^+\,a_{n+2}\\ &\quad + 2(c_n - c_{n-2})\,a_n^2 + 2c_{n+2}^{\;2} + 4c_n^2\\ &\quad + 4c_n(a_{n+2}^+\,a_{n+2} - a_n^+\,a_n)\}\,dx]\,\psi_n\,.\end{aligned}$$

Let us designate by V' the operator defined on n-particle states by the first eight terms of the above integral. If we examine the "four-creation" part of $V - V'$ we find it is of the form

$$\iiiint W''(k,p,q,r)\,a^+(k)\,a^+(p)\,a^+(q)\,a^+(r)\,dk\,dp\,dq\,dr$$

where the kernel W'' is equal to zero when $|k|, |p| < (n+2)^b$ and $|q|, |r| < (n+4)^b$ and equal to the kernel W of V otherwise. By a modification of a proof given by Simon and Höegh-Krohn [3, p. 155], W'' is square integrable and $\|W''\|_2 \leqq \text{const}\,[(n+2)^b]^{-\alpha}$ for a certain α, $0<\alpha<1$. The other terms in $(V-V')$ may be treated similarly with the result that $\|(V-V')\psi_n\| \leqq \text{const}\,n^2 n^{-b\alpha}\|\psi_n\|$ for large n. If we now choose $b = 2/\alpha$ then $\|(V-V')\psi_n\| \leqq \text{const}\,\|\psi_n\|$. We see that $(c_{n+2} - c_n) \leqq \text{const}\,n^{-1}$ and that c_n grows like $\text{const}\log n$ for large n. Since $\|\int a_{n+2}^{+\,2}\,g(x)\,dx\,\psi_n\| \leqq (n+2)\,[\iint |\tilde{g}(k+p)|^2\,w(k)^{-1}\,w(p)^{-1}\,dk\,dp]^{1/2}\,\|\psi_n\| \leqq \text{const}(n+2)\,\|\psi_n\|$ the term involving a_{n+2}^{+2} may be bounded by a constant for large n. The next two terms may be treated similarly. The terms $4c_n^2$ and $2c_{n+2}^{\;2}$ are of order $(\log n)^2$. The last term may be written $4c_n\iint [X_{n+2}(k,p) - X_n(k,p)]\cdot a(k)^+\,a(p)\,dk\,dp\,\psi_n$ where $X_n(k,p)$ equals $\tilde{g}(k-p)\,w(k)^{-1/2}\,w(p)^{-1/2}$ for $|k|, |p| \leqq n^{2/\alpha}$ and equals zero elsewhere. $\iint |X_{n+2} - X_n|^2\,dk\,dp$ may be bounded by a sum of four integrals of the type

$$\sup_p |\tilde{g}(p)|^2\,n^{-2/\alpha}[(n+2)^{2/\alpha} - n^{2/\alpha}] \int_{-(n+2)^{2/\alpha}}^{(n+2)^{2/\alpha}} w(k)^{-1}\,dk$$

which are bounded by $\text{const}\,n^{-1}\log n$ for large n. Hence the last term is bounded by $\text{const}\,n^{1/2}(\log n)^{3/2}\,\|\psi_n\|$ for large n.

We conclude that V differs from the positive operator $\int (Y-2C)^2 \cdot g(x)\,dx$ by an operator A, all of whose nonzero matrix elements $\langle \psi_{n+m} | A\psi_n \rangle$, $m=0$, ± 2, ± 4 are bounded in magnitude by $n^{\varepsilon+1/2} \|\psi_{n+m}\| \, \|\psi_n\|$ for some $\varepsilon > 0$ and large n. It follows that the operator $N^{\varepsilon+1/2} + V$ is bounded below for any $\varepsilon > 0$, which of course implies that the locally correct Hamiltonian $H_0 + V$ is bounded below.

We can improve our estimate on the last term discussed above without essentially changing our estimates on the other terms by using a less sharp momentum cut-off in the definition of a_n. To do this we redefine $f_n(k)$ as the continuous function

$$f_n(k) = \begin{cases} w(k)^{-1/2} & 0 \leqq |k| \leqq n^{2/\alpha} \\ w(n^{2/\alpha})^{-1/2}[1-(k-n^{2/\alpha})n^{\beta-2/\alpha}] & n^{2/\alpha} < |k| < n^{2/\alpha} + n^{2/\alpha-\beta} \\ 0 & |k| \geqq n^{2/\alpha} + n^{2/\alpha-\beta}, \end{cases}$$

where $0 < \beta < 1$.

With this choice of f_n one can show that the integral $\int\int |X_{n+2} - X_n|^2 \, dk \, dp$ may be bounded by $\text{const}\, n^{\beta-2} \log n$ for large n so that our last term is bounded by $\text{const}\, n^{\beta/2} (\log n)^{3/2}$ for large n. By taking β sufficiently small we see that $N^\varepsilon + V$ is bounded below for any $\varepsilon > 0$.

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