Commun. math. Phys. 31, 83–112 (1973)

Axioms for Euclidean Green's Functions

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Received December 18, 1972

Abstract. We establish necessary and sufficient conditions for Euclidean Green's functions to define a unique Wightman field theory.

Contents

1. Introduction

In a relativistic quantum field theory the indefinite metric of Minkowski space causes many problems which could be avoided by replacing the time t by it or the energy E by iE, thereby passing from Minkowski space to Euclidean space. This idea was first used by Dyson [3] in perturbation theory. He continued the Feynman integrands analytically to imaginary energies in order to move the paths of integration away from the mass shell singularities of the causal propagators. Schwinger [21, 22] studied the analytic continuation of time ordered

* Supported by the National Science Foundation under grant GP 31239X.

** Supported in part by the Air Force Office of Scientific Research, contract AF 44620-70-C-0030.

Green's functions to imaginary times and their transformation properties under the Euclidean group. He determined the Euclidean Green's functions (Schwinger functions) as solutions of certain differential equations. In axiomatic field theory it followed from investigations by Wightman [28], by Hall and Wightman [10] and by Jost [13] that the Green's functions (Wightman distributions) are boundary values of functions (Wightman functions) which are analytic in the permuted extended tubes. The Euclidean Green's functions could then be defined as the restriction of the Wightman functions to points with imaginary time and real space components [20, 24]. Symanzik [24, 25] advocated a purely Euclidean approach to quantum field theory: he realized, that given a formal Lagrangian density, the construction of Euclidean Green's functions might be simpler than the direct construction of Wightman distributions. Postponing the problem of continuing back to real time he studied the Euclidean Green's functions for models with boson self-interaction and established a useful connection to classical statistical mechanics. An abstract formulation was introduced by Nelson [15, 17]. He described Euclidean boson quantum field theory as a Markoff process: the Euclidean fields are random variables and the Green's functions are expectations of products of random variables. Starting from a Euclidean Markoff field, which in addition satisfies certain regularity conditions, Nelson reconstructed a relativistic quantum field theory obeying the Wightman axioms.

In this paper we give necessary and sufficient conditions under which Euclidean Green's functions have analytic continuations whose boundary values define a unique set of Wightman distributions. These conditions are

(E 0) Temperedness,
(E 1) Euclidean covariance,
(E 2) Positivity,
(E 3) Symmetry,
(E 4) Cluster property.

Surprisingly, Wightman's spectrum condition is a consequence of (E 0), (E 1) and (E 2). Using (E 2) we construct a Hilbert space $\mathscr{H}$. By (E 1) there exists a semigroup $T^t, t \geqq 0$, on $\mathscr{H}$, whose matrix elements by (E 0) grow at most polynomially as t goes to infinity. Hence T^t is a contraction semigroup and $T^t = e^{-tH}$, where $H \geqq 0$. This gives the spectrum condition. Furthermore $T^\tau, \operatorname{Re}\tau \geqq 0$, defines a holomorphic semigroup which we use to construct the analytic continuation of the Euclidean Green's functions and the Wightman distributions. All the Wightman axioms follow easily. The Hilbert space $\mathscr{H}$ turns out to be the Hilbert space of Wightman's reconstruction theorem. The following chart connecting the Euclidean axioms and the relativistic (Wightman) axioms gives the main theorem of this paper.

Euclidean		*Relativistic*
[Temperedness, Covariance, Positivity]	⟷	[Temperedness, Covariance, Positivity, Spectrum]
[ ] + Symmetry	⟷	[ ] + Locality
[ ] + Cluster	⟷	[ ] + Cluster

Some of our methods have been inspired by Nelson's work [16]. Properties (E1), (E3) and (E4) are obvious "continuations" of the Wightman axioms and have been known for a long time, [21, 24]. Symanzik [25] has introduced a positivity condition which is different from (E2). His condition is necessary for the existence of Euclidean field operators. It is probably true only for a restricted class of models and does not necessarily allow a reconstruction of the Wightman theory.

Our paper is organized as follows. In Chapter 2 we introduce some test function spaces and their duals. In Chapter 3 we formulate the axioms for Euclidean Green's functions and state our main theorems, which we prove in Chapters 4 and 5. Chapters 3–5 deal only with a single hermitean scalar field. The generalization to arbitrary spinor fields is given in Chapter 6. In Chapter 7 we make some remarks about possible applications of our results to constructive field theory. In Chapter 8 we give the proofs of some previously used technical lemmas.

2. Test Functions and Distributions

In this chapter we introduce some test function spaces and their dual spaces. They are all related to Schwartz's space $\mathscr{S}$.

We use the notation $\underset{\sim}{x}$ for a point in $\mathbb{R}^4$ whose coordinates in a fixed frame of reference are given by $(x^0, x^1, x^2, x^3) = (x^0, \underline{x})$. We call the direction of $\underset{\sim}{e}_0 = (1, \underline{0})$ "time direction", all directions orthogonal to $\underset{\sim}{e}_0$ are "space directions". The scalar product $\underset{\sim}{x} \cdot \underset{\sim}{y} = x^0 y^0 - \sum_{k=1}^{3} x^k y^k = x^0 y^0 - \underline{x}\underline{y}$ is always the Minkowski inner product. We also use the standard notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1^0)^{\alpha_1} \dots (\partial x_n^3)^{\alpha_{4n}}}, \qquad \text{where} \qquad \alpha = (\alpha_1, \alpha_2, \dots \alpha_{4n}),$$

$\alpha_k \geqq 0$, $|\alpha| = \Sigma \alpha_k$. For $f \in \mathscr{S}(\mathbb{R}^{4n})$ we define the $\mathscr{S}$-norm $|f|_m$ by

$$|f|_m = \sup_{\substack{x_i \in \mathbb{R}^4 \\ |\alpha| \leqq m}} |(1+|x|^2)^{m/2} f^{(\alpha)}(x_1, \ldots x_n)|,$$

where $|x|^2 = \sum_{i,k} (x_k^i)^2$ and $f^{(\alpha)} = D^\alpha f$.

Let ${}^o\mathscr{S}(\mathbb{R}^{4n})$ be the space of test functions $f \in \mathscr{S}(\mathbb{R}^{4n})$ with the property that $f(x_1, \ldots x_n)$ together with all its partial derivatives $f^{(\alpha)}(x_1, \ldots x_n)$ vanish if $x_i = x_j$ for some $1 \leqq i < j \leqq n$. ${}^0\mathscr{S}(\mathbb{R}^{4n})$ equipped with the induced topology of $\mathscr{S}(\mathbb{R}^{4n})$ is a closed subspace of $\mathscr{S}(R^{4n})$. We also define closed subspaces of ${}^o\mathscr{S}(\mathbb{R}^{4n})$ by $\mathscr{S}_{st}(\mathbb{R}^{4n}) = \{f \in \mathscr{S}(\mathbb{R}^{4n}); f^{(\alpha)}(x_1, \ldots x_n) = 0$ for all α unless $s < x_1^0 < x_2^0 < \cdots < x_n^0 < t\}$,

$$\mathscr{S}_+(\mathbb{R}^{4n}) = \mathscr{S}_{0\,\infty}(\mathbb{R}^{4n}),$$
$$\mathscr{S}_<(\mathbb{R}^{4n}) = \mathscr{S}_{-\infty\,\infty}(\mathbb{R}^{4n}),$$

all with the induced topology of $\mathscr{S}(\mathbb{R}^{4n})$.

We define $\mathscr{S}(\mathbb{R}_+) \subset \mathscr{S}(\mathbb{R})$ to be the space of the functions $f \in \mathscr{S}(\mathbb{R})$ with $\operatorname{supp} f \subset \mathbb{R}_+ = [0, \infty)$. By $\mathscr{S}(\mathbb{R}_+^4)$ we denote the completed topological tensor product $\mathscr{S}(\mathbb{R}_+) \hat{\otimes} \mathscr{S}(\mathbb{R}^3)$: i.e. $f \in \mathscr{S}(\mathbb{R}_+^4)$ if $f \in \mathscr{S}(\mathbb{R}^4)$ and $\operatorname{supp} f \subset \{x; x^0 \geqq 0\}$. We also define $\mathscr{S}(\mathbb{R}_-) = \{f \in \mathscr{S}(\mathbb{R}); \operatorname{supp} f \subset \{x : x \leqq 0\}\}$.

Finally we introduce the topological quotient space $\mathscr{S}(\overline{\mathbb{R}}_+) = \mathscr{S}(\mathbb{R})/\mathscr{S}(\mathbb{R}_-)$, see e.g. [19], Chapter V. Let f be an element in $\mathscr{S}(\mathbb{R})$, then the equivalence class $\{f + \mathscr{S}(\mathbb{R}_-)\}$, denoted by f_+, is an element in $\mathscr{S}(\mathbb{R})/\mathscr{S}(\mathbb{R}_-)$. We think of f_+ as being the restriction $f(x) \upharpoonright \mathbb{R}_+$ of $f(x)$ to the right half line $\mathbb{R}_+$. Because $\mathscr{S}(\mathbb{R})$ is a Frechet space and $\mathscr{S}(\mathbb{R}_-)$ is a closed subspace, $\mathscr{S}(\overline{\mathbb{R}}_+)$ is again a Frechet space and thus complete, [19], Chapter VI, Proposition 13. The topology of $\mathscr{S}(\overline{\mathbb{R}}_+)$ is given by the denumerable set of seminorms

$$|f_+|'_m = \inf_{g \in \mathscr{S}(\mathbb{R}_-)} |f + g|_m. \tag{2.1}$$

The seminorms (2.1) are not very convenient for actual calculations, but by Lemma 8.1 they can always be replaced by the equivalent set of seminorms

$$|f_+|''_m = \sup_{\substack{x \geqq 0 \\ \alpha \leqq m}} (1 + x^2)^{m/2} |f^{(\alpha)}(x)|. \tag{2.2}$$

The space $\mathscr{S}(\overline{\mathbb{R}}_+)$ is isomorphic to the space of all functions defined on $[0, \infty)$ that are infinitely differentiable (right derivatives at $x = 0$) and of fast decrease at infinity, if we equip this space with the topology defined by the seminorms (2.2). Without danger of confusion we may simply identify this space with $\mathscr{S}(\overline{\mathbb{R}}_+)$. An element in the dual space $\mathscr{S}'(\overline{\mathbb{R}}_+)$ is a distribution in $\mathscr{S}'(\mathbb{R})$ with support in $\mathbb{R}_+ = [0, \infty)$. We also define $\mathscr{S}(\overline{\mathbb{R}}_+^4)$ as the completed topological tensor product $\mathscr{S}(\overline{\mathbb{R}}_+) \hat{\otimes} \mathscr{S}(\mathbb{R}^3)$. An element $f_+ \in \mathscr{S}(\overline{\mathbb{R}}_+^4)$ is thus the restriction of some $f \in \mathscr{S}(\mathbb{R}^4)$ to

$\mathbb{R}_+ \times \mathbb{R}^3$. Let $\mathscr{S}_*$ be one of the test function spaces introduced above. Then we denote by $\hat{\otimes}_n \mathscr{S}_*$ the n fold completed topological tensor product of $\mathscr{S}_*$. We abbreviate $\hat{\otimes}_n \mathscr{S}(\mathbb{R}^4_+)$ by $\mathscr{S}(\mathbb{R}^{4 \cdot n}_+)$ and $\hat{\otimes}_n \mathscr{S}(\bar{\mathbb{R}}^4_+)$ by $\mathscr{S}(\bar{\mathbb{R}}^{4 \cdot n}_+)$. All the spaces introduced here are nuclear spaces, [5] I, § 3.6. Thus e.g. any continuous bilinear functional T on $\mathscr{S}(\bar{\mathbb{R}}_+) \times \mathscr{S}(\mathbb{R}^3)$ defines a unique distribution in $\mathscr{S}'(\bar{\mathbb{R}}^4_+)$, etc.

