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Highly Mobile Einstein Spaces in the Large

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Abstract. We consider an Einstein space V of the Petrov type II or III admitting a group of motions G of high order. First we calculate the composition law and topological structure of G. Then V (or its submanifolds of transitivity) is represented as the homogeneous space G/H of G, H being a subgroup of G, and the action G on V and the topology of V are determined. The topologies of the spaces V are as follows: $\mathbb{R}^4$ (space T_2^*), $\mathbb{R}^4$ of $\mathbb{R}^3\,\mathbb{T}^1$ (space T_2), $\mathbb{R}^4$ (space T_3^*), $\mathbb{R}^3$ (submanifolds of transitivity in space T_3).

In two cases (spaces T_2 and T_3) we have obtained metrics free of singularities.

§ 1. Introduction

The aim of this work is to investigate the global structure of some Einstein spaces T_i (satisfying the field equations $R_{\alpha\beta}=0$) and T_i^* (satisfying the field equations $R_{\alpha\beta}=\varkappa g_{\alpha\beta}$, $\varkappa \neq 0$) possessing high mobility. Here $i=1, 2, 3$ is the Petrov type.

The *local* part of this problem has been solved by Petrov ([1], Chapter 5) who has determined Lie algebras of Killing vectors and metrics of these spaces. To obtain a *global* information about the spaces under consideration we shall use the new topological methods based on the idea of homogeneous spaces of Lie groups (see, e.g., [2]).

At present a global investigation of gravitational fields is a matter of current interest. It is particularly interesting to examine the topology of the highly mobile Einstein spaces $T_2(T_2^*)$ and $T_3(T_3^*)$ since:

(i) these spaces are often interpreted as gravitational waves;

(ii) these spaces cannot be asymptotically flat and so they are assumed to have peculiar topology ([1], § 30);

(iii) the boundary conditions for Einstein spaces of the second and the third types are formulated in terms of the highly mobile Einstein spaces of the same types ([1], § 65) so that topology of arbitrary Einstein spaces of the Petrov type II or III is determined by the topology of the corresponding Einstein spaces having maximal mobility.

Taking this into consideration we shall investigate below the global structure of the Einstein spaces T_2, T_2^*, T_3 and T_3^*, possessing maximal mobility.

§ 2. Topology of the Space T_2^*

2.1. The highly mobile Einstein space T_2^* – let us denote it by V – in a special coordinate system has the metric

$$\begin{aligned} g_x(dx) = & -\exp(2x^4)\,[2dx^1\,dx^3 + (dx^2)^2] + a\exp(-x^4)\,(dx^3)^2 \\ & -(3/\varkappa)\,(dx^4)^2, \quad a = \pm 1, \quad \varkappa > 0 \end{aligned} \tag{2.1}$$

and admits the Lie algebra L of Killing vectors spanned by the infinitesimal operators

$$\begin{aligned} & X_i = p_i (i = 1, 2, 3,\ p_\alpha = \partial/\partial x^\alpha = \partial_\alpha),\ X_4 = x^3 p_2 - x^2 p_1, \\ & X_5 = 5x^1 p_1 + 2x^2 p_2 - x^3 p_3 - 2p_4 \end{aligned} \tag{2.2}$$

(see [1], § 30). By the repeated Greek (Latin) indices the summation over the range 1, 2, 3, 4 (resp. 1, 2, 3, 4, 5) is meant.

2.2. Make the commutators of X_i's and find the structural equations of L:

$$\begin{aligned} & [X_1 X_2] = [X_1 X_3] = [X_1 X_4] = [X_2 X_3] = 0,\ [X_2 X_4] = -X_1, \\ & [X_3 X_4] = X_2, \\ & [X_1 X_5] = 5X_1,\ [X_2 X_5] = 2X_2,\ [X_3 X_5] = -X_3,\ [X_4 X_5] = 3X_4. \end{aligned}$$

Obviously the center of L is equal to 0, so that L has the isomorphic adjoint representation in terms of the matrices $a(g) = (a_i^k)$ determined by the equation $[X X_i] = a_i^k X_k$, $X = g^l X_l$:

$$a(g) = \begin{bmatrix} -5g^5 & g^4 & 0 & -g^2 & 5g^1 \\ & -2g^5 & -g^4 & g^3 & 2g^2 \\ & & g^5 & 0 & -g^3 \\ & & & -3g^5 & 3g^4 \\ & & & & 0 \end{bmatrix}.$$

