

Bosons in Thermal Contact: A C^* -Algebraic Model*

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Abstract. A gas of two Boson systems coexisting in \mathbf{R}^3 , and interacting only mutually, is analyzed. The interaction is quadratic, so that the dynamical problem may be solved completely and exactly.

The initial state is taken to be the mutually uncorrelated Gibbs states: $\phi_\beta^{(1)} \otimes \phi_\beta^{(2)} = \psi_\beta$. We find the time evolved state, and its projections onto the separate species and the sub-volumes.

The principle consequences of this model are discussed. In particular we examine the possible occurrence of harmonic oscillations between the species.

§ 1. Introduction

In this note we present a model consisting of two species of Bosons, 1 and 2, coexisting in \mathbf{R}^3 . The species are, by themselves, free; but between them there is an interaction, quadratic in the fields. That being so, a “linear combination of fields” transformation “diagonalizes” the Hamiltonian. In the diagonal frame, there are two types of “quasiparticles” which we denominate a and b; they evolve in time by quasifree evolutions.

The principle consequence of the interaction is that persistent interchanges between the original species occur. A local observable formed from 1 before interaction will be formed from both 1 and 2 after interaction.

As persistent interchange is typical of hyperusasiastic¹ leakage behaviour such as in “superfluid leak”, this model may be relevant to a semi-phenomenological treatment of such phenomena [1, 2].

The paper is organized as follows. In §2 we discuss the necessary configurational geometry, introduce the appropriate Fock spaces, and the pertinent algebras. The next section (§3) discusses the constituent Gibbs states and the initial state of the system constructed from them.

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¹ A Hellenized generic term for super-substances such as Superfluids (J. M. Fergusson).

The dynamics will be introduced in § 4 wherein the dynamical solution is given. The principle consequences are discussed in § 5, including the projections to the local regions. In § 6 we add some further remarks.

§ 2. Model Kinematics

As mentioned in § 1, there are two different species of Bosons, $\underline{1}$ and $\underline{2}$, mutually coexisting in \mathbf{R}^3 , their configuration space. These Boson systems, labelled Σ_1 and Σ_2 , constitute the component subsystems of the model system Σ . As we shall see in § 3, Σ_1 and Σ_2 do not self-interact; they do, however, interact with each other. If the mutual $\Sigma_1 - \Sigma_2$ interaction were removed, Σ_1 and Σ_2 would be infinite ideal Bose gases [3–5].

Our local regions (“volumes”) are elements of the family, \mathbf{I} , of open, relatively compact, and star-shaped subsets of \mathbf{R}^3 . For notational brevity, $\alpha \in \mathbf{I}'$ will mean either $\alpha \in \mathbf{I}$ or that α is \mathbf{R}^3 itself.

Recall [6] that α being relatively compact means that its closure, $\bar{\alpha}$, is compact; by the Borel-Lebesgue theorem, α is therefore bounded, since \mathbf{R}^3 is a Montel space [7].

Recall also that α is said to be star-shaped at the point $\zeta \in \alpha$ iff there is a family $(\eta_a)_{\mathbf{A}}$ where $\eta_a \in \mathbf{R}^3$ for every $a \in \mathbf{A}$ (an index set) such that

$$\alpha = \{[\zeta, \eta_a] \mid a \in \mathbf{A}\}. \quad (1a)$$

By $[\zeta, \eta_a]$ we mean the convex line segment with endpoints ζ and η_a .

Let $r > 1$ be a real number; by α^r we shall mean the r -dilatation of the above set α , namely

$$\alpha^r = \{[\zeta, r\eta_a] \mid a \in \mathbf{A}\}. \quad (1b)$$

In particular, let us choose some $\zeta \in \mathbf{I}$ and impose the convention that one chosen interior point of ζ , which ζ is star-shaped at, will be the *origin* of configuration space.

We also choose some length scale $L \geq 1$, and abbreviate

$$\zeta^{(nL)} = \zeta_n \quad (n \in \mathbf{N}). \quad (2a)$$

Our convention is to take $\mathbf{N} = \{1, 2, 3, \dots\}$. Furthermore, any otherwise unexplained integer sub- or superscript will inevitably refer the corresponding volume $\zeta_{(\cdot)}$.

We assume ζ, L and the origin chosen once and for all, and set

$$\mathbf{I}_L = (\zeta_n)_{\mathbf{N}} \subset \mathbf{I}; \quad (2b)$$

notice that \mathbf{I}_L is an open cover of \mathbf{R}^3 .

To each $\alpha \in \mathbf{I}'$ we associate Boson Fock spaces, one for each species:

$$\mathfrak{F}_j(\alpha) = \sum_{n=0}^{\infty} \oplus \text{symm} \left[\bigotimes_{p=1}^n \mathbf{L}^2(\alpha) \right]; \quad (j=1, 2). \quad (3a)$$

The $n=0$ term is \mathbf{C} as a Euclidean space, by convention; $j=1$ or 2 corresponds to Σ_1 and Σ_2 respectively; and *symm* is the symmetrizing operator.

It is convenient to abbreviate

$$\mathfrak{F}_j(\xi_n) = \mathfrak{F}_j^{(n)} \quad (n \in \mathbf{N}), \quad (3b)$$

and

$$\mathfrak{F}_j(R^3) = \mathfrak{F}_j. \quad (3c)$$

The Fock spaces for the system Σ are constructed from those above by means of the Hilbert space tensor product operation:

$$\mathfrak{F}(\alpha) = \mathfrak{F}_1(\alpha) \otimes \mathfrak{F}_2(\alpha) \quad (\alpha \in \mathbf{I}'); \quad (4a)$$

and we introduce the abbreviations corresponding to (3b) and (3c):

$$\mathfrak{F}^{(n)} = \mathfrak{F}_1^{(n)} \otimes \mathfrak{F}_2^{(n)}, \quad (4b)$$

and

$$\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2. \quad (4c)$$

In what follows it will be very convenient to have a notation for direct sums of certain function spaces, such as $\mathcal{D}_r(\alpha)$, where $\alpha \in \mathbf{I}$ and the r -subscript denotes real-valued. We propose to write, e.g.,

$${}^2\mathcal{D}_r(\alpha) = \mathcal{D}_r(\alpha) \oplus \mathcal{D}_r(\alpha), \quad (5a)$$

$${}^4\mathcal{D}_r(\alpha) = {}^2\mathcal{D}_r(\alpha) \oplus {}^2\mathcal{D}_r(\alpha). \quad (5b)$$

In addition, if $u = u_1 \oplus u_2 \in {}^2\mathcal{D}_r(\alpha)$, we refer to u_1 (resp. u_2) as the first (resp. second) projection of u .