We also introduce the spaces $\underline{\mathscr{S}}, \underline{\mathscr{S}}_+, \underline{\mathscr{S}}_<$ and $\underline{\mathscr{S}}(\bar{\mathbb{R}}^{4\cdot}_+)$. An element $\underline{f}$ in $\underline{\mathscr{S}}$ ($\underline{\mathscr{S}}_{\substack{+\\<}}$, $\underline{\mathscr{S}}(\bar{\mathbb{R}}^{4\cdot}_+)$) is a sequence $\underline{f} = \{f_0, f_1, f_2, \ldots\}$ with $f_0 \in \mathbb{C}$, $f_n \in \mathscr{S}(\mathbb{R}^{4n})$ ($\mathscr{S}_{\substack{+\\<}}(\mathbb{R}^{4n})$, $\mathscr{S}(\bar{\mathbb{R}}^{4 \cdot n}_+)$), $n = 1, 2, \ldots$, and all but finitely many f_n's are equal to zero. We equip the spaces $\underline{\mathscr{S}}, \underline{\mathscr{S}}_+, \underline{\mathscr{S}}_<$ and $\underline{\mathscr{S}}(\bar{\mathbb{R}}^{4\cdot}_+)$ with the direct sum topologies induced by the topologies of $\mathscr{S}(\mathbb{R}^{4n})$, $\mathscr{S}_+(\mathbb{R}^{4n})$, $\mathscr{S}_<(\mathbb{R}^{4n})$ and $\mathscr{S}(\bar{\mathbb{R}}^{4 \cdot n}_+)$ respectively, and we write $\underline{\mathscr{S}} = \bigoplus_n \mathscr{S}(\mathbb{R}^{4n})$, etc. These topologies have the property that a linear map t from $\underline{\mathscr{S}}$ ($\underline{\mathscr{S}}_{\substack{+\\<}}$, $\underline{\mathscr{S}}(\bar{\mathbb{R}}^{4\cdot}_+)$) into a convex space $\mathscr{F}$ is continuous if and only if $[t \circ j_n]$ is a continuous map from $\mathscr{S}(\mathbb{R}^{4n})$ ($\mathscr{S}_{\substack{+\\<}}(\mathbb{R}^{4n})$, $\mathscr{S}(\bar{\mathbb{R}}^{4 \cdot n}_+)$) into $\mathscr{F}$, for $n = 0, 1, 2 \ldots$. We have denoted by j_n the natural injection of $\mathscr{S}(\mathbb{R}^{4n})$ ($\mathscr{S}_{\substack{+\\<}}(\mathbb{R}^{4n})$, $\mathscr{S}(\bar{\mathbb{R}}^{4 \cdot n})$) into $\underline{\mathscr{S}}(\underline{\mathscr{S}}_{\substack{+\\<}}, \underline{\mathscr{S}}(\bar{\mathbb{R}}^{4\cdot}_+))$. See [19], Chapter V, § 6, and also [1], appendix.

On $\underline{\mathscr{S}}$ we define involutions

$$\underline{f} \to \underline{f}^* \quad \text{by} \quad f_n^*(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = \bar{f}_n(\underset{\sim}{x}_n, \ldots \underset{\sim}{x}_1),$$

and

$$\underline{f} \to \Theta \underline{f} \quad \text{by} \quad (\Theta \underline{f})_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = f_n(\underset{\sim}{\vartheta} \underset{\sim}{x}_1, \ldots \underset{\sim}{\vartheta} \underset{\sim}{x}_n),$$

where $\underset{\sim}{\vartheta} \underset{\sim}{x} = (-x^0, \underline{x})$ and $^-$ means complex conjugation. Following [1] for $\underline{f}, \underline{g} \in \underline{\mathscr{S}}$, we define $\underline{f} \times \underline{g} \in \underline{\mathscr{S}}$ by

$$(\underline{f} \times \underline{g})_n = \sum_{k=0}^{n} f_{n-k} \times g_k .$$

In particular for $\underline{f}, \underline{g} \in \underline{\mathscr{S}}_+$, we find that $(\Theta \underline{f}^*) \times \underline{g} \in \underline{\mathscr{S}}_<$.

Finally we introduce some notational conventions. If f is in some space $\mathscr{S}_*$ and T is in the dual space $\mathscr{S}'_*$ of $\mathscr{S}_*$ then we use both $T(f)$ and $\int T(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) f(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)\, d^{4n}x$ to denote the value of T in f.

Let $f \in \mathscr{S}(\mathbb{R}^{4n})$, $R \in SO_4$, $\underset{\sim}{a} \in \mathbb{R}^4$ and let π an element in P_n, the group of permutations of n objects (the letter $\mathfrak{S}_n$ will be used elsewhere). Then we define $f_{(\underset{\sim}{a}, R)}$ and f^π by $f_{(\underset{\sim}{a}, R)}(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = f(R\underset{\sim}{x}_1 + \underset{\sim}{a}, \ldots R\underset{\sim}{x}_n + a)$, and $f^\pi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = f(\underset{\sim}{x}_{\pi(1)}, \ldots \underset{\sim}{x}_{\pi(n)})$. Also we define $f_+(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ to be the restriction of $f(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ to the subset $\{\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n; x_1^0 \geqq 0, \ldots x_n^0 \geqq 0\}$ of $\mathbb{R}^{4n}$, and we sometimes write $f_+(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ as $f(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \upharpoonright \{x_k^0 \geqq 0\}$.

3. The Axioms, Main Theorems

In this chapter we formulate the axioms for the Euclidean Green's functions $\{\mathfrak{S}_n\}$. We also state our main theorems.

We shall assume that $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ is a sequence of distributions $\mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ with the following properties $(n = 1, 2, \ldots)$:

E0: *Distributions*

$$\mathfrak{S}_0 \equiv 1\,, \qquad \mathfrak{S}_n \in {}^0\mathcal{S}'(\mathbb{R}^{4n})\,.$$

E1: *Euclidean Invariance*

$$\mathfrak{S}_n(f) = \mathfrak{S}_n(f_{(\underset{\sim}{a}, R)}), \text{ for all } R \in SO_4, \underset{\sim}{a} \in \mathbb{R}^4, f \in {}^0\mathcal{S}(\mathbb{R}^{4n})\,.$$

E2: *Positivity*

$$\sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times f_m) \geqq 0, \qquad \text{for all} \quad \underline{f} \in \underline{\mathcal{S}}_+\,.$$

E3: *Symmetry*

$$\mathfrak{S}_n(f) = \mathfrak{S}_n(f^\pi)\,, \quad \text{for all permutations} \quad \pi \in P_n\,, \quad \text{all} \quad f \in {}^0\mathcal{S}(\mathbb{R}^{4n})\,.$$

E4: *Cluster Property*

$$\lim_{\lambda \to \infty} \sum_{n,m} \{\mathfrak{S}_{n+m}(\Theta f_n^* \times g_{m(\lambda \underset{\sim}{a}, 1)}) - \mathfrak{S}_n(\Theta f_n^*)\, \mathfrak{S}_m(g_m)\} = 0\,,$$
$$\text{for all} \quad \underline{f}, \underline{g} \in \underline{\mathcal{S}}_+\,, \quad \underset{\sim}{a} = (0, \underline{a})\,, \quad \underline{a} \in \mathbb{R}^3\,.$$

For completeness we also list the axioms for the Wightman distributions $\{\mathfrak{W}_n\}_{n=0}^{\infty}$ as they are given on p. 117 of [23]. The $\mathfrak{W}_n$ are supposed to be distributions with the properties

R0: Temperedness,
R1: Relativistic invariance,
R2: Positivity,
R3: Local commutativity,
R4: Cluster property,
R5: Spectral condition.

We do not list hermiticity as an extra condition, because we do not have to make a distinction between linear and nonlinear conditions.

The main results of this paper are the following theorems.

Theorem E→R. *To a given sequence of Euclidean Green's functions satisfying* E0–E4, *there corresponds a unique sequence of Wightman distributions with the properties* R0–R5.

Theorem R→E. *To a given sequence of Wightman distributions satisfying* R0–R5, *there corresponds a unique sequence of Euclidean Green's functions with the properties* E0–E4.

Remarks

1. The choice of ${}^0\mathcal{S}'(\mathbb{R}^{4n})$ as distribution space for the $\mathfrak{S}_n$ is natural because it makes the correspondence between $\{\mathfrak{S}_n\}$ and $\{\mathfrak{W}_n\}$ unique.

Indeed, two sequences $\{\mathfrak{S}_n\}$ and $\{\mathfrak{S}'_n\}$ of Euclidean Green's functions with the properties that $\mathfrak{S}_n$ and $\mathfrak{S}'_n$ are in $\mathscr{S}'(\mathbb{R}^{4n})$ and that $\mathfrak{S}_n - \mathfrak{S}'_n$ has support in $\{x_1, \ldots x_n; x_i = x_j$ for some $1 \leqq i < j \leqq n\}$, lead to the same set of Wightman distributions. Equivalently we may say that the Wightman distributions tell us nothing about the Euclidean Green's functions in points of coinciding arguments.

2. Even if the properties (E 2) and (E 4) seem to depend on the choice of the "time direction" e_0 in $\mathbb{R}^4$, property (E 1) shows that this dependence is only spurious and that (E 2) and (E 4) have to hold for any choice of e_0. We have chosen the non covariant formulation (E 4) of the cluster property only for convenience. Without any change in our results we could replace (E 4) by the following covariant condition.

$$\text{E4}': \qquad \lim_{\lambda \to \infty} \{\mathfrak{S}_{n+m}(f \times g_{(\lambda a, 1)}) - \mathfrak{S}_n(f)\, \mathfrak{S}_m(g)\} = 0\,, \tag{3.1}$$

$$\text{for all } f \in {}^0\mathscr{S}_c(\mathbb{R}^{4n})\,, \quad g \in {}^0\mathscr{S}_c(\mathbb{R}^{4m})\,, \quad a \in \mathbb{R}^4\,.$$

(${}^0\mathscr{S}_c(\mathbb{R}^{4n})$ is the set of all elements in ${}^0\mathscr{S}(\mathbb{R}^{4n})$ with compact support).

However there seems to be no covariant formulation of the positivity condition, because the "time inversion operator" Θ enters (E 2) in a crucial way.

We remark that exponential vanishing of the expression on the left hand side of (3.1) is equivalent with a mass gap in the relativistic theory, see Ref. [9].

3. It follows from the proofs of our theorems that property (E 0) could be replaced by the weaker – but non covariant – condition

$$\text{E0}': \qquad \mathfrak{S}_n \in \mathscr{S}'_{<}(\mathbb{R}^{4n})\,.$$

This condition becomes important if we want to prove the equivalence of subsets of the axioms, in particular if these subsets do not contain symmetry (E 3) and locality (R 3) respectively. More precisely our theorems remain true if we replace {E 0–E 4 | R 0–R 5} by either of

(a) {E 0′, E 1, E 2 | R 0, R 1, R 2, R 5},

(b) {E 0–E 3 | R 0–R 3, R 5},

(c) {E 0′, E 1, E 2, E 4 | R 0, R 1, R 2, R 4, R 5}.

(See also chart on p. 85). This follows from Chapters 4 and 5.

4. Obviously the translation scheme E↔R may be extended to the case where the theory has additional symmetry properties. In particular the symmetries P, C and T of the relativistic theory can be expressed as symmetry properties of the Euclidean Green's functions. The symmetry property of the Euclidean Green's functions corresponding to PCT is an immediate consequence of axioms (E 1) and (E 3). This is the well known PCT theorem.

4. Theorem E → R

In this chapter we prove theorem E→R. Given a sequence of Euclidean Green's functions satisfying (E0)–(E4) we construct explicitly a sequence of Wightman distributions satisfying (R0)–(R5). The main steps in this construction are the following:

a) Using (E0) and (E2) we define a positive semi-definite form $(\underline{f}, \underline{g}) = \sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times g_m)$ on $\underline{\mathscr{S}}_+ \times \underline{\mathscr{S}}_+$. After dividing out the vectors of norm zero we obtain a pre Hilbert space, whose completion we denote by $\mathscr{H}$.

b) By (E0) and (E1) there exists a one parameter semigroup $\{T^t = e^{-tH}\}_{t \geqq 0}$ of self-adjoint contractions on $\mathscr{H}$, which can be extended to a holomorphic semigroup T^τ, $\operatorname{Re}\tau \geqq 0$. With the help of this holomorphic semigroup we show that the Euclidean Green's functions are the Fourier-Laplace transforms of distributions $\tilde{\mathfrak{W}}_n$ in $\mathscr{S}'$ which have certain support properties. We define $\mathfrak{W}_n$, the Fourier transforms of $\tilde{\mathfrak{W}}_n$, to be the Wightman distributions. (R0) follows immediately.

c) Euclidean invariance (E1) of $\mathfrak{S}_n$ now implies relativistic invariance (R1) of $\mathfrak{W}_n$, and the support properties of $\tilde{\mathfrak{W}}_n$ give the spectrum condition (R5).

d) Using positivity (E2) we prove positivity (R2), and we will show that the Hilbert space $\mathscr{H}$, constructed at the beginning, is the Hilbert space of Wightman's reconstruction theorem.

e) Using the cluster property (E4) we prove the cluster property (R4) and

f) Using symmetry (E3) and a theorem in Ref. [12], p. 83, we show local commutativity (R3).