Find the connected and simply connected Lie group G with L as its Lie algebra. For this purpose one may use the exponential mapping:

$$a(g) \mapsto A(g) = \exp a(g), \tag{2.3}$$

$$A(g) = \begin{bmatrix} E^5 & \frac{1}{3}\bar{g}^4(E^2 - E^3) & \frac{1}{18}(\bar{g}^4)^2(2E^2 - E^{-1} - E^5) & f_1(g) & f_2(g) \\ & E^2 & \frac{1}{3}\bar{g}^4(E^2 - E^{-1}) & \bar{g}^3(E^2 - E^3) & f_3(g) \\ & & E^{-1} & 0 & \bar{g}^3(1 - E^{-1}) \\ & & & E^3 & \bar{g}^4(1 - E^3) \\ & & & & 1 \end{bmatrix}$$

$$E=\exp(-g^5),\ \bar{g}^i=g^i/g^5,$$
$$f_1(g)=\tfrac{1}{2}\bar{g}^2(E^5-E^3)+\tfrac{1}{6}\bar{g}^3\bar{g}^4(E^5+2E^2-3E^3),$$
$$f_2(g)=\bar{g}^1(1-E^5)+\tfrac{1}{6}\bar{g}^2\bar{g}^4(3E^3-2E^2-E^5)$$
$$+\tfrac{1}{18}\bar{g}^3(\bar{g}^4)^2(9E^3+E^{-1}-2E^5-8E^2),$$
$$f_3(g)=\bar{g}^2(1-E^2)+\tfrac{1}{3}\bar{g}^3\bar{g}^4(3E^3+E^{-1}-4E^2).$$

The set G of matrices $A(g)$ forms the Lie group with the Lie algebra L. The correspondence $A(g)\leftrightarrow g=(g^1,g^2,g^3,g^4,g^5)$ is analytical homeomorphism between G and the Euclidean space $\mathbb{R}^5$ so that G is really connected and simply connected. Coordinates g^i of $A(g)$ are canonical ([2], § 42, Example 77). The center of G consists of the only unit matrix, hence there are no other Lie groups with the Lie algebra L ([2], § 51).

Make the transformation of coordinates in G:

$$a^1=f_2(g),\ a^2=f_3(g)/2,\ a^3=\bar{g}^3(1-E^{-1}),\quad a^4=\bar{g}^4(1-E^3)/6,\ a^5=g^5. \tag{2.4}$$

In the new coordinates a^i $A(g)$ takes the form:

$$B(a)=\begin{bmatrix} F^5 & 2a^4F^2 & -2(a^4)^2F^{-1} & -(a^2+3a^3a^4)F^3 & a^1 \\ & F^2 & -2a^4F^{-1} & -a^3F^3 & 2a^2 \\ & & F^{-1} & 0 & a^3 \\ & & & F^3 & 6a^4 \\ & & & & 1 \end{bmatrix}$$
$$F=\exp(-a^5).$$

From the equation $B(ab)=B(a)B(b)$ one may obtain the composition law in G (in the coordinates a^i):

$$(a,b)\mapsto ab\qquad(a,b\in G),$$
$$(ab)^1=a^1-6(a^2+3a^3a^4)b^4\exp(-3a^5)-2(a^4)^2b^3\exp(a^5)$$
$$+4a^4b^2\exp(-2a^5)+b^1\exp(-5a^5),$$
$$(ab)^2=a^2+b^2\exp(-2a^5)-3a^3b^4\exp(-3a^5)-a^4b^3\exp(a^5),$$
$$(ab)^3=a^3+b^3\exp(a^5),$$
$$(ab)^4=a^4+b^4\exp(-3a^5),$$
$$(ab)^5=a^5+b^5.$$

2.3. Determine topology of V. G acts on V transitively so that V may be represented as the homogeneous space G/H of left cosets gH, $g\in G$, H being a subgroup of G. Obviously H has the dimensionality $\dim H=\dim G-\dim V=1$.