We now introduce the Weyl operators pertinent to this model. For $\alpha \in \mathbf{I}'$, the $\mathbf{L}_r^2(\alpha)$ – inner product is denoted $(\cdot, \cdot)_\alpha$.

Let

$${}^2\sigma: {}^2[\mathbf{L}_r^2(\alpha)] \times {}^2[\mathbf{L}_r^2(\alpha)] \rightarrow \mathbf{R} \quad (6a)$$

be the symplectic form given through the formula

$${}^2\sigma(u, v) = (u_1, v_2)_\alpha - (u_2, v_1)_\alpha. \quad (6b)$$

If H is any Hilbert space, by $\mathbf{B}(H)$ we shall mean the set of all bounded operators on H . Unless otherwise specified, it will be assumed to be equipped with the weak *-topology $\sigma(\mathbf{B}(H), \mathbf{B}(H)_*)$, whence it is a W^* -algebra [9].

For Σ_1 and Σ_2 we shall need the CCR representations associated with the above symplectic form [Eq. (6)]:

$$\begin{aligned} W_j &: {}^2\mathcal{D}_r(\alpha) \rightarrow \mathfrak{F}_j(\alpha), \\ W_j(u) W_j(v) &= \exp[i^2\sigma(u, v)] \cdot W_j(u + v), \end{aligned} \quad (7)$$

for $\forall u, v \in {}^2\mathcal{D}_r(\alpha); \alpha \in \mathbf{I}$; and $j = 1, 2$.

Our usual abbreviation shall hold:

$$W_j(\xi_n) = W_j^{(n)}. \quad (8)$$

For Σ we must double up by direct sum and by tensor product. Let us write

$${}^4\sigma: {}^4[\mathbf{L}_r^2(\alpha)] \times {}^4[\mathbf{L}_r^2(\alpha)] \rightarrow \mathbf{R} \quad (9a)$$

for the symplectic form

$${}^4\sigma(u \oplus v, u' \oplus v') = {}^2\sigma(u, u') + {}^2\sigma(v, v'). \quad (9b)$$

Now we are in a position to define the CCR Weyl operator for Σ constructed from W_1 and W_2 , namely: to each $\alpha \in \mathbf{I}$, let

$$W: {}^4\mathcal{D}_r(\alpha) \rightarrow \mathfrak{F}(\alpha) = \mathfrak{F}_1(\alpha) \otimes \mathfrak{F}_2(\alpha), \quad (10a)$$

be defined by

$$W(u \oplus v) = W_1(u) \otimes W_2(v). \quad (10b)$$

Thus, for every $z', z \in {}^4\mathcal{D}_r(\alpha)$, the corresponding Weyl relations are:

$$W(z) W(z') = \exp[i^4\sigma(z, z')] W(z + z'). \quad (10c)$$

Next we shall introduce the quasiparticle Weyl operators, although there is no justification for the ‘‘quasiparticle’’ nomenclature at this point. For $p = \underline{a}, \underline{b}$, $\alpha \in \mathbf{I}$, let the Weyl operators

$$Q_p: {}^2\mathcal{D}_r(\alpha) \rightarrow \mathfrak{F}(\alpha) \quad (11a)$$

be defined by

$$Q_a(u) = W(u \oplus u), \quad (11b)$$

and

$$Q_b(u) = W(u \oplus -u). \quad (11c)$$

It then follows from (11b–c) and (10c) that the corresponding Weyl relations are

$$Q_p(u) Q_p(v) = \exp[2i^2\sigma(u, v)] Q_p(u + v). \quad (11d)$$

The following formulae enable us to transform between the $(\underline{1}, \underline{2})$ description and the $(\underline{a}, \underline{b})$ description. They constitute, therefore, the quasiparticle transformation for this model:

$$W_1(u) \otimes W_2(v) = Q_a(\frac{1}{2}u + \frac{1}{2}v) Q_b(\frac{1}{2}u - \frac{1}{2}v), \quad (12)$$

$$Q_a(u) Q_b(v) = W_1(u + v) \otimes W_2(u - v). \quad (13)$$

From the Weyl operators we construct the C^* -algebras. It might be worth emphasizing that our algebras are not W^* -algebras, which choice is forced upon us by the instability of $\mathcal{D}_r(\mathbf{R}^3)$ under time translations.

The Boson local C^* -algebras are taken to be

$$\mathfrak{A}_j(\alpha) = uc - \langle W_j(u) | \forall u \in {}^2\mathcal{D}_r(\alpha) \rangle, \quad (14)$$

where the angular brackets denote the free polynomial algebra formed from the Weyl operators, modulo the (canonical commutation) relations (7). The uc -denotes uniform closure, that is, closure in the operator norm on $\mathfrak{F}_j(\alpha)$.