4.1. Construction of the Wightman Distributions

Let $\{\mathfrak{S}_n\}_{n=0}^{\infty}$ be a sequence of distributions satisfying conditions (E0)–(E4). As a preliminary step of our construction we use (E0) and the translation invariance (E1) of $\mathfrak{S}_n$ to conclude that there exists a unique sequence of distributions $S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n)$ in $\mathscr{S}'(\mathbb{R}_+^{4 \cdot n})$ such that for $f(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \mathscr{S}_<(\mathbb{R}^{4 \cdot n})$

$$\mathfrak{S}_n(f) = \int f(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)\, S_{n-1}(\underset{\sim}{x}_2 - \underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n - \underset{\sim}{x}_{n-1})\, d^{4n}x\,.$$

In the sense of distributions this means

$$\mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = S_{n-1}(\underset{\sim}{x}_2 - \underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n - \underset{\sim}{x}_{n-1})\,, \quad \text{for} \quad x_1^0 < x_2^0 < \cdots < x_n^0\,. \tag{4.1}$$

The distributions S_n are invariant under SO_4 in a restricted sense:

$$S_n(\mathrm{R}\underset{\sim}{\xi}_1, \ldots \mathrm{R}\underset{\sim}{\xi}_n) = S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n)\,, \tag{4.2}$$

provided $\xi_k^0 > 0$ and $(\mathrm{R}\underset{\sim}{\xi}_k)^0 > 0$, $k = 1, \ldots n$.

Now we define on $\underline{\mathscr{S}}_+ \times \underline{\mathscr{S}}_+$ a sesquilinear form by

$$(\underline{f}, \underline{g}) = \sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times g_m) . \tag{4.3}$$

This form is positive semidefinite due to (E 2), it is linear in $\underline{g}$, antilinear in $\underline{f}$ and it satisfies $(\underline{f}, \underline{g}) = \overline{(\underline{g}, \underline{f})}$, for all $\underline{f}, \underline{g} \in \underline{\mathscr{S}}_+$. Denoting by $\mathscr{N}$ the set of vectors in $\underline{\mathscr{S}}_+$ of norm zero, $\mathscr{N} = \{\underline{f} \in \underline{\mathscr{S}}_+ ; \|\underline{f}\|^2 = (\underline{f}, \underline{f}) = 0\}$, we define $\mathscr{H}$ to be the Hilbert space completion of the quotient space $\underline{\mathscr{S}}_+ / \mathscr{N}$. By v we denote the canonical injection of $\underline{\mathscr{S}}_+$ into $\mathscr{H}$. Then for $\underline{f}, \underline{g} \in \underline{\mathscr{S}}_+$,

$$(v(\underline{f}), v(\underline{g}))_{\mathscr{H}} = (\underline{f}, \underline{g}) , \tag{4.4}$$

and the range of v, denoted by $\mathscr{D}_0$ is a dense subset of $\mathscr{H}$. By (E 0), v is a continuous map from $\underline{\mathscr{S}}_+$ onto $\mathscr{D}_0$.

For $\underline{f} \in \underline{\mathscr{S}}_+$ and $\underset{\sim}{a} = (0, \underline{a})$, we define $\hat{U}_s(\underline{a}) \underline{f}$ by

$$(\hat{U}_s(\underline{a}) \underline{f})_n (\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = f_n(\underset{\sim}{x}_1 - \underset{\sim}{a}, \ldots \underset{\sim}{x}_n - \underset{\sim}{a}) .$$

By (E 1), for $\underline{f}, \underline{g} \in \underline{\mathscr{S}}_+$,

$$(\underline{f}, \hat{U}_s(\underline{a}) \underline{g}) = (\hat{U}_s(-\underline{a}) \underline{f}, \underline{g}) ,$$

and

$$(\hat{U}_s(\underline{a}) \underline{f}, \hat{U}_s(\underline{a}) \underline{g}) = (\underline{f}, \underline{g}) .$$

Thus extension of

$$U_s(\underline{a})\, v(\underline{f}) = v(\hat{U}_s(\underline{a}) \underline{f}) \tag{4.5}$$

by continuity leads to a unitary representation $U_s(\underline{a})$ in $\mathscr{H}$ of the three dimensional translation group (translations in space directions).

For translations in the time direction the situation is different because the sesquilinear form (4.3) and hence the scalar product in $\mathscr{H}$ involves a time inversion Θ. For $t \geqq 0$ we define the map $\hat{T}^t$ from $\underline{\mathscr{S}}_+$ into itself by

$$(\hat{T}^t \underline{f})_n (\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = f_n(\underset{\sim}{x}_1 - \underset{\sim}{t}, \ldots \underset{\sim}{x}_n - \underset{\sim}{t}) , \tag{4.6}$$

where $\underset{\sim}{t} = (t, \underline{0})$. By (E 1) and definition (4.3), for $\underline{f}, \underline{g}, \in \underline{\mathscr{S}}_+$ and $t \geqq 0$,

$$(\underline{f}, \hat{T}^t \underline{g}) = (\hat{T}^t \underline{f}, \underline{g}) . \tag{4.7}$$

Furthermore for $s, t \geqq 0$ we have

$$\hat{T}^t \hat{T}^s = \hat{T}^{t+s} .$$

By (4.1), we find that for $\underline{f} \in \underline{\mathscr{S}}_+$,

$$\begin{aligned} |(\underline{f}, \hat{T}^t \underline{f})| = \Big| \sum_{n,m} \int & \bar{f}_n(\vartheta \underset{\sim}{x}_n, \ldots \vartheta \underset{\sim}{x}_1) f_m(\underset{\sim}{y}_1, \ldots \underset{\sim}{y}_m) \\ & \cdot S_{m+n-1}(\underset{\sim}{x}_2 - \underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n - \underset{\sim}{x}_{n-1}, \underset{\sim}{y}_1 + \underset{\sim}{t} - \underset{\sim}{x}_n, \underset{\sim}{y}_2 - \underset{\sim}{y}_1, \\ & \ldots \underset{\sim}{y}_m - \underset{\sim}{y}_{m-1})\, d^{4n}x\, d^{4m}y \Big| \\ \leqq & P(t) , \end{aligned} \tag{4.8}$$

for some polynomial $P(t)$, which depends on $\underline{f}$. This follows from the facts that S_{m+n-1} is in $\mathscr{S}'(\mathbb{R}_+^{4(m+n-1)})$, and that $\underline{f}$ has only a finite number of nonvanishing components. We can improve inequality (4.8) by a repeated application of the Schwarz inequality and of (4.7) and (4.8).

$$\begin{aligned} |(\underline{f}, \hat{T}^t \underline{f})| &\leqq \|\underline{f}\| \, \|\hat{T}^t \underline{f}\| \\ &= \|\underline{f}\| \, (\underline{f}, \hat{T}^{2t} \underline{f})^{\frac{1}{2}} \\ &\leqq \|\underline{f}\|^{\sum_{k=0}^{n-1} 2^{-k}} (\underline{f}, \hat{T}^{2^n t} \underline{f})^{2^{-n}} \\ &\leqq \|\underline{f}\|^{\sum_{k=0}^{n-1} 2^{-k}} (P(2^n t))^{2^{-n}}, \end{aligned}$$

for all $n = 1, 2, \ldots$ Taking the limit as $n \to \infty$ we obtain

$$|(\underline{f}, \hat{T}^t \underline{f})| \leqq \|\underline{f}\|^2 . \tag{4.9}$$

It follows from (4.9) that $\hat{T}^t$ maps one equivalence class mod $\mathcal{N}$ onto another equivalence class mod $\mathcal{N}$. Thus for $t \geqq 0$

$$T_0^t v(\underline{f}) = v(\hat{T}^t \underline{f})$$

defines a continuous one paremater semigroup $\{T_0^t\}_{t \geqq 0}$ of operators on $\mathscr{D}_0 \subset \mathscr{H}$ and T_0^t is positive, symmetric and has norm smaller or equal to one. Therefore T_0^t has a self-adjoint extension, denoted by T^t, and $\{T^t\}_{t \geqq 0}$ is a weakly continuous one parameter semigroup of self-adjoint contractions on $\mathscr{H}$. Let H be the infinitesimal generator of T^t. It is a positive self-adjoint operator on $\mathscr{H}$ and we can define the one parameter group of unitary operators $T^{is} = e^{isH}$, $-\infty < s < \infty$. This is the unitary representation of the time translation group and we set

$$U(\underset{\sim}{a}) = T^{ia^0} U_s(\underline{a}), \quad \underset{\sim}{a} = (a^0, \underline{a}) \in \mathbb{R}^4 .$$

This defines a unitary representation of the four dimensional translation group in $\mathscr{H}$.

The family $T^\tau = T^t T^{is}$, $\tau = t + is$, is a holomorphic semigroup for $\mathrm{Re}\,\tau = t > 0$, uniformly bounded and strongly continuous for $\mathrm{Re}\,\tau \geqq 0$. We can use this holomorphic semigroup to construct the analytic continuation of the Euclidean Green's functions.

Let $\underline{f}_m = (0, \ldots, f_m, 0 \ldots)$ and $\underline{g} = (0, \ldots, g_n, 0 \ldots)$ be vectors in $\underline{\mathscr{S}}_+$. Then $f_m \in \mathscr{S}_+(\mathbb{R}^{4m})$, $g_n \in \mathscr{S}_+(\mathbb{R}^{4n})$ and $\Theta f_m^* \times g_n \in \mathscr{S}_<(\mathbb{R}^{4(m+n)})$. Thus for $t \geqq 0$ fixed and $h \in \mathscr{S}(\mathbb{R})$, the mapping

$$(\Theta f_m^* \times g_n, h) \to \int (v(\underline{f}_m), T^{t+is} v(\underline{g}_n))_{\mathscr{H}} \, h(s) \, ds \tag{4.10}$$

defines a continuous linear functional on $\mathscr{S}_<(\mathbb{R}^{4(m+n)}) \hat{\otimes} \mathscr{S}(\mathbb{R})$, by the nuclear theorem. Moreover, for any $\underline{a} \in \mathbb{R}^3$, $a^0 \geqq 0$,

$$\big(v(\underline{f}_m), T^{t+is}(T^{a^0} U_s(\underline{a})) v(\underline{g}_n)\big)_{\mathscr{H}} = \big(T^{a^0} U_s(-\underline{a}) v(\underline{f}_m), T^{t+is} v(\underline{g}_n)\big)_{\mathscr{H}} .$$

We conclude that the right hand side of (4.10) can be written as

$$\begin{aligned} &\int (v(f_m), T^{t+is} v(g_n))_{\mathscr{H}} \, h(s) \, ds \\ &\quad = \int \bar{f}_m(\vartheta \underset{\sim}{x}_m, \ldots \vartheta \underset{\sim}{x}_1) \, g_n(\underset{\sim}{y}_1, \ldots \underset{\sim}{y}_n) h(s) \qquad (4.11) \\ &\quad\quad S^{(m)}_{n+m-1}(\underset{\sim}{x}_2 - \underset{\sim}{x}_1, \ldots, \underset{\sim}{y}_1 + \underset{\sim}{t} - \underset{\sim}{x}_m, \ldots \underset{\sim}{y}_n - \underset{\sim}{y}_{n-1} \,|\, s) \, d^{4m}x \, d^{4n}y \, ds \,, \end{aligned}$$

where $S^{(m)}_{n+m-1}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{m+n-1} \,|\, s)$ is a distribution in the dual space of $\mathscr{S}(\mathbb{R}_+^{4\cdot(m+n-1)}) \hat{\otimes} \mathscr{S}(\mathbb{R})$ and a continuous function of s when smeared in all the other variables. Furthermore

$$S^{(m)}_{m+n-1}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{m+n-1} \,|\, 0) = S_{m+n-1}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{m+n-1}) \,.$$

Now let $h_m(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{m-1}, \underline{\xi}_m, \underset{\sim}{\xi}_{m+1}, \ldots \underset{\sim}{\xi}_{m+n-1})$ be an element in $\mathscr{S}(\mathbb{R}_+^{4\cdot(m-1)}) \hat{\otimes} \mathscr{S}_{\underline{\xi}_m}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}(\mathbb{R}_+^{4\cdot(n-1)})$ and set for $t > 0$

$$\begin{aligned} S^{(m)}(t = \xi^0_m, s \,|\, h_m) = \int S^{(m)}_{m+n-1}(\underset{\sim}{\xi}_1, \ldots \,|\, s) \, h_m(\underset{\sim}{\xi}_1, \ldots \underline{\xi}_m, \ldots \underset{\sim}{\xi}_{m+n-1}) \\ d^4 \xi_1, \ldots d^3 \xi_m \ldots d^4 \xi_{m+n-1} \,. \end{aligned}$$