Let L_H be the Lie algebra of H. L_H can be taken as the subalgebra of L consisting of all infinitesimal transformations $X(x) \in L$ with the property $X(0)=0$. We obtain from (2.2) that $X(0)=0$ implies $X = \text{const.}\ X_4$ so that L_H is spanned by X_4.

Let first H be connected. L_H lies on the axis g^4 in the vector space L and so H in the canonical coordinates g^i lies on the same axis g^4. It is immediately verified that under the transformation (2.4) the axis g^4 is transformed into the axis a^4, hence in the new coordinates a^i H consists of the elements $h=(0,0,0,c,0)$, c is real.

Calculate the homogeneous space $V=G/H$. For any $g \in G$ the left coset gH consists of the elements:

$$gh = (g^1 - 6c(g^2 + 3g^3 g^4)\exp(-3g^5),\ g^2 - 3cg^3 \exp(-3g^5),\ g^3,\ g^4 + c\exp(-3g^5),\ g^5).$$

It is easy to see that for the every $g \in G$ there exists in gH the unique element with zero fourth coordinate (the corresponding value of c is equal to $c = -g^4 \exp(3g^5)$). This element

$$(g^1 + 6g^4(g^2 + 3g^3 g^4),\ g^2 + 3g^3 g^4,\ g^3,\ 0,\ g^5) \tag{2.5}$$

will be considered as the one representing the left coset $x = gH$. We shall also denote it by

$$(x^1, x^2,\ -5x^3,\ 0,\ x^4/2) \tag{2.6}$$

and so the defined numbers $x^\alpha (\alpha = 1, 2, 3, 4)$ will be considered as coordinates of gH in G/H. Comparing (2.5) and (2.6) we obtain the coordinate expression for the canonical projection of G onto G/H:

$$\begin{aligned} &p: g \mapsto gH,\ G \to G/H,\\ &p^1(g) = g^1 + 6g^4(g^2 + 3g^3 g^4),\ p^2(g) = g^2 + 3g^3 g^4,\\ &p^3(g) = -g^3/5,\ p^4(g) = 2g^5. \end{aligned}$$

The homogeneous space $V = G/H$ consists of all points $x = (x^1, x^2, x^3, x^4)$. If g runs over $\mathbb{R}^5$ then x^α run over all real numbers so that $V = p(G) = p(\mathbb{R}^5) = \mathbb{R}^4$ is homeomorphic to the Euclidean space $\mathbb{R}^4$.

Determine the action of G on V. For this purpose one must take an element $g \in G$ and a point $x = aH \in V$ and calculate $p(ga)$:

$$\begin{aligned} &(g, x) \mapsto g(x) = g(aH) = (ga)H = p(ga),\\ &g^1(x) = x^1 \exp(-5g^5) + 10x^2 g^4 \exp(-2g^5) - 50x^3 (g^4)^2 \exp(g^5)\\ &\qquad + g^1 + 6g^2 g^4 + 18g^3 (g^4)^2,\\ &g^2(x) = x^2 \exp(-2g^5) - 10x^3 g^4 \exp(g^5) + g^2 + 3g^3 g^4,\\ &g^3(x) = x^3 \exp(g^5) - g^3/5,\\ &g^4(x) = x^4 + 2g^5. \end{aligned}$$

If g^i's are infinitesimal then

$$g(x) = x + \delta x\,,$$
$$\delta x^1 = g^1 + 10x^2 g^4 - 5x^1 g^5\,, \qquad \delta x^2 = g^2 + 10x^3 g^4 - 2x^2 g^5\,, \tag{2.7}$$
$$\delta x^3 = -(g^3/5) + x^3 g^5\,, \qquad \delta x^4 = 2g^5\,.$$

On the other hand we must have

$$\delta x^\alpha = \xi_i^\alpha g^i \tag{2.8}$$

where ξ_i are Killing vectors. Comparing (2.7) and (2.8) we find $\xi_i^\alpha(x)$ and then the infinitesimal operators $X_i(x) = \xi_i^\alpha(x)\, p_\alpha$. These X_i coincide with (2.2) so that coordinates x^α in (2.2) and (2.6) are the same and we may use the Eq. (2.1) for the metric of $V = G/H$.