The $\mathfrak{A}_j(\alpha)$ act on the $\mathfrak{F}_j(\alpha)$ and are irreducible [10]:

$$\mathfrak{A}_j(\alpha)'' = \mathbf{B}(\mathfrak{F}_j(\alpha)). \quad (15)$$

The family \mathbf{I}_L [Eq. (2b)] is upwards directed by inclusion ; and for $\forall n \geq m$ there is an injective mapping, $f_{nm}^{(i)}$, of $\mathfrak{A}_j^{(m)} \equiv \mathfrak{A}_j(\xi_m)$ into $\mathfrak{A}_j^{(n)}$, defined by

$$\begin{aligned} f_{nm}^{(j)}(\mathbf{I}_m) &= \mathbf{I}_n, \\ f_{pn}^{(j)} \circ f_{nm}^{(j)} &= f_{pm}^{(j)} \quad (p \geq n \geq m). \end{aligned}$$

Then $(\mathfrak{A}_j^{(m)}, f_{nm}^{(j)} : n \geq m)_{m \in \mathbf{N}}$ forms an *inductive family*. The inductive limit exists [11] and is called the *quasilocal algebra* for Σ_j :

$$\mathfrak{A}_j = \lim_{n \rightarrow \infty} \{ \mathfrak{A}_j^{(m)}, f_{m,n}^{(j)} | m, n \in \mathbf{N}; m \geq n \}. \quad (16)$$

The quasilocal algebra \mathfrak{A}_j acts on $\mathfrak{F}_j = \mathfrak{F}_j(\mathbf{R}^3)$ and is a proper subalgebra of $\mathbf{B}(\mathfrak{F}_j)$. Note that \mathfrak{A}_j does not depend upon \mathbf{I} .

For the system Σ we take the local C^* -algebras to be

$$\mathfrak{A}(\alpha) = \mathfrak{A}_1(\alpha) \otimes \mathfrak{A}_2(\alpha) \quad (\forall \alpha \in \mathbf{I}), \quad (17)$$

where the symbol \otimes denotes the C^* -tensor product, the completion of the algebraic tensor product in the α_0 -cross norm [12].

For $n \geq m$, the mappings

$$\mathfrak{A}_1^{(m)} \otimes [I_2]_m \rightarrow \mathfrak{A}_1^{(n)} \otimes [I_2]_n, \quad (18a)$$

$$[I_1]_m \otimes \mathfrak{A}_2^{(m)} \rightarrow [I_1]_n \otimes \mathfrak{A}_2^{(n)} \quad (18b)$$

are bounded and extend continuously to injections of $\mathfrak{A}^{(m)} \rightarrow \mathfrak{A}^{(n)}$. Denote these injections by

$$f_{nm} = f_{nm}^{(1)} \otimes f_{nm}^{(2)} : \mathfrak{A}^{(m)} \rightarrow \mathfrak{A}^{(n)}. \quad (18c)$$

Then just as for the Σ_j , the quasilocal algebra for Σ will be taken to be

$$\mathfrak{A} = \lim \{ \mathfrak{A}^{(m)}; f_{mn} | m, n \in \mathbf{N}; m \geq n \}. \quad (19)$$

The algebra \mathfrak{A} is a proper subalgebra of $\mathbf{B}(\mathfrak{F})$ acting on \mathfrak{F} .

§ 3. The Initial State

Let $\phi_\beta^{(1)}$ and $\phi_\beta^{(2)}$ denote the Gibbs states for Σ_1 and Σ_2 at temperature $\beta = (kT)^{-1}$; it is supposed that they were calculated using rigid wall boundary conditions and the family \mathbf{I}_L [5].

Given the Gibbs states, the GNS construction associates with them certain canonical triples, the details of which need not concern us here [5]:

$$\phi_\beta^{(j)} \leftrightarrow (\mathcal{H}_\beta^{(j)}, \pi_\beta^{(j)}, \Omega_\beta^{(j)}). \quad (1)$$

We now define the tensor product space:

$$\mathcal{H}_\beta = \mathcal{H}_\beta^{(1)} \otimes \mathcal{H}_\beta^{(2)} \quad (2a)$$

and vector:

$$\Omega_\beta = \Omega_\beta^{(1)} \otimes \Omega_\beta^{(2)}. \quad (2b)$$

And let $\pi_\beta = \pi_\beta^{(1)} \otimes \pi_\beta^{(2)}$ denote the unique C^* -representation [13]

$$\pi_\beta: \mathfrak{A} \rightarrow \mathbf{B}(\mathcal{H}_\beta), \quad (2c)$$

which is the continuous extension of

$$\pi_\beta^{(1)} \otimes \pi_\beta^{(2)}(a \otimes b) = [\pi_\beta^{(1)}(a) \otimes I_{\mathcal{H}_\beta^{(2)}}] [I_{\mathcal{H}_\beta^{(1)}} \otimes \pi_\beta^{(2)}(b)]. \quad (2d)$$

Note that Ω_β is a cyclic vector for the representation π_β .

We shall take the state of Σ at time $t=0$ to be that state of \mathfrak{A} ,

$$\psi_\beta \in \mathfrak{S}(\mathfrak{A}),$$

constructed from the triple $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$ as follows:

$$\psi_\beta(a) = (\Omega_\beta, \pi_\beta(a) \Omega_\beta) \quad (\forall a \in \mathfrak{A}). \quad (3)$$

The extension of ψ_β to the weak closure $\pi_\beta(\mathfrak{A})''$ is denoted

$$\psi_\beta^\sim \in \mathfrak{S}[\pi_\beta(\mathfrak{A})''],$$

and is given by

$$\psi_\beta^\sim(A) = (\Omega_\beta, A \Omega_\beta) \quad [\forall A \in \pi_\beta(\mathfrak{A})'']. \quad (4)$$

Let $\psi_\beta^{(\alpha)}$ be the restriction of ψ_β to the local C^* -subalgebra $\mathfrak{A}(\alpha)$ ($\alpha \in \mathbf{I}$).