It follows from (4.11) that for fixed h_m, $S^{(m)}(t, s \,|\, h_m)$ is a distribution in the dual space of $\mathscr{S}(\mathbb{R}_+) \hat{\otimes} \mathscr{S}(\mathbb{R})$ and satisfies the Cauchy-Riemann equation for $t > 0$, $-\infty < s < \infty$. Hence by Lemma 8.7 there is a distribution $\check{S}^{(m)}(\alpha \,|\, h_m)$ in $\mathscr{S}'(\overline{\mathbb{R}}_+)$ such that $S^{(m)}$ is the Fourier-Laplace transform of $\check{S}^{(m)}$. $S^{(m)}(t, s \,|\, h_m) = \int e^{-\alpha(t+is)} \check{S}^{(m)}(\alpha \,|\, h_m) \, d\alpha$, and $S^{(m)}_{m+n-1}(\underset{\sim}{\xi}_1, \ldots (t, \underline{\xi}_m), \ldots \underset{\sim}{\xi}_{m+n-1} \,|\, s)$ is the analytic continuation of $S_{m+n-1}(\underset{\sim}{\xi}_1, \ldots (t, \underline{\xi}_m), \ldots \underset{\sim}{\xi}_{m+n-1})$ in the time component of the m-th variable. Lemma 8.8 shows that under these circumstances we can analytically continue S_{m+n-1} simultaneously in the time components of all variables and that there exists a uniquely determined distribution $\tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n)$ in $\mathscr{S}'(\overline{\mathbb{R}}_+^{4\cdot n})$ such that

$$S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n) = \int e^{-\sum_{k=1}^{n} (\xi^0_k q^0_k - i \underline{\xi}_k \underline{q}_k)} \tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n) \, d^{4n} q \,. \qquad (4.12)$$

Now we define the Wightman distributions $\mathfrak{W}_n$ by

$$\mathfrak{W}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = \int e^{-i \sum_{k=1}^{n-1} (\underset{\sim}{x}_{k+1} - \underset{\sim}{x}_k) \underset{\sim}{q}_k} \tilde{W}_{n-1}(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_{n-1}) \, d^{4(n-1)} q \,. \qquad (4.13)$$

According to the results of Chapter 2, [see the discussion following Eq. (2.2)] $\tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n)$ is a distribution in $\mathscr{S}'(\mathbb{R}^{4n})$ with support in $\{\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n; q^0_1 \geqq 0, \ldots q^0_n \geqq 0\}$. In Section 4.2 we prove that for $\Lambda \in L^{\uparrow}_+$, $\tilde{W}_n(\Lambda \underset{\sim}{q}_1, \ldots \Lambda \underset{\sim}{q}_n) = \tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n)$. Hence the support of $\tilde{W}_n$ is in $\{\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n; \underset{\sim}{q}_1 \in V_+, \ldots \underset{\sim}{q}_n \in V_+\}$, where V_+ is the forward light cone. This is the spectrum condition (R 5). Temperedness (R 0) follows from (4.13).

4.2. Lorentz Covariance and Spectrum Condition

Translation invariance of $\mathfrak{W}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ follows from definition (4.13). Thus relativistic invariance (R 1) and also the spectrum condition (R 5) follow if we can prove that for all $\Lambda \in L_+^\uparrow$,

$$\tilde{W}_n(\Lambda \underset{\sim}{q}_1, \ldots \Lambda \underset{\sim}{q}_n) = \tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n),$$

or equivalently if we can prove that for $0 \leqq i < j \leqq 3$,

$$X_{ij} \tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n) = 0, \tag{4.14}$$

where X_{ij} are the operators

$$X_{0j} = \sum_{k=1}^{n} \left(q_k^0 \frac{\partial}{\partial q_k^j} + q_k^j \frac{\partial}{\partial q_k^0} \right), \qquad j = 1, 2, 3$$

$$X_{ij} = \sum_{k=1}^{n} \left(q_k^i \frac{\partial}{\partial q_k^j} - q_k^j \frac{\partial}{\partial q_k^i} \right), \qquad 1 \leqq i < j \leqq 3.$$

From Eq. (4.2) we conclude that for $0 \leqq i < j \leqq 3$,

$$Y_{ij} S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n) = 0, \tag{4.15}$$

where

$$Y_{ij} = \sum_{k=1}^{n} \left(\xi_k^i \frac{\partial}{\partial \xi_k^j} - \xi_k^j \frac{\partial}{\partial \xi_k^i} \right).$$

On the other hand we get from Eq. (4.12)

$$Y_{ij} S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n) = \int e^{-\sum_{k=1}^{n} (\xi_k^0 q_k^0 - i \underset{\sim}{\xi}_k \underset{\sim}{q}_k)} X_{ij} \tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n)\, d^{4n} q, \tag{4.16}$$

for $1 \leqq i < j \leqq 3$, and

$$Y_{0j} S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n) = i \int e^{-\sum_{k=1}^{n} (\xi_k^0 q_k^0 - i \underset{\sim}{\xi}_k \underset{\sim}{q}_k)} X_{0j} \tilde{W}_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n)\, d^{4n} q, \tag{4.17}$$

for $j = 1, 2, 3$. [To prove Eq. (4.16) we use the definitions of the Fourier transform and of the derivative of a distribution; to prove (4.17) we also need Lemma 8.4.]

Eq. (4.14) is now a consequence of Eqs. (4.15–17) and of the uniqueness theorem for Laplace and Fourier transforms of distributions.

4.3. Positivity

In this section we prove the positivity condition (R 2). We have to show that for all $\underset{\sim}{f} \in \underset{\sim}{\mathscr{S}}$

$$\sum_{n,m} \mathfrak{W}_{n+m}(f_n^* \times f_m) \tag{4.18}$$

is nonnegative. The idea of the proof is simple. We show that any element $\underline{f} \in \underline{\mathcal{S}}$ defines a vector $u(\underline{f})$ in the Hilbert space $\mathcal{K}$, constructed in Section 4.1, and that the norm of $u(\underline{f})$ is given by (4.18). We also show that the set $\mathcal{D}_1 = \{u(\underline{f}); \underline{f} \in \underline{\mathcal{S}}\}$ is dense in $\mathcal{K}$. Hence $\mathcal{K}$ is the Hilbert space of Wightman's reconstruction theorem.

For $f \in \mathcal{S}_+(\mathbb{R}^{4n})$ we define $f^\dagger \in \mathcal{S}(\mathbb{R}_+^{4n})$ by

$$f^\dagger(\underset{\sim}{x}_1, \underset{\sim}{x}_2 - \underset{\sim}{x}_1, \dots \underset{\sim}{x}_n - \underset{\sim}{x}_{n-1}) = f(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) . \tag{4.19}$$

Then for $\underline{f} \in \underline{\mathcal{S}}_+$ we define $\check{\underline{f}}$ by

$$\check{f}_n(\underset{\sim}{q}_1, \dots \underset{\sim}{q}_n) = \int f_n^\dagger(\underset{\sim}{\xi}_1, \dots \underset{\sim}{\xi}_n)\, e^{-\sum\limits_{k=1}^{n} (\xi_k^0 q_k^0 - i \underset{\sim}{\xi}_k \underset{\sim}{q}_k)} d^{4n}\xi \restriction \{q_k^0 \geqq 0\} . \tag{4.20}$$

Lemma 4.1. *The map $\underline{f} \to \check{\underline{f}}$ for $\underline{f} \in \underline{\mathcal{S}}_+$ is a continuous map of $\underline{\mathcal{S}}_+$ onto a dense subset $\check{\underline{\mathcal{S}}}$ of $\underline{\mathcal{S}}(\overline{\mathbb{R}}_+^{4\cdot})$. The kernel of this map is $\{0\}$.*

Proof. It suffices to show that for $n = 1, 2, \dots, f_n \to \check{f}_n$ is a continuous map of $\mathcal{S}_+(\mathbb{R}^{4n})$ onto a dense subset $\check{\mathcal{S}}_+(\mathbb{R}^{4n})$ of $\mathcal{S}(\overline{\mathbb{R}}_+^{4\cdot n})$ and that the kernel of this map is $\{0\}$. The map $f_n \to f_n^\dagger$, defined by (4.19) is an isomorphism of $\mathcal{S}_+(\mathbb{R}^{4n})$ onto $\mathcal{S}(\mathbb{R}_+^{4\cdot n})$, therefore the lemma follows if we can prove that $f_n^\dagger \to \check{f}_n$ is a continuous map of $\mathcal{S}(\mathbb{R}_+^{4\cdot n})$ onto a dense subset $\check{\mathcal{S}}_+(\mathbb{R}^{4n})$ of $\mathcal{S}(\overline{\mathbb{R}}_+^{4\cdot n})$ with kernel $\{0\}$. But this is an easy consequence of Lemma 8.2 and the fact that the Fourier transform is a homomorphism of $\mathcal{S}(\mathbb{R})$ onto itself.

In Section 4.1 we have introduced a continuous map v from $\underline{\mathcal{S}}_+$ onto a dense subset $\mathcal{D}_0$ of $\mathcal{K}$. By Lemma 4.1, the map w defined by

$$w(\check{\underline{f}}) = v(\underline{f}), \quad \text{for} \quad \underline{f} \in \underline{\mathcal{S}}_+ , \tag{4.21}$$

is a map from the dense subset $\check{\underline{\mathcal{S}}}_+$ of $\underline{\mathcal{S}}(\overline{\mathbb{R}}_+^{4\cdot})$ onto $\mathcal{D}_0 \subset \mathcal{K}$. We want to show that w can be extended to a continuous map $\overline{w}$ from all of $\underline{\mathcal{S}}(\overline{\mathbb{R}}_+^{4\cdot})$ onto a dense subset $\mathcal{D}_1$ of $\mathcal{K}$. This follows from

Lemma 4.2. *The map w is continuous.*

Proof. Let $\underline{f}, \underline{g} \in \underline{\mathcal{S}}_+$. Then, by (4.21) and by (4.3),

$$\begin{aligned}(w(\check{\underline{f}}), w(\check{\underline{g}}))_{\mathcal{K}} &= (v(\underline{f}), v(\underline{g}))_{\mathcal{K}} \\ &= (\underline{f}, \underline{g}) \\ &= \sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times f_m) .\end{aligned} \tag{4.22}$$

We rewrite the last expression of (4.22) in terms of $\check{f}_n$ and $\check{g}_m$. By (4.1) and (4.12) we find that this expression equals

$$\sum_{n,m} \int \bar{f}_n^\dagger(\vartheta \underset{\sim}{x}_n, -\vartheta \underset{\sim}{\xi}_{n-1}, \dots, -\vartheta \underset{\sim}{\xi}_1)\, g_m^\dagger(\underset{\sim}{\xi}_n + \underset{\sim}{x}_n, \underset{\sim}{\xi}_{n+1}, \dots \underset{\sim}{\xi}_{n+m-1}) \tag{4.23}$$

$$\cdot \left[\int e^{-\sum\limits_{k=1}^{n+m-1} (\xi_k^0 q_k^0 - i \underset{\sim}{\xi}_k \underset{\sim}{q}_k)} \tilde{W}_{n+m-1}(\underset{\sim}{q}_1, \dots \underset{\sim}{q}_{n+m-1})\, d^{4(n+m-1)}q\right] d^{4(n+m-1)}\xi\, d^4 x_n .$$

Interchanging the order of integration in (4.23) we obtain

$$(w(\bar{\underset{\sim}{f}}), w(\check{\underset{\sim}{g}}))_{\mathscr{H}} = \sum_{n,m} \int \bar{\bar{f}}_n(\underset{\sim}{q}_n, \underset{\sim}{q}_{n-1}, \dots \underset{\sim}{q}_1)\, \check{g}_m(\underset{\sim}{q}_n, \underset{\sim}{q}_{n+1}, \dots \underset{\sim}{q}_{n+m-1}) \cdot \tilde{W}_{n+m-1}(\underset{\sim}{q}_1, \dots \underset{\sim}{q}_{n+m-1})\, d^{4(n+m-1)}q\,. \tag{4.24}$$

For the space components $\xi_k^{1,2,3}$ and $q_k^{1,2,3}$ the change in the order of integration is justified by the definition of the Fourier transform of a distribution. For the time components ξ_k^0 and q_k^0 we refer to Lemma 8.4.

By Lemma 4.1, $\bar{\bar{f}}_n$ and $\check{g}_m$ are elements in $\mathscr{S}(\overline{\mathbb{R}}_+^{4\cdot n})$ and $\mathscr{S}(\overline{\mathbb{R}}_+^{4\cdot m})$ respectively, and thus $\bar{\bar{f}}_n(\underset{\sim}{q}_n, \dots \underset{\sim}{q}_1)\, \check{g}_m(\underset{\sim}{q}_n, \dots \underset{\sim}{q}_{n+m-1})$ is in $\mathscr{S}(\overline{\mathbb{R}}_+^{4\cdot(n+m-1)})$. On the other hand $\tilde{W}_{n+m-1}$ is a distribution in $\mathscr{S}'(\overline{\mathbb{R}}_+^{4\cdot(n+m-1)})$. This proves Lemma 4.2.