2.4. Investigate whether H may be nonconnected. Let $\tilde{H}$ be a nonconnected subgroup of G. Then $\tilde{H}$ contains H as its connected component of the unit element and $G/\tilde{H}$ is obviously a factorspace of the space $G/\tilde{H}$. If $\tilde{H}$ is closed then the corresponding homogeneous space $\tilde{V} = G/\tilde{H}$ always exists, but it admits a factorstructure of the Riemannian structure of the space $V = G/H$ if and only if the equation

$$g_0(dx) = g_0(f_h(dx)) \tag{2.9}$$

is satisfied for all $h \in \tilde{H}$ (see [3]). Here 0 is the point $H \in V$, f_h is the automorphism of the tangent space T_0, induced by the inner automorphism $g \mapsto h^{-1} g h$ of G, $dx \in T_0$ is any.

In the case under consideration the Eq. (2.9) has only the trivial solution $h = (0, 0, 0, c, 0)$, c being any real number, i.e. $h \in H$. So $h \in \tilde{H}$ implies $h \in H$ and this gives $\tilde{H} = H$. Therefore, H may not be nonconnected and V is always homeomorphic to $\mathbb{R}^4$.

§ 3. Topology of the Space T_2

3.1. The symmetric Einstein space V of the Petrov type II has the metric

$$g_x(dx) = 2dx^1\, dx^4 - \mathrm{ch}^2 x^4 (dx^2)^2 - \cos^2(x^4 + c)\,(dx^3)^2 \qquad c = \mathrm{const} \tag{3.1}$$

and admits the Lie algebra of Killing vectors L spanned by the infinitesimal operators

$$X_1 = p_1\,,\ X_2 = p_2\,,\ X_3 = x^2 p_1 - \mathrm{th}\, x^4 p_2\,,\ X_4 = p_3\,,$$
$$X_5 = x^3 p_1 - \mathrm{tg}(x^4 + c)\, p_3\,, \tag{3.2}$$
$$X_6 = \tfrac{1}{2}[(x^2)^2 - (x^3)^2]\, p_1 - x^2\, \mathrm{th}\, x^4 p_2 + x^3\, \mathrm{tg}(x^4 + c)\, p_3 - p_4$$

(see [1], § 30); note that the second metric (30.26) given in [1] under the name of the symmetric one does not admit a 6-parametric Lie group of motions and does not satisfies the field equations.

3.2. As usual, for the global investigation of the metric (3.1) we first find the structural equations of L:

$$[X_2 X_3] = [X_4 X_5] = X_1, \; [X_2 X_6] = X_3, \; [X_3 X_6] = X_2,$$
$$-[X_4 X_6] = X_5, \; [X_5 X_6] = X_4, \quad \text{other} \quad [X_i X_j] = 0.$$

The center of L is nontrivial and is spanned by X_1 so that we may not use the adjoint representation. To determine the Lie group G with the Lie algebra L we may proceed as follows.

The composition law $(a, b) \mapsto ab = f(a, b)$ in every Lie group satisfies the equations

$$v_k^i(f)\,(\partial f^k/\partial a^j) = v_j^i(a), \qquad f(0, b) = b$$

where $v_k^i(a)$'s can be obtained from the Maurer equations

$$\partial_k v_j^i - \partial_j v_k^i = c_{mn}^i v_j^m v_k^n, \qquad v_k^i(0) = \delta_k^i,$$

c_{mn}^i being the structural constants of L. In our case:

$$v_i^i = 1 \; (1 \leqq i \leqq 6), \; v_6^2 = -a^3, \; v_6^3 = -a^2, \; v_6^4 = -a^5, v_6^5 = a^4,$$
$$v_2^1 = a^3/2, \; v_3^1 = -a^2/2, \; v_4^1 = a^5/2, \; v_5^1 = -a^4/2,$$
$$v_6^1 = [(a^2)^2 - (a^3)^2 - (a^4)^2 - (a^5)^2]/2, \quad \text{other} \quad v_j^i = 0.$$