Proposition 3.1. *The restriction mappings $f_\alpha^*: \mathfrak{S}(\mathfrak{A}) \rightarrow \mathfrak{S}(\mathfrak{A}(\alpha))$ are relatively weak*-continuous. The f_α^* are the adjoints of the injections of Eq. (218c).*

Proof. This is a special case of a general situation encountered in algebraic statistical mechanics, namely, let [11] \mathcal{A} be a C^* -inductive limit:

$$\mathcal{A} = \lim_{\alpha \rightarrow} \{\mathcal{A}_\alpha; f_{\beta\alpha} | (\beta, \alpha) \in \mathbf{I} \times \mathbf{I}, \beta \geq \alpha\}. \quad (5)$$

Let \mathcal{A}_α^* be the dual of \mathcal{A}_α as a Banach space, but equipped with the weak*-topology $\sigma(\mathcal{A}_\alpha^*, \mathcal{A}_\alpha)$. By construction, \mathcal{A} is equipped with the finest topology under which the $f_{\beta\alpha}$ are continuous [14]. Define the transpose of these maps by

$$f_{\beta\alpha}^* : \mathcal{A}_\beta^* \rightarrow \mathcal{A}_\alpha^* \quad (\beta \geq \alpha), \quad (6)$$

$$\begin{aligned} f_{\beta\alpha}^* \Phi(a) &= \Phi(f_{\beta\alpha} a) \\ (\forall a \in \mathcal{A}_\alpha, \forall \Phi \in \mathcal{A}_\beta^*). \end{aligned} \quad (7)$$

It is a known theorem [15] that the dual of \mathcal{A} , \mathcal{A}^* , equipped with its weak*-topology $\sigma(\mathcal{A}^*, \mathcal{A})$, is the projective limit of the $[\mathcal{A}_\alpha^*, \sigma(\mathcal{A}_\alpha^*, \mathcal{A}_\alpha)]$ with respect to the mappings $f_{\beta\alpha}^*$. But the projective limit topology is the coarsest topology under which the $f_{\beta\alpha}$ are continuous [16]. As the $f_{\beta\alpha}^*$ are the desired local projections on the states considered as a subset of the weak duals \mathcal{A}_α^* — and consequently equipping the $\mathfrak{S}(\mathcal{A}_\alpha)$ with the (\mathcal{A}_α^*) -relative topology — they are relatively weak*-continuous.

As the C^* -tensor product operation is sufficiently algebraically continuous so as to allow an inductive limit algebra (2.19) for the system Σ , these considerations apply to \mathfrak{A} .

The point of this lemma is to ensure that the restricted states $f_\alpha^*(\psi_\beta)$ do not give rise to any unexpected continuity pathologies.

Note that although ψ_β will be seen to be locally normal, the fact that the $\mathfrak{A}(\alpha)$ are not W^* -algebras prevents us from proving that the $\psi_\beta^{(\alpha)}$ are such that the corresponding representations $\pi_\beta^{(\alpha)}$ are normal representations on all of \mathcal{H}_β . That is, we have no analogue of Lemma 3.1 of Ref. [17].

The state ψ_β is well-behaved, as seen from the following lemma.

Lemma 3.2. *ψ_β is translationally invariant and has the finite mean density property (locally normal).*

Proof. Let $\delta : \mathbf{R}^3 \rightarrow {}^2[\mathbf{L}_r^2(\mathbf{R}^3)]$ be the unitary translation group, and ${}^2\delta : \mathbf{R}^3 \rightarrow {}^4[\mathbf{L}_r^2(\mathbf{R}^3)]$ be defined by ${}^2\delta_a = \delta_a \oplus \delta_a$. Then

$$\begin{aligned} \psi_\beta \{ W[{}^2\delta_a(u \oplus v)] \} &= \phi_\beta^{(1)} [W_1(\delta_a u)] \phi_\beta^{(2)} [W_2(\delta_a v)] \\ &= \psi_\beta [W(u \oplus v)] \end{aligned} \quad (8)$$

$[\forall u, v \in {}^2\mathcal{D}_r(\mathbf{R}^3)]$ proves the translation invariance.

The generators of the Weyl operators, the fields, are related by, in an obvious notation,

$$\phi(u \oplus v) = [\phi_1(u) + \pi_1(v)] \otimes I_2, \quad (9a)$$

$$\pi(u \oplus v) = I_1 \otimes [\phi_2(u) + \pi_2(v)] \quad [u, v \in {}^2\mathcal{D}_r(\alpha)]. \quad (9b)$$

By using the 2×2 -matrix $\mathbf{J} = (2)^{-1/2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, we can transform from the fields to the annihilation and creation operators, e.g.

$$\mathbf{J} \begin{pmatrix} \phi(z) \\ \pi(z) \end{pmatrix} = \begin{pmatrix} a^*(z) \\ a(z) \end{pmatrix} \quad (z \in {}^4\mathcal{D}_r(\alpha)). \quad (10)$$

This leads to the following expressions

$$N(u \oplus 0) = N_1(u) \otimes I_2, \quad [u, v \in {}^2\mathcal{D}_r(\alpha)] \quad (11a)$$

$$N(0 \oplus v) = I_1 \otimes N_2(v) \quad (11b)$$

for the number operators, and thence to

$$N(\alpha) = N_1(\alpha) \otimes I_2 + I_1 \otimes N_2(\alpha) \quad (11c)$$

on $\mathfrak{F}(\alpha)$.