From Eq. (4.24) we conclude that for any $\underline{h}, \underline{k} \in \underline{\mathscr{S}}(\overline{\mathbb{R}}_+^{4\cdot})$

$$(\overline{w}(\underline{h}), \overline{w}(\underline{k}))_{\mathscr{H}} \tag{4.25}$$

$$= \sum_{n,m} \int \bar{h}_n(\underset{\sim}{q}_n, \dots \underset{\sim}{q}_1)\, k_m(\underset{\sim}{q}_n, \dots \underset{\sim}{q}_{n+m-1})\, \tilde{W}_{n+m-1}(\underset{\sim}{q}_1, \dots \underset{\sim}{q}_{n+m-1})\, d^{4(n+m-1)}q\,.$$

Now let $\underline{f} \in \underline{\mathscr{S}}$ and define $\hat{\underline{f}}$ by $(\hat{\underline{f}})_n = \hat{f}_n$ and

$$\hat{f}_n(\underset{\sim}{q}_1, \dots \underset{\sim}{q}_n) = \int e^{-i\sum_{k=1}^{n} q_k \xi_k} f_n^\dagger(\underset{\sim}{\xi}_1, \dots \underset{\sim}{\xi}_n)\, d^{4n}\xi \restriction \{q_k^0 \geqq 0\}\,, \tag{4.26}$$

where $f_n^\dagger$ is defined as in (4.19). Obviously $\hat{f}_n$ is the restriction to $\{q_k^0 \geqq 0\}$ of a test function in $\mathscr{S}(\mathbb{R}^{4n})$ and is therefore an element in $\mathscr{S}(\overline{\mathbb{R}}^{4\cdot n})$. Furthermore the set $\{\hat{\underline{f}}; \underline{f} \in \underline{\mathscr{S}}\}$ is equal to $\underline{\mathscr{S}}(\overline{\mathbb{R}}_+^{4\cdot})$. Therefore we may define a map $u: \underline{\mathscr{S}} \to \mathscr{H}$ by

$$\underline{f} \to u(\underline{f}) = \overline{w}(\hat{\underline{f}}) \in \mathscr{H}\,. \tag{4.27}$$

The range of u is $\mathscr{D}_1$ (= range of $\overline{w}$), which is a dense subset of $\mathscr{H}$.

Substituting (4.13) and (4.26) in (4.18) we find that for $\underline{f} \in \underline{\mathscr{S}}$,

$$\sum_{n,m} \mathfrak{W}_{n+m}(f_n^* \times f_m)$$

$$= \sum_{n,m} \int \bar{\hat{f}}_n(\underset{\sim}{q}_n, \dots \underset{\sim}{q}_1)\, \hat{f}_m(\underset{\sim}{q}_n, \dots \underset{\sim}{q}_{n+m-1})\, \tilde{W}_{n+m-1}(\underset{\sim}{q}_1, \dots \underset{\sim}{q}_{n+m-1})\, d^{4(n+m-1)}q$$

$$= (\overline{w}(\hat{\underline{f}}), \overline{w}(\hat{\underline{f}}))_{\mathscr{H}} = (u(\underline{f}), u(\underline{f}))_{\mathscr{H}}\,, \tag{4.28}$$

where the second Eq. in (4.28) follows from (4.25). From Eq. (4.28) we now get the positivity condition (R 2). The density of $\mathscr{D}_1$ in $\mathscr{H}$ implies that $\mathscr{H}$ is the Hilbert space of Wightman's reconstruction theorem [23].

4.4. Cluster Property

In this section we derive the cluster property (R 4) from the cluster property (E 4). Using (4.22) we rewrite (E 4) in vector notation:

$$\lim_{\lambda \to \infty} (w(\check{\underline{f}}), U_s(\lambda \underline{a})\, w(\check{\underline{g}}))_{\mathscr{H}} = (w(\check{\underline{f}}), \Omega)_{\mathscr{H}}\, (\Omega, w(\check{\underline{g}}))_{\mathscr{H}}\,, \tag{4.29}$$

for $\underline{f}, \underline{g} \in \mathscr{S}_+$. The vacuum vector Ω is defined to be $v(\{1, 0, 0, \ldots\})$. By continuity (4.29) remains true if we replace $w(\check{\underline{f}})$ and $w(\check{\underline{g}})$ by arbitrary vectors in $\mathscr{K}$, in particular we find that for all $\underline{h}, \underline{k} \in \mathscr{S}$

$$\lim_{\lambda \to \infty} (u(\underline{h}), U_s(\lambda \underline{a})\, u(\underline{k}))_{\mathscr{K}} = (u(\underline{h}), \Omega)_{\mathscr{K}}\, (\Omega, u(\underline{k}))_{\mathscr{K}} . \tag{4.30}$$

Eq. (4.30) is the cluster property (R 4) in vector notation.

4.5. Locality

For $\operatorname{Re} z_k^0 = 0, \operatorname{Im} \underline{z}_k = 0$ and $\underline{z}_k - \underline{z}_{k'} \neq 0$, if $k \neq k'$, we define the Wightman function by

$$\mathfrak{W}_n(z_1, \ldots z_n) = \mathfrak{S}_n((iz_1^0, \underline{z}_1), \ldots (iz_n^0, \underline{z}_n)) .$$

Then $\mathfrak{W}_n(z_1 \ldots z_n)$ is symmetric in its arguments and has an $L_+^\uparrow$ invariant, single valued, symmetric analytic continuation into the domain $\mathfrak{S}_n = \{z_1, \ldots z_n | (z_{\pi(k)} - z_{\pi(k-1)}) \in \mathfrak{T}, k = 1, \ldots n,$ for some permutation $\pi(1), \ldots \pi(n)$ of $1, \ldots n\}$. $\mathfrak{T}$ is the forward tube $\{z; \operatorname{Im} z \in V_+\}$. This follows easily from the symmetry property (E 2) for the Euclidean Green's functions and Eqs. (4.1), (4.12) and (4.14). Using the Bargmann Hall Wightman theorem, [10], we conclude that $\mathfrak{W}_n(z_1, \ldots z_n)$ allows even a single valued, symmetric $L_+(\mathbb{C})$ invariant analytic continuation into the domain $\mathfrak{S}_n^P = \bigcup_{A \in L_+(\mathbb{C})} A\mathfrak{S}_n$. Now we use a theorem in Ref. [12] (p. 83, second theorem) to conclude that the boundary distributions $\mathfrak{W}_n(x_1, \ldots x_n)$ of $\mathfrak{W}_n(z_1, \ldots z_n)$ satisfy the locality condition (R 3).

5. Theorem R → E

In this chapter we start from a relativistic field theory given by a sequence $\{\mathfrak{W}_n\}_{n=0}^\infty$ of Wightman distributions, satisfying axioms (R 0)–(R 5), and we construct a sequence $\{\mathfrak{S}_n\}_{n=0}^\infty$ of Euclidean Green's functions with the properties (E 0)–(E 4).

It is well known that the Wightman distribution $\mathfrak{W}_n$ is the boundary value of the Wightman function $\mathfrak{W}_n(z_1, \ldots, z_n)$ which is analytic, single valued, symmetric and $iL_+(\mathbb{C})$ invariant for $(z_1, \ldots, z_n) \in \mathfrak{S}_n^P$, see [12], p. 83. $\mathfrak{S}_n^P$ contains the set

$$\mathscr{E}_n^0 = \{(z_1, \ldots z_n) : \operatorname{Im}(z_k - z_{k-1}) \in V_+ \quad \text{for all} \quad 1 < k \leqq n\} ,$$

and it is invariant under permutations of $(z_1, \ldots z_n)$. Hence it contains the set $\mathscr{E}_n = \{(z_1, \ldots z_n) : \operatorname{Re} z_k^0 = 0, \operatorname{Im} \underline{z}_k = 0, z_k \neq z_{k'}$ for all $1 \leqq k < k' \leqq n\}$. Points in $\mathscr{E}_n$ are called Euclidean points. The restriction of the Wightman functions to Euclidean points defines the Euclidean Green's functions. We set $\mathfrak{S}_0 = \mathfrak{W}_0 = 1$ and

$$\mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = \mathfrak{W}_n((-ix_1^0, \underline{x}_1), \ldots, (-ix_n^0, \underline{x}_n)), \tag{5.1}$$

for $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \Omega_n \equiv \{\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_n; \underset{\sim}{x}_i \neq \underset{\sim}{x}_j \text{ for all } 1 \leqq i < j \leqq n\}$.

The $\mathfrak{S}_n$, $n = 0, 1, \ldots$, are the Euclidean Green's functions and we have to verify that they have the properties (E0)–(E4). By (5.1), $\mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ is an analytic function, invariant under permutations of the arguments $\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n$, which is (E3). It is invariant under translations, because $\mathfrak{W}_n$ is. It is furthermore invariant under the action of the subgroup of those elements in $L_+(\mathbb{C})$ which map Euclidean points in Euclidean points. But this subgroup of $L_+(\mathbb{C})$ is just the group of Euclidean rotations. This proves (E1).

Now we prove (E0). Obviously the linear space ${}^0\mathscr{S}_c(\mathbb{R}^{4n})$ of all functions in $\mathscr{S}(\mathbb{R}^{4n})$ with compact support in Ω_n is dense in ${}^0\mathscr{S}(\mathbb{R}^{4n})$, and $\mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ is a continuous, uniformly bounded function on any compact subset of Ω_n. Hence the Riemannian integral

$$\mathfrak{S}_n(f) = \int \mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) f(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \, d^{4n}x \tag{5.2}$$

defines a linear functional on ${}^0\mathscr{S}_c(\mathbb{R}^{4n})$, which is symmetric under permutations of the arguments $\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n$ and invariant under iSO_4. (Note that $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \to (R\underset{\sim}{x}_1 + \underset{\sim}{a}, \ldots R\underset{\sim}{x}_n + \underset{\sim}{a})$, $R \in SO_4$, $\underset{\sim}{a} \in \mathbb{R}^4$, maps a compact set K in Ω_n onto another compact set K' in Ω_n.)

Proposition 5.1. *For $n = 1, 2, \ldots$, there exists a constant c and a number m such that for all $f \in {}^0\mathscr{S}_c(\mathbb{R}^{4n})$*

$$|\mathfrak{S}_n(f)| \leqq c |f|_m . \tag{5.3}$$

Postponing the proof of the proposition, we use it together with the density of ${}^0\mathscr{S}_c(\mathbb{R}^{4n})$ in ${}^0\mathscr{S}(\mathbb{R}^{4n})$ to extend $\mathfrak{S}_n$ to a distribution in ${}^0\mathscr{S}'(\mathbb{R}^{4n})$, proving (E0). By continuity this distribution, again denoted by $\mathfrak{S}_n$, has the invariance property (E1) and the symmetry property (E3). Positivity (E2) follows from the arguments given in Section 4.3. Let $\underset{\sim}{f} \in \underset{\sim}{\mathscr{S}}_+$. Then according to (4.22) and (4.24)

$$\begin{aligned} &\sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times f_m) \\ &= \sum_{n,m} \int \bar{\check{f}}_n(\underset{\sim}{q}_n, \ldots \underset{\sim}{q}_1) \, \check{f}_m(\underset{\sim}{q}_n, \ldots \underset{\sim}{q}_{n+m-1}) \, \tilde{W}_{n+m-1}(q_1, \ldots q_{n+m-1}) \, d^{4(n+m-1)}q . \end{aligned} \tag{5.4}$$

Axiom (R2) implies that (5.4) is nonnegative because for $\underset{\sim}{f} \in \underset{\sim}{\mathscr{S}}_+$, $\check{f}_n$ is in $\mathscr{S}(\overline{\mathbb{R}}_+^{4 \cdot n})$ and can therefore be interpreted as the restriction to $\{q_k^0 \geqq 0\}$ of some function $g_n(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n)$ in $\mathscr{S}(\mathbb{R}^{4n})$. The cluster property (E4) follows from (R4) by the arguments of Section 4.4: (R4) implies that

$\lim_{\lambda \to \infty} [(F, U_s(\lambda \underset{\sim}{a}) G) - (F, \Omega)(\Omega, G)] = 0$ for all $F, G \in \mathscr{K}$, and this implies (E 4).

Now we return to Proposition 5.1. Its proof has a geometrical and an analytical part. Let us do geometry first.