The composition law in G is as follows:

$$\begin{aligned}
(ab)^1 &= a^1 + b^1 + [(a^3 b^2 - a^2 b^3)\operatorname{ch} a^6 + (a^3 b^3 - a^2 b^2)\operatorname{sh} a^6 \\
&\qquad + (a^5 b^4 - a^4 b^5)\cos a^6 + (a^4 b^4 + a^5 b^5)\sin a^6]/2, \\
(ab)^2 &= a^2 + b^2 \operatorname{ch} a^6 + b^3 \operatorname{sh} a^6, \\
(ab)^3 &= a^3 + b^2 \operatorname{sh} a^6 + b^3 \operatorname{ch} a^6, \\
(ab)^4 &= a^4 + b^4 \cos a^6 + b^5 \sin a^6, \\
(ab)^5 &= a^5 - b^4 \sin a^6 + b^5 \cos a^6, \\
(ab)^6 &= a^6 + b^6.
\end{aligned} \tag{3.3}$$

The group G consists of all points $a = (a^1, \ldots, a^6)$ with the composition law (3.3) and so is homeomorphic to $\mathbb{R}^6$ and is connected and simply connected. Note that the center of G is nontrivial so that there are other Lie groups with the algebra L, but we loosing no generality may restrict ourselves to considering only simply connected group G.

3.3. Determine the topology of V. G is transitive on V so that V is a homogeneous space G/H. The Lie algebra L_H of H is spanned by X_3 and $X_5 + \operatorname{tg} c X_4$.

One-parametric subgroup of G with tangent vector $X = e^i X_i$ can be calculated from the equation ([2], § 42)

$$dg^i(t)/dt = w^i_k(g(t))\, e^k, \;\; g^i(0) = 0, \;\; w^i_k(a) = \partial f^i(a, 0)/\partial b^k,$$

t being canonical parameter. We obtain

$$X = X_3: \qquad g(t) = (0, 0, t, 0, 0, 0),$$
$$X = X_5 + X_4 \operatorname{tg} c: \quad g(s) = (0, 0, 0, s \operatorname{tg} c, s, 0).$$

The connected subgroup $H \subset G$ with the algebra L_H can be obtained by making all products of $g(t)$ and $g(s)$:

$$h \in H \Rightarrow h = (0, 0, t, s \operatorname{tg} c, s, 0),$$

s and t being any real numbers.

Calculate G/H. For the every $g \in G$ the left coset gH consists of the elements

$$\begin{aligned} gh = (&g^1 + \tfrac{1}{2}[t(-g^2 \operatorname{ch} g^6 + g^3 \operatorname{sh} g^6) + s(-g^4 + g^5 \operatorname{tg} c) \cos g^6 \\ &+ s(g^4 \operatorname{tg} c + g^5) \sin g^6],\; g^2 + t \operatorname{sh} g^6,\; g^3 + t \operatorname{ch} g^6, \\ &g^4 + s(\operatorname{tg} c \cos g^6 + \sin g^6),\; g^5 + s(-\operatorname{tg} c \sin g^6 + \cos g^6),\; g^6). \end{aligned}$$

Choose an element $\tilde{g} \in gH$ representing gH as the unique element satisfying the conditions:

$$(gh)^3 = 0, \qquad ((gh)^4)^2 + ((gh)^5)^2 = \text{minimum}.$$

It follows from this that

$$\begin{aligned} &t = -g^3/\operatorname{ch} g^6, \;\; s = -(g^4 \sin(g^6 + c) + g^5 \cos(g^6 + c)) \cos c, \\ &\tilde{g} = (g^1 + (g^3/2)(g^2 - g^3 \operatorname{th} g^6) + (N/2)(g^4 \sin(g^6 + c) + g^5 \cos(g^6 + c)), \\ &\quad g^2 - g^3 \operatorname{th} g^6,\; 0,\; N \cos(g^6 + c),\; -N \sin(g^6 + c),\; g^6), \\ &N = g^4 \cos(g^6 + c) - g^5 \sin(g^6 + c). \end{aligned}$$

$\tilde{g}$ also will be denoted by

$$(x^1, x^2, 0, x^3 \cos(x^4 + c), -x^3 \sin(x^4 + c), x^4)$$

and so defined x^α's will be considered as coordinates of gH. Therefore, the analytic expression of the canonical projection $G \to G/H$ is as follows:

$$\begin{aligned} &p^1(g) = g^1 + (g^3/2)(g^2 - g^3 \operatorname{th} g^6) + (N/2)(g^4 \sin(g^6 + c) + g^5 \cos(g^6 + c)), \\ &p^2(g) = g^2 - g^3 \operatorname{th} g^6, \;\; p^3(g) = N, \;\; p^4(g) = g^6. \end{aligned}$$

Obviously $p(\mathbb{R}^6) = \mathbb{R}^4$ so that $V = G/H$ is homeomorphic to $\mathbb{R}^4$.