From this it follows that

$$\psi_\beta(N(\alpha)) = \phi_\beta^{(1)}(N_1(\alpha)) + \phi_\beta^{(2)}(N_2(\alpha)), \quad (12)$$

or in terms of densities,

$$\varrho_\beta(\alpha) = \varrho_\beta^{(1)}(\alpha) + \varrho_\beta^{(2)}(\alpha). \quad \square \quad (13)$$

Corollary 3.3 (Lanford and Robinson) [18]. (i) ψ_β extends continuously to a CCR representation from ${}^4\mathcal{D}_r(\mathbf{R}^3)$ to $\mathbf{B}(\mathcal{H}_\beta)$; (ii) if

$$\Gamma : {}^4\mathcal{D}_r(\mathbf{R}^3) \rightarrow {}^4\mathcal{S}_r(\mathbf{R}^3) \quad (14)$$

is a bounded automorphism of ${}^4\mathbf{L}_r^2(\mathbf{R}^3)$, then it induces an automorphism of $\pi_\beta(\mathfrak{A})'$ defined by

$$\psi_\beta \circ \pi_\beta[W(z)] \rightarrow \psi_\beta \circ \pi_\beta[W(\Gamma z)]. \quad \square \quad (15)$$

For $\phi_\beta^{(j)}$ ($j = 1, 2$) a similar result is valid.

§ 4. Model Dynamics

Let us establish some notation. The one-particle kinetic energy operator, denoted by κ , may be defined by the bilinear form [19]

$$\begin{aligned} \mathfrak{R} : \mathbf{L}^2(\mathbf{R}^3) \times \mathbf{L}^2(\mathbf{R}^3) &\rightarrow \mathbf{C}, \\ \mathfrak{R}(\bar{f}, g) &= \int_{\mathbf{R}^3} \bar{f}(\mathbf{k}) g(-\mathbf{k}) \frac{k^2}{2m} d\mathbf{k}, \end{aligned} \quad (1)$$

where \bar{f} is the Fourier image of $f \in \mathbf{L}^2(\mathbf{R}^3)$. The domain of \mathfrak{R} is

$$\mathbf{D}(\kappa) = \{f \in \mathbf{L}^2(\mathbf{R}^3) \mid \mathfrak{R}(f, f) < \infty\}. \quad (2)$$

One also writes

$$\mathfrak{R}(f, g) = (f, \kappa g). \quad (3)$$

We shall define Hamiltonians associated with the local regions. Then we write

$$\kappa|\gamma$$

for κ restricted to $\gamma \in \mathbf{I}$; in particular we write

$$\kappa_n = \kappa|\xi_n. \quad (4)$$

If $\mathfrak{a} \in \mathbf{B}(\mathbf{L}^2(\gamma))$ is a γ -localized one-particle operator, then the corresponding Fock space operator is constructed by means of the Fock-Cook biquantization operator [20]

$$\begin{aligned} \mathbf{B}(\mathbf{L}^2(\gamma)) &\rightarrow \mathfrak{F}(\gamma), \\ \omega(\mathfrak{a}) &= 0 \oplus \mathfrak{a} \oplus [\mathfrak{a} \otimes 1 + 1 \otimes \mathfrak{a}] \oplus \dots, \end{aligned} \quad (5)$$

which we formally extend to the closed densely-defined operators on $\mathbf{L}^2(\gamma)$ such as $\kappa|\gamma$. We also write ω_1 and ω_2 for the biquantization on \mathfrak{F}_1 and \mathfrak{F}_2 .

Let v be a potential operator on $\mathbf{L}^2(\mathbf{R}^3)$ with kernel $V(\mathbf{x})$; we shall ordinarily have V of compact support, bounded and reasonably smooth.

In heuristic terms we shall be interested in local Hamiltonians of the form

$$\begin{aligned} H(\gamma) &= \omega_1(\kappa|\gamma) \otimes I_2 + I_1 \otimes \omega_2(\kappa|\gamma) \\ &\quad + (a_1^* \otimes a_2 + a_1 \otimes a_2^*)(v). \end{aligned} \quad (6)$$

The a_j, a_j^* are the annihilation and creation operators following from the fields as in Eq. (3.10).

Equation (6) is *not* the axiomatic way that we shall define the model dynamics [see Eq. (18) below]. It does convey the essential idea behind the axiomatic definition, however. In fact, (6) points the way towards the diagonalization procedure. For $H(\gamma)$ is quadratic; one suspects, therefore, that completing the square will diagonalize it. Of course one must be careful about the kernels [e.g. $(\kappa + v)|\gamma$] used to complete the square.

If one formally follows this idea through, one is led to those fields which generate Q_a and Q_b defined in Eq. (2.11). In terms of these \underline{a} and \underline{b} fields, $H(\gamma)$ is equal to $H_a(\gamma) + H_b(\gamma)$, where each term is quadratic and depends upon \underline{a} and \underline{b} alone, respectively.

For mathematical simplicity it is easier to start with the \underline{a} and \underline{b} mode description, and to show that $H(\gamma)$ follows. In this manner one may even generalize $H(\gamma)$ slightly, at no extra effort.

Clearly the evolution in terms of \underline{a} and \underline{b} is separately quasifree, since $H_a(\gamma), H_b(\gamma)$ are quadratic. We start, therefore, by introducing one-particle evolutions for Q_a and Q_b .

For $p = \underline{a}, \underline{b}$, let $\mu_p^{(n)}: \mathbf{R} \rightarrow \mathbf{B}(\mathbf{L}^2(\xi_n))$ be strongly continuous unitary groups such that

$$\mu_p^{(n)}(t) \mathcal{D}(\xi_n) \subset \mathcal{D}(\xi_n); \quad (7)$$

The generator of $\mu_p^{(n)}$ is written $a_p^{(n)}$. By ${}^2\mu_p^{(n)}$ we mean $\mu_p^{(n)} \oplus \mu_p^{(n)}$ as usual. We use $\mu_p^{(n)}$ to define an automorphism group of $\mathfrak{A}^{(n)}$ as follows:

$$\begin{aligned} \tau^{(n)}: \mathbf{R} &\rightarrow \text{Aut}[\mathfrak{A}^{(n)}], \\ \tau^{(n)}(t): W(u \oplus v) & \\ &\rightarrow \tau^{(n)}(t) [Q_a(\tfrac{1}{2}u + \tfrac{1}{2}v) Q_b(\tfrac{1}{2}u - \tfrac{1}{2}v)] \\ &= Q_a[{}^2\mu_a^{(n)}(t)(\tfrac{1}{2}u + \tfrac{1}{2}v)] Q_b[{}^2\mu_b^{(n)}(t)(\tfrac{1}{2}u - \tfrac{1}{2}v)]; \end{aligned} \quad (8)$$

or converting back:

Definition 4.1. Let

$$\tau^{(n)}: \mathbf{R} \rightarrow \text{Aut}[\mathfrak{A}^{(n)}]$$

be defined by

$$\tau^{(n)}(t): W(u \oplus v) \rightarrow W\{[{}^2\mu_+^{(n)}(t)u + {}^2\mu_-^{(n)}(t)v] + [{}^2\mu_-^{(n)}(t)u + {}^2\mu_+^{(n)}(t)v]\}, \quad (9)$$

where ${}^2\mu_{\pm}^{(n)} = \mu_a^{(n)} \pm \mu_b^{(n)}$; $\forall u, v \in {}^2\mathcal{D}_r(\xi_n)$. \square

Let us demonstrate that $\tau^{(n)}$ is related to $H(\gamma)$ of Eq. (6). This demonstration is entirely heuristic although there is no reason to suspect the result. We are content with 4.1 above as the *definition* of the local time translations.

If we write

$$\begin{aligned} a_a^{(n)} &= g_1 \kappa_n + g_2 v_n \\ a_b^{(n)} &= g_1 \kappa_n - g_2 v_n \end{aligned} \quad (10)$$

then taking $(g_1 = 1, g_2 = 0)$ we can isolate the kinetic energy part of the evolution; and $(g_1 = 0, g_2 = 1)$ isolates the interaction part.

Then for $(g_1 = 1, g_2 = 0)$, $\mu_+^{(n)}$ is the kinetic energy evolution operator $\exp(it\kappa_n)$; and (9) reduces to

$$\begin{aligned} \tau^{(n)}(t; g_1 = 1; g_2 = 0) W(u \oplus v) \\ = W[{}^2\mu_+^{(n)}(t)u \oplus {}^2\mu_-^{(n)}(t)v]. \end{aligned} \quad (11)$$

This automorphism group is generated by

$$\omega_1(\kappa_n) \otimes I_2 + I_1 \otimes \omega_2(\kappa_n), \quad (12)$$

the kinetic energy part of $H(\gamma)$ [Eq. (6)].

In order to complete the demonstration, we must be able to recognize the action of the interaction part of $H(\gamma)$. Using the formal multiple

commutator expansion of Magnus, the exponential of the $H(\gamma)$ interaction leads to the evolution, e.g.:

$$\phi_1(f) \otimes I_2 \rightarrow \phi_1[\cos(tv) f] \otimes I_2 + I_1 \otimes \phi_2[\sin(tv) f] \quad (13)$$

for $\forall t \in \mathbf{R}, \forall f \in \mathcal{D}_r(\gamma)$. And for example, if we set $g_1 = 0, g_2 = 1$ and $v = 0, u = f \oplus 0$ in (9), we find that

$$\begin{aligned} \tau^{(n)}(t; g_1 = 0; g_2 = 1) W(u \oplus 0) \\ = W_1[\cos(tv) f \oplus 0] \otimes W_2[\sin(tv) f \oplus 0]. \end{aligned} \quad (14)$$

The relation between (13) and (14) is obvious. This ends our examination of the relation between our axiomatic definition of time translations for this model (Definition 4.1) and the heuristic Hamiltonian $H(\gamma)$ [Eq. (6)].

Our next task is the consideration of the limit $n \rightarrow \infty$ for the time translations (9). To do this requires knowledge of the behaviour of the $\mu_p^{(n)}$ in the limit. Using the ideal Bose gas as an example, and considering $\alpha_p^{(n)}$ of (10) in this light, we shall demand that $\mu_p^{(n)}$ satisfy the following conditions [4, 5, 21].

Definition 4.2. There exists a strongly continuous unitary group $\mu_p: \mathbf{R} \rightarrow \mathbf{B}(\mathbf{L}^2(\mathbf{R}^3))$ ($p = \underline{a}, \underline{b}$) satisfying

$$\mu_p(t): \mathcal{D}(\mathbf{R}^3) \rightarrow \mathcal{S}(\mathbf{R}^3) \quad (15)$$

and defined by

$$s - \mathbf{L}^2(\mathbf{R}^3) \lim_{n \rightarrow \infty} \mu_p^{(n)}(t) f = \mu_p(t) f \quad (16)$$

for every $f \in \mathbf{L}_0^2(\mathbf{R}^3)$, elements of $\mathbf{L}^2(\mathbf{R}^3)$ of compact support. We shall denominate the generators of $\mu_p(t)$ as $\alpha_p(p = \underline{a}, \underline{b})$; by Stone's theorem,

$$\mu_p(t) = \exp(it \alpha_p). \quad \square \quad (17)$$

The limit $n \rightarrow \infty$ for Eq. (9) is, in view of (15) and (16), identical to the similar problem for the ideal Bose gas. The solution is [4, 5, 21]

$$\begin{aligned} s - \mathfrak{F}(\mathbf{R}^3) \lim_{n \rightarrow \infty} \tau^{(n)}(t) W(u \oplus v) \\ = W \{ [\mu_+(t) u + \mu_-(t) v] \oplus [\mu_-(t) u + \mu_+(t) v] \}, \end{aligned} \quad (18)$$

with

$$2\mu_{\pm}(t) = \mu_a(t) \pm \mu_b(t); \forall u, v \in {}^2\mathcal{D}_r(\mathbf{R}^3).$$

Equation (18) gives the dynamical solution for our model; it is to be interpreted as in Corollary 3.3, in view of our choice of initial state. Namely, as an automorphism group of $\pi_\beta(\mathfrak{A})''$.