Lemma 5.2. *For $n = 2, 3, \ldots$ there exist $N = N(n) < \infty$ unit vectors $\underset{\sim}{e}_1, \ldots \underset{\sim}{e}_N$ in $\mathbb{R}^4$ and a constant $A > 0$, depending on n, such that for all $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \Omega_n$*

$$\max_k \min_{i<j} |\langle \underset{\sim}{e}_k, \underset{\sim}{x}_i - \underset{\sim}{x}_j \rangle| \geqq A \varrho(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)^{-1}, \tag{5.5}$$

where $\langle , \rangle$ is the Euclidean scalar product, and

$$\varrho(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) = \Big(\sum_{i<j} |\underset{\sim}{x}_i - \underset{\sim}{x}_j|^{-2} \Big)^{\frac{1}{2}}.$$

Proof. Pick $N > 3 \cdot 2^{-1} n(n-1)$ vectors $\underset{\sim}{e}_1, \ldots \underset{\sim}{e}_N$ such that any four of them span $\mathbb{R}^4$. Now take $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \Omega_n$ and define $2^{-1} n(n-1)$ unit vectors

$$\underset{\sim}{f}_{ij} = \frac{\underset{\sim}{x}_i - \underset{\sim}{x}_j}{|\underset{\sim}{x}_i - \underset{\sim}{x}_j|}, \qquad 1 \leqq i < j \leqq n.$$

Then each of the vectors $\underset{\sim}{f}_{ij}$ is orthogonal to at most 3 vectors $\underset{\sim}{e}_r, \underset{\sim}{e}_s, \underset{\sim}{e}_t$, and because of the choice of N, $\langle \underset{\sim}{e}_M, \underset{\sim}{f}_{ij} \rangle \neq 0$ for at least one $M \in \{1, 2, \ldots N\}$ and all $\underset{\sim}{f}_{ij}$, i.e.

$$h(f) = \max_k \min_{ij} |\langle e_k, f_{ij} \rangle| > 0 \quad \text{for all} \quad f = \{f_{12}, \ldots f_{n-1\,n}\}.$$

As $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)$ varies over Ω_n, f varies over the $2^{-1} n(n-1)$ fold tensor product of the unit sphere in $\mathbb{R}^4$, which is a compact set, and we have $h(f) > 0$. Thus there is a constant $A > 0$ such that $h(f) \geqq A$, for all f, i.e. for all $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \Omega_n$

$$\max_k \min_{ij} \left| \left\langle \underset{\sim}{e}_k, \frac{\underset{\sim}{x}_i - \underset{\sim}{x}_j}{|\underset{\sim}{x}_i - \underset{\sim}{x}_j|} \right\rangle \right| \geqq A.$$

Using the inequality $\varrho(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_n) \geqq |\underset{\sim}{x}_i - \underset{\sim}{x}_j|^{-1}$ for all $1 \leqq i < j \leqq n$, we prove (5.5).

We choose N unit vectors as in Lemma 5.3. These vectors remain fixed for the remainder of the proof. Then we define $N \cdot n!$ open subsets of Ω_n by

$$\Omega_{k\pi} = \{(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n); \langle \underset{\sim}{e}_k, \underset{\sim}{x}_{\pi(j+1)} - \underset{\sim}{x}_{\pi(j)} \rangle > 0 \quad \text{for all} \quad 1 \leqq j < n\},$$

where π is an element in the permutation group P_n of n elements and $k = 1, \ldots N$. The $\Omega_{k\pi}$ obviously cover Ω_n.

For the analytic part of the proof of Proposition 5.1 we have to define a partition of unity on Ω_n: Let $\tau \in C^\infty(\mathbb{R})$ be such that $0 \leqq \tau \leqq 1$

and $\tau(x)=0$ for $x<\frac{1}{2}$; $\tau(x)=1$, for $x \geqq 1$. Then we define $\hat{\chi}_{k\pi} \geqq 0$ on Ω_n by

$$\hat{\chi}_{k\pi}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) = \prod_{j=1}^{n-1} \tau(A^{-1} \varrho(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) \langle \underset{\sim}{e}_k, \underset{\sim}{x}_{\pi(j+1)} - \underset{\sim}{x}_{\pi(j)} \rangle),$$

where A is as in (5.5). Obviously $supp\, \hat{\chi}_{k\pi} \subset \Omega_{k\pi}$. From Lemma 5.2 we now conclude that for any $(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) \in \Omega_n$ there exists k, π such that $\hat{\chi}_{k\pi}(\underset{\sim}{x}_1 \dots \underset{\sim}{x}_n) = 1$, and therefore

$$\sum_{k,\pi} \hat{\chi}_{k\pi}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) \geqq 1$$

on Ω_n. Now we can define

$$\chi_{k\pi}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) = \hat{\chi}_{k\pi}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) \left[\sum_{k',\pi'} \hat{\chi}_{k'\pi'}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) \right]^{-1}, \tag{5.6}$$

and $\chi_{k\pi}$ is well defined and nonnegative on Ω_n; $\sum_{k,\pi} \chi_{k\pi} = 1$ on Ω_n, i.e. $\chi_{k\pi}$ defines a partition of unity on Ω_n. For $f \in {}^0\mathscr{S}_c(\mathbb{R}^{4n})$ we can now write

$$\mathfrak{S}_n(f) = \sum_{k,\pi} \mathfrak{S}_n(\chi_{k\pi} f), \tag{5.7}$$

and for a proof of Proposition 5.1 it suffices to estimate a single term in the sum on the right hand side of (5.7). Using (5.2), and the symmetry and SO_4 invariance of $\mathfrak{S}_n$ we write

$$\begin{aligned} \mathfrak{S}_n(\chi_{k\pi} f) &= \int \mathfrak{S}_n(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n)(\chi_{k\pi} f)(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n)\, d^{4n}x \\ &= \int \mathfrak{S}_n(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n)(\chi_{k\pi} f)^{\pi'}_{(\underset{\sim}{0},R)}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n)\, d^{4n}x, \end{aligned} \tag{5.8}$$

where π' is the inverse permutation of π and $R \in SO_4$ is chosen such that $R^{-1}\underset{\sim}{e}_k$ is the unit vector in the time direction, i.e. $R^{-1}\underset{\sim}{e}_k = (1, \underline{0})$. As $\chi_{k\pi}$ has its support in $\Omega_{k\pi}$, $(\chi_{k\pi} f)^{\pi'}_{(\underset{\sim}{0},R)}$ has its support in $\Omega_< = \{\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n;\ x^0_{j+1} - x^0_j > 0, j = 1, \dots n-1\}$, and is therefore in $\mathscr{S}_<(\mathbb{R}^{4n}) \cap {}^0\mathscr{S}_c(\mathbb{R}^{4n})$. To finish the proof of Proposition 5.1 we need two more lemmas

Lemma 5.3. *The restriction of* $\mathfrak{S}_n(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n)$ *to* $\Omega_<$ *defines a distribution in* $\mathscr{S}'_<(\mathbb{R}^{4n})$.

Lemma 5.4. *For all* $m_1 \geqq 0$ *there exist constants* c *and* m *depending only on* m_1*, such that*

$$|\chi_{k\pi} f|_{m_1} < c |f|_m,$$

for all $k = 1, \dots N$, $\pi \in P_n$, $f \in {}^0\mathscr{S}_c(\mathbb{R}^{4n})$.

We use Lemma 5.3 to show that there exist constants c and m_1 such that (using 5.8)

$$\begin{aligned} |\mathfrak{S}_n(\chi_{k\pi} f)| &= |\mathfrak{S}_n((\chi_{k\pi} f)^{\pi'}_{(\underset{\sim}{0},R)})| \\ &\leqq c |(\chi_{k\pi} f)^{\pi'}_{(\underset{\sim}{0},R)}|_{m_1} = c |\chi_{k\pi} f|_{m_1}, \end{aligned} \tag{5.9}$$

and Proposition 5.1 follows from (5.7), (5.9) and Lemma 5.4.

Proof of Lemma 5.3: For $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \Omega_<$ we can write

$$\begin{aligned}\mathfrak{S}_n(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) &= S_{n-1}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{n-1}) \\ &= \int e^{-\sum_{k=1}^{n-1} (\xi_k^0 q_k^0 - i \underset{\sim}{\xi}_k \underset{\sim}{q}_k)} \tilde{W}_{n-1}(\underset{\sim}{q}_1 \ldots \underset{\sim}{q}_{n-1})\, d^{4(n-1)} q\,,\end{aligned}$$

where $\underset{\sim}{\xi}_k = \underset{\sim}{x}_{k+1} - \underset{\sim}{x}_k$ and $\xi_k^0 > 0$ for $(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n) \in \Omega_<$. It suffices to prove that $S_{n-1}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{n-1})$ is in $\mathscr{S}'(\mathbb{R}_+^{4 \cdot n})$. By the nuclear theorem and the support properties of $\tilde{W}_{n-1}(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_{n-1})$ we only need to show that for $W(q) \in \mathscr{S}'(\overline{\mathbb{R}}_+)$ and $g_\xi(q) = e^{-\xi q} (q \geqq 0)$, the function $S(\xi) = W(g_\xi) \restriction (\xi > 0)$ defines a distribution in $\mathscr{S}'(\mathbb{R}_+)$. But for $\xi > 0$

$$\begin{aligned}|W(g_\xi)| &\leqq c_1 |g_\xi|''_m \\ &\leqq c_1 \sup_{\substack{q \geqq 0 \\ \alpha \leqq m}} (1+q)^m \left| \frac{d^\alpha}{dq^\alpha} e^{-\xi q} \right| \\ &\leqq c_2 (1 + \xi^m) \sup_{q \geqq 0} (1+q)^m e^{-\xi q} \\ &\leqq c_3 (1 + \xi^m)(1 + \xi^{-m})\,,\end{aligned}$$

for some constants c_i and m. Now let $f \in \mathscr{S}(\mathbb{R}_+)$. Because $f(x)$ vanishes together with all its derivatives $f^{(\alpha)}(x)$ at $x = 0$, we can write it as $f(x) = \frac{x^n}{n!} f^{(n)}(y_x)$ for any $n \geqq 0$ and some y_x, $0 \leqq y_x \leqq x$. Then we find that for some constants c_i

$$\begin{aligned}\left| \int_0^\infty f(\xi) W(g_\xi)\, d\xi \right| &\leqq c_3 \int_0^\infty (1+\xi^m)(1+\xi^{-m}) |f(\xi)|\, d\xi \\ &\leqq c_4 \sup_{\xi > 0} (1 + \xi^{m+2} + \xi^{-m}) |f(\xi)| \\ &\leqq c_4 \left(\sup_{\xi > 0} \{(1 + \xi^{m+2}) |f(\xi)|\} + \sup_{\xi > 0} \left\{ \xi^{-m} \left| \frac{\xi^m}{m!} f^{(m)}(y_\xi) \right| \right\} \right) \\ &\leqq c_5 |f|_{m+2}\,.\end{aligned}$$

This proves the assertion and thus Lemma 5.3.

Proof of Lemma 5.4. We use the following estimates on derivatives of $\chi_{k\pi}$:

$$\begin{aligned}|\chi_{k\pi}^{(\alpha)}(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)| &\leqq c_\alpha \varrho(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n)^{|\alpha|} \\ &\leqq c'_\alpha \sum_{i<j} |\underset{\sim}{x}_i - \underset{\sim}{x}_j|^{-|\alpha|}\,.\end{aligned}$$

Then the lemma follows from arguments similar to those given at the end of the proof of Lemma 5.3.

6. Arbitrary Spinor Fields

In this chapter we generalize the axioms (E0)–(E4) to include arbitrary spinor fields. We also show how to modify the arguments of Chapters 4 and 5 to prove theorems E→R and R→E in this general situation.

The Wightman distributions are given by

$$\mathfrak{W}_{\nu k}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) = (\Omega, \psi_{\nu_1}^{(k_1)}(\underset{\sim}{x}_1), \dots \psi_{\nu_n}^{(k_n)}(\underset{\sim}{x}_n)\Omega). \tag{6.1}$$

Here $\psi_{\nu_i}^{(k_i)}(\underset{\sim}{x}_i)$ stands for one of the finite number of fields that describe the theory. ν_i represents a set of dotted and undotted indices $(\alpha_1, \dots \alpha_{m_{k_i}}, \dot{\beta}_1, \dots \dot{\beta}_{n_{k_i}})$ and describes the spinor character of the field labeled by k_i. Finally ν stands for the set $\nu_1, \dots \nu_n$ and k stands for the set $k_1, \dots k_n$.