G acts on V as follows:

$$g^1(x) = x^1 + g^1 + (g^3/2)(g^2 - g^3 \operatorname{th}(g^6 + x^4)) + x^2 g^3 \operatorname{ch} x^4/\operatorname{ch}(g^6 + x^4)$$
$$+ (x^2)^2 \operatorname{sh} g^6 \operatorname{ch} x^4/2\operatorname{ch}(g^6 + x^4)$$
$$+ [g^4 \sin(g^6 + x^4 + c) + g^5 \cos(g^6 + x^4 + c)]$$
$$\cdot [x^3 + (g^4/2)\cos(g^6 + x^4 + c) - (g^5/2)\sin(g^6 + x^4 + c)],$$
$$g^2(x) = x^2 \operatorname{ch} x^4/\operatorname{ch}(g^6 + x^4) + g^2 - g^3 \operatorname{th}(g^6 + x^4),$$
$$g^3(x) = x^3 + g^4 \cos(g^6 + x^4 + c) - g^5 \sin(g^6 + x^4 + c),$$
$$g^4(x) = x^4 + g^6.$$

When g^i's are infinitesimal we obtain:

$$g(x) = x + \delta x:$$
$$\delta x^1 = g^1 + x^2 g^3 + x^3 (g^4 \sin(x^4 + c) + g^5 \cos(x^4 + c)),$$
$$\delta x^2 = g^2 - g^3 \operatorname{th} x^4 - g^6 x^2 \operatorname{th} x^4,$$
$$\delta x^3 = g^4 \cos(x^4 + c) - g^5 \sin(x^4 + c),$$
$$\delta x^4 = g^6,$$
$$\delta x^\alpha = \xi_i^\alpha(x)\, g^i,\ X_i = \xi_i^\alpha(x)\, p_\alpha:$$
$$X_1 = p_1,\ X_2 = p_2,\ X_3 = x^2 p_1 - \operatorname{th} x^4 p_2,$$
$$X_4 = x^3 \sin(x^4 + c)\, p_1 + \cos(x^4 + c)\, p_3,$$
$$X_5 = x^3 \cos(x^4 + c)\, p_1 - \sin(x^4 + c)\, p_3,$$
$$X_6 = (1/2)(x^2)^2 p_1 - x^2 \operatorname{th} x^4 p_2 + p_4.$$

To obtain the metric of V one can solve the Killing equations for these operators X_i:

$$g_x(dx) = 2dx^1 dx^4 - \operatorname{ch}^2 x^4 (dx^2)^2 - (dx^3)^2 + [(x^3)^2 + c](dx^4)^2 \qquad c = \text{const}. \tag{3.4}$$

This metric automatically satisfies the field equations $R_{\alpha\beta} = 0$ and belongs to the Petrov type II.

3.4. Let us compare (3.1) and (3.4). Note first that the Lie group G is an analytic manifold; hence G induces an analytic structure in its homogeneous space $V = G/H$ so that V is also an analytic manifold and $g_x(dx)$ must be an analytic function of x and must not possess any singularity. It is the result (3.4) that we have obtained.

It follows from this that the singularities in the metric (3.1) (at $x^4 = (n + 1/2)\pi - c$, n being any integer, $\det g_x = 0$) are connected with a "bad" choice of the coordinates. It should be noted that coordinates x^α in (3.1) and in (3.4) are transformed into each other by functions of the class C^0 (i.e. continuous but without continuous derivatives). The metric (3.1) admits only a local group of motions (with the Lie algebra L) whereas the metric (3.4) admits the full group of motions G.

So our method enables one to determine global structure of the Einstein space V, to remove the coordinate singularities in the metric (3.1), and to obtain the correct global expression (3.4) for the metric of V.