Whereas μ_a and μ_b are unitary groups, μ_{\pm} are not. In the $(\underline{a}, \underline{b})$ -mode description, the evolution is unitarily implemented on \mathcal{H}_β . As it is the

($\underline{1}, \underline{2}$) – mode description which is of primary interest, let us define the family of automorphisms (not unitarily implemented!):

Definition 4.3. Let

$$T_\beta: \mathbf{R} \rightarrow \text{Aut}[\pi_\beta(\mathfrak{A}'')] \quad (19)$$

be defined by

$$\begin{aligned} s - \mathcal{H}_\beta \lim_{n \rightarrow \infty} \pi_\beta[\tau^{(n)}(t) W(u \oplus v)] \\ = T_\beta(t) \pi_\beta[W(u \oplus v)] \quad [u, v \in {}^2\mathcal{D}_r(\mathbf{R}^3)] \\ = \pi_\beta^{(1)}\{W_1[{}^2\mu_+(t)u + {}^2\mu_-(t)v]\} \\ \otimes \pi_\beta^{(2)}\{W_2[{}^2\mu_-(t)u + {}^2\mu_+(t)v]\}. \quad [u, v \in {}^2\mathcal{D}_r(\mathbf{R}^3)] \end{aligned} \quad (20)$$

The last expression in (20) is in terms of W_1 and W_2 , and follows from substituting Eq. (2.10b) in Eq. (18).

§ 5. Consequences

Let $(\Gamma_{\beta, \lambda})$ be the family of Bose-Einstein operators on $\mathbf{L}^2(\mathbf{R}^3)$ associated with the Gibbs state for the ideal Bose gas: $(\beta, \mu) \in \mathbf{R}_+ \times \mathbf{R}_-$ with inverse temperature β , and chemical potential μ . Each $\Gamma_{\beta, \mu}$ defines a bilinear form (see Eqs. (4.1–4.3 for example) on $\mathbf{L}^2(\mathbf{R}^3)$ through the expression

$$(\bar{f}, \Gamma_{\beta, \mu} g) = \int_{\mathbf{R}^3} \bar{f}(\mathbf{k}) g^*(-\mathbf{k}) \mathfrak{S}(\beta, \mu, ; \mathbf{k}) d\mathbf{k}, \quad (1)$$

the kernel for $\Gamma_{\beta, \mu}$ is the Bose-Einstein distribution

$$\mathfrak{S}(\beta, \mu; \mathbf{k}) = \begin{cases} \left[1 - \exp\left(\beta \frac{k^2}{2m} - \beta\mu\right)\right]^{-1} & 0 < \beta < \beta_c; \mu < 0 \\ \left[1 - \exp\left(\beta \frac{k^2}{2m}\right)\right]^{-1} + \lambda \delta^{(3)}(\mathbf{k}) & \beta \geq \beta_c; \mu = 0, \end{cases} \quad (2)$$

with λ proportional to the mean superfluid density $\bar{\rho} - \rho_c$ [4, 5].

If $u = u_1 \oplus u_2, v = v_1 \oplus v_2 \in {}^2\mathbf{L}_0^2(\mathbf{R}^3)$, we shall abbreviate

$$\langle u, {}^2\Gamma v \rangle = (u_1, \Gamma v_1) + (u_2, \Gamma v_2), \quad (3)$$

omitting the β, μ indices when no confusion is likely to result. In the same spirit we shall write

$$\langle u, v \rangle = (u_1, v_1) + (u_2, v_2), \quad (4)$$

which is relevant to the Fock state component of the Gibbs state. The Gibbs states for the algebras \mathfrak{A}_1 and \mathfrak{A}_2 are given by

$$\phi_\beta^{(j)}[W_j(u)] = \exp\left\{-\frac{1}{4}\langle u, u \rangle - \frac{1}{2}\langle u, {}^2\Gamma u \rangle\right\} \quad (j = 1, 2), \quad (5)$$

so that

$$\varphi_\beta[W(u \oplus v)] = \exp\left\{-\frac{1}{4}\langle u, u \rangle - \frac{1}{4}\langle v, v \rangle - \frac{1}{2}\langle u, {}^2\Gamma u \rangle - \frac{1}{2}\langle v, {}^2\Gamma v \rangle\right\}. \quad (6)$$

Hereafter, we shall write $\langle u, u \rangle = [u]_0^2$ and $\langle u, {}^2Fu \rangle = \{u\}_0^2$. Substituting Eq. (4.20) in this gives:

$$\begin{aligned} & \psi_\beta \circ T_\beta(t) [W(u \oplus v)] \\ &= \exp(-\frac{1}{4}[{}^2\mu_+(t)u + {}^2\mu_-(t)v]_0^2 - \frac{1}{4}[{}^2\mu_-(t)u + {}^2\mu_+(t)v]_0^2) \quad (7) \\ & \quad - \frac{1}{2}\{{}^2u_+(t)u + {}^2\mu_-(t)v\}_0^2 - \frac{1}{2}\{{}^2\mu_-(t)u + {}^2\mu_+(t)v\}_0^2, \end{aligned}$$

for the principle equation of the model. In the full generality of (7), the quasiparticle transfer is not particularly emphasized. It is, however, if we specialize to $v=0$, say. If in addition we specialize to $u \in {}^2\mathcal{D}_r(\alpha)$, $\alpha \in \mathbf{I}$, we shall be able to consider the full time evolution of the state ψ_β at time t , restricted to $\alpha \in \mathbf{I}$ and to the algebra \mathfrak{A}_1 . This partial state will be denoted $\sigma_1(t; \alpha) \psi_\beta$ and is a state on $\pi_\beta^{(1)}(\mathfrak{A}_1)''$.