Let $S(A, B)$ be a finite dimensional analytic representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, the universal covering group of $L_+(\mathbb{C})$. Then $S(A, \bar{A})$ is a representation of $SL(2,\mathbb{C})$, the universal covering group of $L_+^\uparrow$, and all finite dimensional continuous representations of $SL(2,\mathbb{C})$ may be obtained in this way. The transformation properties of the fields are given by

$$U(\{\underset{\sim}{a}, A\})\, \psi_{\nu_i}^{(k_i)}(\underset{\sim}{x})\, U(\{\underset{\sim}{a}, A\})^{-1} = \sum_{\mu_i} S^{(k_i)}(A^{-1}, \bar{A}^{-1})_{\nu_i}^{\mu_i}\, \psi_{\mu_i}^{(k_i)}(\Lambda(A, \bar{A})\underset{\sim}{x} + \underset{\sim}{a}),$$

$\underset{\sim}{a} \in \mathbb{R}^4$, $A \in SL(2,\mathbb{C})$. U is a unitary representation of the inhomogeneous $SL(2,\mathbb{C})$ and $S^{(k_i)}(A, B)$ is a finite dimensional analytic representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Furthermore $\Lambda(A, B)$ is given by $\widetilde{\Lambda(A,B)z} = A\tilde{z}B^T$ where $\tilde{z} = \sum_{\mu=0}^{3} z^\mu \sigma^\mu$ and $\underset{\sim}{z} \in \mathbb{C}^4$. For notational convenience we denote the adjoint fields $\psi_{\nu_i}^{(k_i)}(\underset{\sim}{x})^*$ by $\psi_{\nu_i^*}^{(-k_i)}(\underset{\sim}{x})$, where $\nu_i^* = (\beta_1, \dots \beta_{n_{k_i}}, \dot{\alpha}_1, \dots \dot{\alpha}_{m_{k_i}})$. We have to require that $\overline{S^{(k_i)}(A, B)_{\mu_i}^{\nu_i}} = S^{(-k_i)}(A, B)_{\mu_i^*}^{\nu_i^*}$. Relativistic covariance leads to

(R 1) $$\mathfrak{W}_{\nu k}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) = \sum_{\mu} S_k(A^{-1}, \bar{A}^{-1})_\nu^\mu\, \mathfrak{W}_{\mu k}(\Lambda\underset{\sim}{x}_1 + \underset{\sim}{a}, \dots \Lambda\underset{\sim}{x}_n + \underset{\sim}{a}),$$

for all $A \in SL(2,\mathbb{C})$, $\underset{\sim}{a} \in \mathbb{R}^4$. Here $\Lambda = \Lambda(A, \bar{A})$ and $S_k(A, B)$ is a finite dimensional analytic representation of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Again using formula (5.1) to define the Euclidean Green's function we find that the covariance axiom (E 1) becomes

(E 1) $$\mathfrak{S}_{\nu k}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) = \sum_{\mu} S_k(U^{-1}, V^{-1})_\nu^\mu\, \mathfrak{S}_{\mu k}(R\underset{\sim}{x}_1 + \underset{\sim}{a}, \dots R\underset{\sim}{x}_n + \underset{\sim}{a}),$$

for all $U, V \in SU(2)$, $\underset{\sim}{a} \in \mathbb{R}^4$. Here $R = R(U, V)$ is defined by $\Lambda(U, V)(-ix^0, \underline{x}) = (-i(Rx)^0, \underline{Rx})$ and is a homomorphism of $SU(2) \times SU(2)$ onto SO_4. Note that $SU(2) \times SU(2)$ is the universal covering group of SO_4. Hence $S_k(U, V)$ defines a continuous finite dimensional (one or two valued) representation of SO_4. Axiom (E0) remains of course unchanged and its derivation from the Wightman axioms is as in Chapter 5. Axiom (E3) becomes

$$\text{(E3)} \quad \mathfrak{S}_{\nu k}(x_1, \ldots x_n) = \sigma \mathfrak{S}_{\hat{\nu}\hat{k}}(x_{\pi(1)}, \ldots x_{\pi(n)})$$

for all permutations π. Here $\sigma = \pm 1$ and $\hat{\nu} = (\nu_{\pi(1)}, \ldots \nu_{\pi(n)})$, $\hat{k} = (k_{\pi(1)}, \ldots k_{\pi(n)})$, [14]. A reformulation of the nonlinear axioms (E2) and (E4) requires some modifications of the notation introduced in Chapter 2. In the following $\mathscr{S}_+$ will denote the set of all finite sequences $\underline{f} = (f_0, f_1, f_2, \ldots)$ where $f_0 \in \mathbb{C}$ and each f_n is a sequence of elements $f_{n,\nu k} \in \mathscr{S}_+(\mathbb{R}^{4n})$, with νk as before. Also for $\underline{f} \in \mathscr{S}_+$ we redefine $\underline{f}^*$ by

$$(\underline{f}^*)_{n,\nu k}(x_1, \ldots x_n) = \bar{f}_{n,\nu^* k^*}(x_n, \ldots x_1), \tag{6.2}$$

where $\nu^* = (\nu_n^*, \ldots \nu_1^*)$ and $k^* = (-k_n, \ldots -k_1)$. Furthermore for $a \in \mathbb{R}^4$ we define $\underline{f}_{(a)}$ by

$$(\underline{f}_{(a)})_{n,\nu k}(x_1, \ldots x_n) = f_{n,\nu k}(x_1 - a, \ldots x_n - a).$$

The remaining axioms can now be written as

$$\text{(E2)} \quad \sum_{\substack{n,m \\ \nu k \\ \mu \ell}} \mathfrak{S}_{\nu\mu k\ell}(\Theta(\underline{f}^*)_{n,\nu k} \times f_{m,\mu\ell}) \geqq 0, \quad \text{for all} \quad \underline{f} \in \mathscr{S}_+.$$

$$\text{(E4)} \quad \lim_{\lambda \to \infty} \sum_{\substack{n,m \\ \nu k \\ \mu \ell}} \{\mathfrak{S}_{\nu\mu k\ell}(\Theta(\underline{f}^*)_{n,\nu k} \times g_{m,\mu\ell(\lambda a)}) - \mathfrak{S}_{\nu k}(\Theta(\underline{f}^*)_{n,\nu k})\, \mathfrak{S}_{\mu\ell}(g_{m,\mu\ell})\} = 0,$$

for all $\underline{f}, \underline{g} \in \mathscr{S}_+$, $a = (0, \underline{a})$, $\underline{a} \in \mathbb{R}^3$.

(E2) and (E4) follow from the Wightman axioms as in Chapter 5.

To reconstruct the Wightman distributions from a set of Euclidean Green's functions obeying (E0)–(E4) we proceed as in Chapter 4. The only step which has to be modified is the derivation of Lorentz covariance of $\mathfrak{W}_{\nu k}$ from Euclidean covariance of $\mathfrak{S}_{\nu k}$. Given the Euclidean covariance of $S_{\nu k}(\xi_1, \ldots \xi_{n-1}) = \mathfrak{S}_{\nu k}(x_1, \ldots x_n)$, $\xi_i = x_{i+1} - x_i$, $\xi_i^0 > 0$, we have to prove the Lorentz covariance of $\tilde{W}_{\nu k}(q_1, \ldots q_n)$, where

$$S_{\nu k}(\xi_1, \ldots \xi_n) = \int e^{-\sum_{k=1}^{n} (\xi_k^0 q_k^0 - i \underline{\xi}_k \underline{q}_k)} \tilde{W}_{\nu k}(q_1, \ldots q_n)\, d^{4n} q. \tag{6.3}$$

First we note that any element A in $SL(2,\mathbb{C})$ can be written as $A = UH$, where $U \in SU(2)$ and H is positive. Hence it is sufficient to prove that

$$\tilde{W}_{\nu k}(q_1, \ldots q_n) = \sum_{\mu} S_k(U, \bar{U})_\nu^\mu\, \tilde{W}_{\mu k}(\Lambda^{-1}(U, \bar{U}) q_1, \ldots \Lambda^{-1}(U, \bar{U}) q_n), \tag{6.4}$$

and

$$\tilde{W}_{\nu k}(q_1, \ldots q_n) = \sum_{\mu} S_k(H, \bar{H})_\nu^\mu\, \tilde{W}_{\mu k}(\Lambda^{-1}(H, \bar{H}) q_1, \ldots \Lambda^{-1}(H, \bar{H}) q_n), \tag{6.5}$$

for all $U \in SU(2)$ and all positive H. Eq. (6.4) follows immediately from (6.3) and (E 1) with V replaced by $\bar{U}$, because $\Lambda(U, \bar{U})$ leaves the zero component invariant. To prove Eq. (6.5) we consider the one parameter group $H(\varphi) = \mathrm{Cosh}\,\frac{\varphi}{2} + \mathrm{Sinh}\,\frac{\varphi}{2}\,\underline{e}\cdot\underline{\sigma}$ of hermitian operators, where $\underline{e}$ is a fixed unit vector in $\mathbb{R}^3$ and $-\infty < \varphi < \infty$. Then it suffices to show that for $\Lambda(\varphi) = \Lambda(H(\varphi), \bar{H}(\varphi))$, the expression

$$\begin{aligned} &\frac{d}{d\varphi}\sum_{\mu} S_k(H(\varphi), \bar{H}(\varphi))^{\mu}_{\nu}\, \tilde{W}_{\mu k}(\Lambda^{-1}(\varphi)\underset{\sim}{q}_1, \ldots \Lambda^{-1}(\varphi)\underset{\sim}{q}_n)|_{\varphi=0} \\ &= \sum_{\substack{\mu\\ \alpha,\beta}}\left(\frac{dH_{\alpha\beta}}{d\varphi}\,\frac{\partial S_k}{\partial A_{\alpha\beta}}(A,B)^{\mu}_{\nu} + \frac{d\bar{H}_{\alpha\beta}}{d\varphi}\,\frac{\partial S_k}{\partial B_{\alpha\beta}}(A,B)^{\mu}_{\nu}\right)\Bigg|_{\substack{A=B=1\\ \varphi=0}} \tilde{W}_{\mu k}(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n) \\ &\qquad + \left(\sum_{i,\alpha}\frac{d(\Lambda^{-1}\underset{\sim}{q}_i)^{\alpha}}{d\varphi}\,\frac{\partial}{\partial q_i^{\alpha}}\Bigg|_{\varphi=0}\right)\tilde{W}_{\mu k}(\underset{\sim}{q}_1, \ldots \underset{\sim}{q}_n) \end{aligned} \tag{6.6}$$

vanishes.

Let $V(\varphi) = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\,\underline{e}\cdot\underline{\sigma}$ be a one parameter group in $SU(2)$ with $\underline{e}$ fixed, as above. Then Euclidean covariance of $S_{\nu k}$ implies that for $R(\varphi) = R(V(\varphi), {}^tV(\varphi))$

$$\begin{aligned} 0 &= \frac{d}{d\varphi}\sum_{\mu} S_k(V(\varphi), {}^tV(\varphi))^{\mu}_{\nu}\, S_{\mu k}(R^{-1}(\varphi)\underset{\sim}{\xi}_1, \ldots R^{-1}(\varphi)\underset{\sim}{\xi}_n)|_{\varphi=0} \\ &= \sum_{\substack{\mu\\ \alpha,\beta}}\left(\frac{dV_{\alpha\beta}}{d\varphi}\,\frac{\partial S_k}{\partial A_{\alpha\beta}}(A,B)^{\mu}_{\nu} + \frac{d\,{}^tV_{\alpha\beta}}{d\varphi}\,\frac{\partial S_k}{\partial B_{\alpha\beta}}(A,B)^{\mu}_{\nu}\right)\Bigg|_{\substack{A=B=1\\ \varphi=0}} S_{\mu k}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n) \\ &\quad + \left(\sum_{i,\alpha}\frac{d(R^{-1}\underset{\sim}{\xi}_i)^{\alpha}}{d\varphi}\,\frac{\partial}{\partial \xi_i^{\alpha}}\Bigg|_{\varphi=0}\right) S_{\mu k}(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n)\,. \end{aligned} \tag{6.7}$$

From the definition of $H(\varphi)$ and $V(\varphi)$ we get

$$\frac{dH_{\alpha\beta}}{d\varphi}\Bigg|_{\varphi=0} = -i\,\frac{dV_{\alpha\beta}}{d\varphi}\Bigg|_{\varphi=0} = \tfrac{1}{2}(\underline{e}\,\underline{\sigma})_{\alpha\beta}\,,$$

and

$$\frac{d\bar{H}_{\alpha\beta}}{d\varphi}\Bigg|_{\varphi=0} = -i\,\frac{d\,{}^tV_{\alpha\beta}}{d\varphi}\Bigg|_{\varphi=0} = \tfrac{1}{2}\overline{(\underline{e}\,\underline{\sigma})}_{\alpha\beta}\,. \tag{6.8}$$

Furthermore

$$\sum_{i,\alpha}\frac{d(\Lambda^{-1}q_i)^{\alpha}}{d\varphi}\,\frac{\partial}{\partial q_i^{\alpha}}\Bigg|_{\varphi=0} = -\sum_{\substack{i\\ 1\le j\le 3}} e^j\left(q_i^0\frac{\partial}{\partial q_i^j} + q_i^j\frac{\partial}{\partial q_i^0}\right),$$

and

$$\sum_{i,\alpha}\frac{d(R^{-1}\xi_i)^{\alpha}}{d\varphi}\,\frac{\partial}{\partial \xi_i^{\alpha}}\Bigg|_{\varphi=0} = -\sum_{\substack{i\\ 1\le j\le 3}} e^j\left(\xi_i^0\frac{\partial}{\partial \xi_i^j} - \xi_i^j\frac{\partial}{\partial \xi_i^0}\right). \tag{6.9}$$

Now we insert (6.8) and (6.9) in (6.6) and (6.7) respectively. Then the vanishing of (6.6) follows from (6.3) and (6.7) by the arguments of Section 3.