3.5. Solving Eq. (2.9) we obtain:

$$h = (p, 0, q, r\sin c, r\cos c, 0)\,.$$

Here the parameters q and r describe H (Eq. (2.9) is trivially satisfied for all $h \in H$), the parameter p describes the center $C(G)$ of G. Choosing a nontrivial discrete subgroup $N \subset C(G)$ we obtain the group $\tilde{G} = G/N$ which is homeomorphic to $\mathbb{R}^5\,\mathbb{T}^1$ (topological product of the Euclidean space $\mathbb{R}^5$ and circle $\mathbb{T}^1$). It is immediately stated that the corresponding homogeneous space $\tilde{V}$ has topology $\mathbb{R}^3\,\mathbb{T}^1$, since now $x^1 \in \mathbb{T}^1$, i.e. the axis x^1 is rolled up into a circle.

§ 4. Topology of the Space T_3^*

4.1. The highly mobile Einstein space V of the Petrov type III ($\varkappa \neq 0$) has the metric

$$\begin{aligned} g_x(dx) = \exp(-2x^4)\,(2e_1\,dx^1\,dx^3 - (dx^2)^2) + 2e_2\exp(x^4)\,dx^2\,dx^3 \\ -(1/2)\exp(4x^4)\,(dx^3)^2 - (3/\varkappa)\,(dx^4)^2\,, \\ e_1, e_2 = \pm 1, \qquad \varkappa > 0 \end{aligned} \tag{4.1}$$

and admits the Lie algebra L of Killing vectors spanned by the operators

$$X_i = p_i\ (i=1,2,3),\ \ X_4 = p_4 + \sum_i n_i x^i p_i\ (n_1 = 4, n_2 = 1, n_3 = -2) \tag{4.2}$$

(see [1], § 30).

4.2. L has the structure

$$[X_i X_j] = 0,\ \ [X_i X_4] = n_i X_i\ (i,j = 1,2,3)\,.$$

Its center is trivial and so to find the group G with the algebra L one may use the adjoint representation:

$$a(g) = \begin{bmatrix} -n_1 g^4 & & & n_1 g^1 \\ & -n_2 g^4 & & n_2 g^2 \\ & & -n_3 g^4 & n_3 g^3 \\ & & & 0 \end{bmatrix}$$

and the exponential mapping:

$$A(g) = \exp a(g) = \begin{bmatrix} \exp(-n_1 g^4) & & & n_1 g^1 E(n_1 g^4) \\ & \exp(-n_2 g^4) & & n_2 g^2 E(n_2 g^4) \\ & & \exp(-n_3 g^4) & n_3 g^3 E(n_3 g^4) \\ & & & 1 \end{bmatrix}$$

$$E(x) = (1 - \exp(-x))/x .$$

After the transformation

$$a^i = n_i g^i E(n_i g^4) \ (i = 1, 2, 3), \quad a^4 = -g^4$$

we obtain the more simple matrix representation of G:

$$B(g) = \begin{bmatrix} \exp(n_1 a^4) & & & a^1 \\ & \exp(n_2 a^4) & & a^2 \\ & & \exp(n_3 a^4) & a^3 \\ & & & 1 \end{bmatrix}$$

and (from the relation $B(ab) = B(a)B(b)$) the following composition law:

$$(ab)^i = a^i + b^i \exp(n_i a^4), \quad (ab)^4 = a^4 + b^4 .$$

G is homeomorphic to $\mathbb{R}^4$.

4.3. Construct now $V = G/H$ (G is transitive in V). We have: $\dim H = \dim G - \dim V = 0$, so that H is discrete. The connected discrete subgroup $H \subset G$ is just the trivial subgroup $H_0 = \{e\}$ consisting of the only unit element $e \in G$. The corresponding homogeneous space $V = G/H_0$ coincides (as manifold) with G so that V is also homeomorphic to $\mathbb{R}^4$.

G acts on V as follows:

$$(g, x) \mapsto g(x) = gx \quad (g \in G, x \in V = G) .$$

When g^α's are infinitesimal we obtain:

$$gx = x + \delta x ,$$

$$\delta x^i = g^i + n_i g^4 x^i, \quad \delta x^4 = g^4 .$$

From here we find the infinitesimal operators. They coincide with (4.2) so that coordinates x^α in $V = G/H_0$ are the same as in (4.2) and we may use the expression (4.1) for the metric of V.