Then a simple computation gives:

$$\begin{aligned} & \sigma_1(t; \alpha) \psi_\beta \{ \pi_\beta^{(1)} [W_1(u)] \} \\ &= \exp(-\frac{1}{4}[{}^2\mu_+(t)u]_0^2 - \frac{1}{4}[{}^2\mu_-(t)u]_0^2) \quad (8) \\ & \quad - \frac{1}{2}\{{}^2\mu_+(t)u\}_0^2 - \frac{1}{2}\{{}^2\mu_-(t)u\}_0^2. \end{aligned}$$

Let us note here that the generalized master equation of Ref. [17] is an equation for the first (strong-) time derivative of a partial state analogue to $\sigma_1(t; \alpha) \psi_\beta$. There is the difference that the $\hat{\cdot}$ -system of Ref. [17] is finite and no additional projection to a finite subvolume is necessary there.

That (8) is associated with an interchange between the two species of Bosons is seen by viewing (8) in conjunction with the expression

$$\begin{aligned} & T_\beta(t) \circ \pi_\beta [W(u \oplus 0)] \\ &= \pi_\beta^{(1)} [W_1({}^2\mu_+(t)u)] \otimes \pi_\beta^{(2)} [W_2({}^2\mu_-(t)u)]. \quad (9) \end{aligned}$$

It is worth noting that the time dependence in Eqs. (7)–(9) is not spurious; it does not generally cancel out. One may in fact choose the potential so that the spectra of $\kappa \pm v$ is of the form

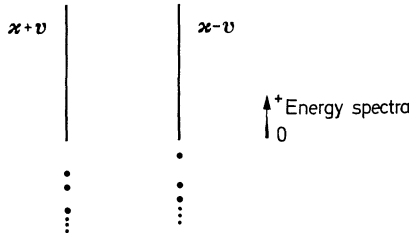


Fig. 1

That is, there is a discrete spectrum outside the essential spectrum in both cases. The discrete spectra must have proper values below zero. These values need not be the same in both cases. We have not investigated the conditions on v for the negative part of the discrete spectra to be finite or even consist of one proper value, if that is possible. For it is sufficient for our purposes that such proper values exist. A one-dimensional potential showing such behaviour is an easy example to construct; *viz.* a well-barrier:

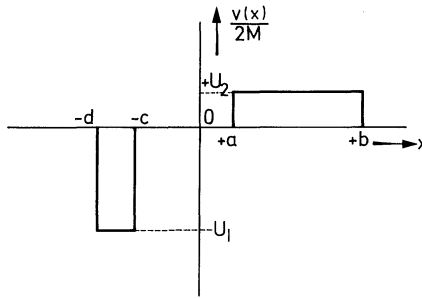


Fig. 2

If $|U_1| = U_2$, $c = a$, and $d = b$ in this example, the discrete spectra of $\kappa \pm v$ are found to be identical, and both are found just as for one finite square well.

In cases such as Figure 1 illustrates, a general observable can exhibit *harmonic response*. By this we mean that resonances corresponding to the discrete spectra below zero can be extracted from the noisy response by a spectral analyzer. This implies that a system described by our model may be triggered externally – by a transducer? – to show resonance. Herein lies a possible connection to the Josephson effect in superfluids [1] which is currently thought *not to have been observed*: the effect described in Reference [1] is now thought to be an experimental error, but the effect described is still expected to occur [22]!

In this regard we point out that these time dependent effects are *persistent* rather than *damped*.

§ 6. Further Remarks

a) One could alter the geometry of this model formally so as to initially confine the two species 1 and 2 to contiguous half-spaces. The

details are similar to this model. The only essential computational change is that the ascending cover replacing \mathbf{I}_L is as follows:

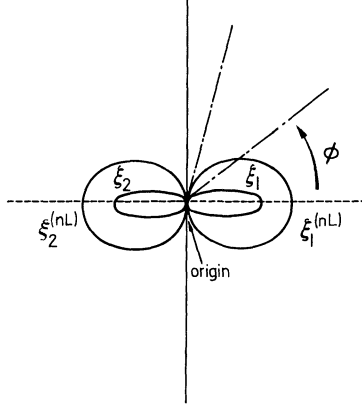


Fig. 3

In terms of polar coordinates, one such dilatation has $[r, \theta] \in \xi_1$ correspond to the point $[nLr, g_n \theta] \in \xi_1^{(nL)}$, where

$$g_n = \frac{\pi}{2\phi} + \left(1 - \frac{\pi}{2\phi}\right) e^{1-n}.$$

The major difficulty with such a model lies in the interpretation of time translation formulae; space translational invariance is clearly broken, and it is not clear what to make of v . The formal calculations are identical, in any event.

b) A comparison of this model with the abstract model of Ref. [17] shows a great similarity. One suspects that if the local algebras of a model were type I W^* -algebras, with both components, $\underline{1}$ and $\underline{2}$ giving rise to Type I funnels:

$$\mathcal{A}_j = \lim_{n \rightarrow \infty} \{\mathcal{A}_j^{(n)}\}, \quad (j = 1, 2) \quad (6.1)$$

then the composite system

$$\mathcal{A} = \lim_{n \rightarrow \infty} \{\mathcal{A}_1^{(n)} \bar{\otimes} \mathcal{A}_2^{(n)}\} \quad (6.2)$$

is also a Type I funnel. This is obviously so, when $\bar{\otimes}$ is a W^* -tensor product.

Moreover, the two crucial lemmata, 2.3 and 3.1 of [17], concerning the uniform convergence of states of Σ defined by states of Σ_1 and Σ_2 which are themselves uniform limits of states; and the natural behaviour

of local normality and normal representation under such W^* -tensor products, both still hold. It follows that the principle result of [17], the generalized master equation for sufficiently good interaction between Σ_1 and Σ_2 may still be valid for two Type I funnels.

c) The proof of Proposition 3.1 seems to point out that the duality between inductive and projective limit topologies is an important technical point when considering composite systems in thermal contact. It might possibly prove useful in considering the behaviour of subsystems of homogeneous systems, which are not usual thought of as composite. A model along these lines has been discussed by Emch and Radin [24].

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