4.4. Solving Eq. (2.9) we obtain the trivial result: $h = (0, 0, 0, 0) = e$. Hence homogeneous spaces G/H with discrete nontrivial H do not admit the Riemannian structure defined by the scalar product (4.1).

§ 5. Topology of the Space T_3

5.1. The highly mobile Einstein space T_3 is not determined yet. The previous result of Petrov ([1], § 30) has turned out to be wrong. Einstein spaces T_3 possessing groups of motions have been considered in [4–6] and it has been shown that there exists a space V with the metric

$$\begin{aligned} g_x(dx) = &-(x^3)^2/(x^2)^3 \cdot ((dx^1)^2 + (dx^2)^2) + 2dx^3\,dx^4 \\ &-(3x^2/2)\,(dx^4)^2 \end{aligned} \tag{5.1}$$

and with the Lie algebra L of Killing vectors

$$X_1 = p_1,\ X_2 = p_4,\ X_3 = -2x^1 p_1 - 2x^2 p_2 - x^3 p_3 + x^4 p_4\,.$$

Make the transformation:

$$x^4 \to x^1,\ x^1 \to x^2,\ \ln x^2 \to -2x^3,\ 2\ln x^3 - \ln x^2 \to 2x^4\,,$$

after which we have:

$$\begin{aligned} g_x(dx) = &-(3/2)\exp(-2x^3)\,(dx^1)^2 - \exp(4x^3 + 2x^4)\,(dx^2)^2 \\ &-4\exp(2x^4)\,(dx^3)^2 + 2\exp(x^4 - x^3)\,dx^1(dx^4 - dx^3)\,, \end{aligned} \tag{5.2}$$

$$X_1 = p_1,\ X_2 = p_2,\ X_3 = x^1 p_1 - 2x^2 p_2 + p_3\,. \tag{5.3}$$

In new coordinates the equations of the transitivity manifolds are $x^4 = \text{const}$. Besides, this new metric (5.2) does not possess coordinate singularities (the old one (5.1) has such singularities on the axes x^2 and x^3 and, besides this, has incorrect signature in the region $x^2 < 0$).

5.2. The theory of topological groups enables one to determine the global structure of the transitivity manifolds $V_3 \subset V$. We proceed as before.

L has structure

$$[X_1 X_2] = 0,\ [X_1 X_3] = X_1,\ [X_2 X_3] = -2X_2\,.$$

Its center is trivial so that we can use the adjoint representation:

$$a(g) = \begin{bmatrix} -g^3 & 0 & g^1 \\ & 2g^3 & -2g^2 \\ & & 0 \end{bmatrix}$$

and the exponential mapping:

$$A(g) = \exp a(g) = \begin{bmatrix} \exp(-g^3) & 0 & g^1 E(g^3) \\ & \exp(2g^3) & -2g^2 E(-2g^3) \\ & & 1 \end{bmatrix}$$

It will be convenient to transform the coordinates g^i:

$$B(a)=\begin{bmatrix}\exp(a^3) & 0 & a^1\\ & \exp(-2a^3) & a^2\\ & & 1\end{bmatrix}$$

The composition law in group G with the algebra L is as follows:

$$(ab)^1=a^1+b^1\exp(a^3),\ (ab)^2=a^2+b^2\exp(-2a^3),\ (ab)^3=a^3+b^3\,.$$

Here $V_3=G/H$ and $\dim H=0$ so that we must take the trivial subgroup $H_0=\{e\}$ as discrete connected subgroup of G. Hence V_3 coincides (as manifold) with G and we have (cf. § 4):

$$(g,x)\mapsto g(x)=gx \qquad (g\in G,\ x\in V_3=G)\,.$$

Taking infinitesimal g^i's we obtain the same operators (5.3) so we can use the metric (5.2).

G is obviously homeomorphic to $\mathbb{R}^3$, and so V_3. The Eq. (2.9) in this case has the only trivial solution $h=e$, hence the topology of V_3 is unique.

5.3. The whole space V can have, in principle, very complicated topology. To determine it one must introduce some additional demands about V (e.g. geodesic completeness).

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