7. Application

By theorem E→R, a relativistic quantum field theory model can be obtained by constructing a set of Euclidean Green's functions, satisfying (E 0)–(E 4). Formally the Euclidean Green's functions are given by

$$\mathfrak{S}_{n\,\mathrm{formal}}(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n) = (\Omega_E, A^{(1)}(\underset{\sim}{x}_1) \dots A^{(n)}(\underset{\sim}{x}_n) e^{-V} \Omega_E)/(\Omega_E, e^{-V} \Omega_E),$$

where Ω_E is the Euclidean no particle state, $A^i(\underset{\sim}{x}_i)$ are free Euclidean fields and V is the Euclidean action, [21] and [18]. By introducing a volume cutoff h and an ultraviolet cutoff $\varkappa$ we make $\mathfrak{S}_{n,h,\varkappa}$ well defined objects. We define $\mathfrak{S}_n$ as the limit of $\mathfrak{S}_{n,h,\varkappa}$ as $\varkappa \to \infty$, and $h \to 1$. To complete the program we would have to show that this limit exists and has properties (E 0)–(E 4). Euclidean covariance (E 1) follows once the uniqueness of the limit is established. All the other axioms are expected to hold for the cutoff Green's functions, independently of $\varkappa$ and h. Estimates are needed to prove this assertion for (E 0) (distribution property) and (E 4) (cluster property). For superrenormalizable models the necessary techniques have been developed by Glimm and Jaffe [7, 8] and by Dimock, Glimm and Spencer [9, 2]. On the other hand, for models involving boson-fermion interactions, (E 2) (symmetry) and (E 3) (positivity) are trivially satisfied for a large class of cutoffs. The symmetry property follows from the fact that free Euclidean bose and fermi fields commute, respectively anticommute, with or without ultraviolet cutoff. If there is no ultraviolet cutoff in time direction then positivity (E 3) follows from the Feynman-Kac formula and the relation connecting the Euclidean action V with its adjoint, [18].

8. Technicalities

In this chapter we state and – where necessary – prove some technical lemmas, used in earlier chapters.

Lemma 8.1. (see p. 86). *The set of seminorms on* $\mathscr{S}(\overline{\mathbb{R}}_+)$

$$|f_+|''_m = \sup_{\substack{x \geqq 0 \\ \alpha \leqq m}} (1 + x^2)^{m/2} \, |f^{(\alpha)}(x)|,$$

$m = 1, 2, \dots$ *is equivalent to the set of seminorms* $|f_+|'_m$.

Proof. Obviously for $f \in \mathscr{S}(\mathbb{R})$,

$$|f_+|'_m = \inf_{g \in \mathscr{S}(\mathbb{R}_-)} |f + g|_m \geqq |f_+|''_m .$$

Thus it remains to prove that for a given m we can find c and n such that for all $f \in \mathscr{S}(\mathbb{R})$,

$$|f_+|'_m \leqq c|f_+|''_n . \tag{8.1}$$

Let $\mathscr{S}^{(m)}(\mathbb{R}_-)$ be the closure of $\mathscr{S}(\mathbb{R}_-)$ with respect to the $|\ \ |_m$-norm. Then

$$|f_+|'_m = \inf_{g \in \mathscr{S}^{(m+1)}(\mathbb{R}_-)} |f+g|_m . \tag{8.2}$$

Let $\varphi(x)$ be a C^∞ function such that $0 \leqq \varphi(x) \leqq 1$, $\varphi(x)=0$ for $x \leqq -1$, $\varphi(x)=1$ for $x \geqq -\frac{1}{2}$. Now we define

$$g(x) = \begin{cases} \varphi(x) \sum_{\alpha=0}^{m+1} \left(f^{(\alpha)}(0) \dfrac{x^\alpha}{\alpha!} \right) - f(x), & \textit{for} \quad x \leqq 0, \\ 0, & \textit{for} \quad x > 0. \end{cases}$$

Certainly $g \in \mathscr{S}^{(m+1)}(\mathbb{R}_-)$, and

$$\begin{aligned} |f+g|_m &\leqq \sup_{\substack{x \leqq 0 \\ \beta \leqq m}} (1+x^2)^{m/2} \left| D^\beta \left(\varphi(x) \sum_{\alpha=0}^{m+1} f^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right) \right| \\ &\quad + \sup_{\substack{x \geqq 0 \\ \beta \leqq m}} (1+x^2)^{m/2} |f^{(\beta)}(x)| \\ &\leqq c_1 \sup_{\alpha \leqq m+1} f^{(\alpha)}(0) + |f_+|''_m \\ &\leqq c|f_+|''_{m+1} , \end{aligned} \tag{8.3}$$

for some constants c_1 and c, depending on φ and m but not on f. Ineq. (8.1) follows from (8.2) and (8.3). Our proof of (8.3) is an adapted version of Hörmanders proof of Whitney's extension theorem see [27] and [11].

Lemma 8.2. *Suppose $g(x) \in \mathscr{S}(\mathbb{R}_+)$ and define $\check{g}$ by*

$$\check{g}(q) = \int e^{-qx} g(x)\, dx \restriction \{q \geqq 0\} .$$

Then $\check{g} \in \mathscr{S}(\overline{\mathbb{R}}_+)$ and $g \to \check{g}$ is a continuous map of $\mathscr{S}(\mathbb{R}_+)$ onto a dense subset $\check{\mathscr{S}}$ of $\mathscr{S}(\overline{\mathbb{R}}_+)$. The kernel of this map is $\{0\}$.

Proof. The integral $\int e^{-qx} g(x)\, dx$ is uniformly convergent for $q \geqq 0$, thus $(1+q)^m D^\alpha \check{g}(q) = \int e^{-qx} \left(1 + \dfrac{d}{dx}\right)^m ((-x)^\alpha g(x))\, dx \restriction \{q \geqq 0\}$. In particular (see (2.2))

$$\begin{aligned} |\check{g}|''_m &\leqq \sup_{\substack{q \geqq 0 \\ \alpha \leqq m}} |(1+q)^m D^\alpha \check{g}(q)| \\ &\leqq c_1 |g|_{m+2} \sup_{q \geqq 0} \int_0^\infty e^{-qx}(1+x^2)^{-1}\, dx \leqq c_2 |g|_{m+2} , \end{aligned} \tag{8.4}$$

for some constants c_1 and c_2. Ineq. (8.4) proves that $\check{g} \in \mathscr{S}(\overline{\mathbb{R}}_+)$ and that the map $g \to \check{g}$ is continuous. In order to prove that the range $\check{\mathscr{S}}$ of this map is dense in $\mathscr{S}(\overline{\mathbb{R}}_+)$ we take a distribution $W \in \mathscr{S}'(\overline{\mathbb{R}}_+)$ with the property that $W(\check{g}) = 0$ for all $g \in \mathscr{S}(\mathbb{R}_+)$ and show that this implies $W \equiv 0$. As W is in $\mathscr{S}'(\overline{\mathbb{R}}_+)$ it is a distribution in $\mathscr{S}'(\mathbb{R})$ with support in $[0, \infty)$ and its Fourier-Laplace transform $\int e^{izq} W(q)\, dq = \tilde{W}(z)$ is an analytic function in $\{z : \operatorname{Im} z > 0\}$. Now we need the following well known lemma.

Lemma 8.3 ([26], p. 23). *Suppose $W \in \mathscr{S}'(\mathbb{R})$ and $\operatorname{supp} W \subset [0, \infty)$. Then*

$$W = \sum_{\alpha \leq M} D^\alpha \mu_\alpha , \tag{8.5}$$

where $\mu_\alpha(q)$ are measures of power increase with support in $[0, \infty)$.

Remark. We say a measure μ is of power increase of order α if $\int |d\mu(x)| (1 + |x|)^{-\alpha} < \infty$ for some $\alpha \geq 0$.

Using (8.5) and the definition of $\check{g}$ we now can write for $g \in \mathscr{S}(\mathbb{R}_+)$

$$W(\check{g}) = \sum_{\alpha \leq M} \int \left(\int e^{-qx} x^\alpha g(x)\, dx \right) d\mu_\alpha(q) . \tag{8.6}$$

We claim that

$$\int \left(\int |e^{-qx} x^\alpha g(x)|\, dx \right) |d\mu_\alpha(q)| < \infty , \tag{8.7}$$

thus by Fubini's theorem we can change the order of integration in (8.6) and obtain

$$\begin{aligned} W(\check{g}) &= \sum_{\alpha \leq M} \int_0^\infty \left(\int e^{-qx} D^\alpha\, d\mu_\alpha(q) \right) g(x)\, dx \\ &= \int_0^\infty \tilde{W}(ix)\, g(x)\, dx , \end{aligned} \tag{8.8}$$

and we suppose (8.8) to vanish for all $g \in \mathscr{S}(\mathbb{R}_+)$. As $\tilde{W}(ix)$ is a real analytic function of $x > 0$, (8.8) implies that $\tilde{W}(ix) = 0$ for $x > 0$ and hence $\tilde{W}(z) = 0$ for $\operatorname{Im} z > 0$. By the uniqueness of the Laplace-Fourier transform we conclude that the distribution W is identically zero, which proves the density of $\check{\mathscr{S}}$ in $\mathscr{S}(\overline{\mathbb{R}}_+)$. The last statement of the lemma is obvious. It remains to prove Ineq. (8.7). Suppose μ_α is of increase of order $\beta > 0$. Then

$$\begin{aligned} &\int \left(\int e^{-qx} |x^\alpha g(x)|\, dx \right) |d\mu_\alpha(q)| \\ &\leq \sup_{q \geq 0} \left((1+q)^\beta \int e^{-qx} |x^\alpha g(x)|\, dx \right) \int |d\mu_\alpha(q)| (1+q)^{-\beta} \\ &\leq c_1 \int_0^\infty \left(\sup_{q \geq 0} (1+q)^\beta e^{-qx} \right) |x^\alpha g(x)|\, dx \\ &\leq c_2 \int_0^\infty |x^{-\beta+\alpha} g(x)|\, dx \leq c_3 \sup_{x \geq 0} (1 + x^2)\, x^{-\gamma} |g(x)| , \end{aligned} \tag{8.9}$$

for some constants c_ι, γ. Hence it suffices to show that $\sup_{x\geqq 0}|x^k g(x)|$ is finite for any integer k. For nonnegative k this is trivial as $g\in\mathscr{S}(\mathbb{R}_+)$. For negative k we use the fact that $g(x)$ vanishes with all its derivatives at $x=0$ and can be written as $g(x)=\dfrac{x^{-k}}{(-k)!}g^{(-k+1)}(y_x)$, where $y_x\in[0,x)$. Hence

$$\sup_{x\geqq 0}|x^k g(x)|=(|k|!)^{-1}\sup_{x\geqq 0}|g^{(-k+1)}(y_x)|<\infty\,.$$

This proves inequality (8.7).

The first part of the following lemma is equivalent to Eq. (8.8). We leave out the proof of the second part.

Lemma 8.4 *Suppose $W\in\mathscr{S}'(\mathbb{R})$ and $\operatorname{supp}W\subset[0,\infty)$ and denote by $f_x(q)$ the restriction of e^{-qx} to $\{q\geqq 0\}$. Then $W(\int f_x(\cdot)\,g(x)\,dx)=\int g(x)\,W(f_x(\cdot))\,dx$ for all $g\in\mathscr{S}(\mathbb{R}_+)$. Furthermore $\frac{d}{dx}W(f_x(\cdot))=W\left(\frac{d}{dx}f_x(\cdot)\right)$.*

Remark. Lemmas 8.1–8.4 can immediately be generalized to the case of several variables.

Lemma 8.5 ([26], p. 31). *Let $T(t,s)$ be a distribution in the dual space of $\mathscr{S}(\mathbb{R}_+)\hat{\otimes}\mathscr{S}(\mathbb{R})$ and suppose that for $t>0$ its real and imaginary parts satisfy the Cauchy-Riemann conditions*

$$\frac{\partial}{\partial t}\operatorname{Re}T=\frac{\partial}{\partial s}\operatorname{Im}T,\quad \frac{\partial}{\partial t}\operatorname{Im}T=-\frac{\partial}{\partial s}\operatorname{Re}T\,.$$

Then $T(t,s)=G(\tau)$, $\tau=t+is$, for some function G which is analytic in the open right half plane $\{\tau:\operatorname{Re}\tau>0\}$.

Lemma 8.6 ([26], p. 239). *Let $G(\tau)$ be a function, analytic in the open right half plane, satisfying the inequality*

$$|G(\tau)|\leqq M(1+|\tau|)^\beta t^{-\alpha}\,, \tag{8.10}$$

for some positive constants M, α, β and for $t=\operatorname{Re}\tau>0$. Then $G(\tau)$ is the Fourier-Laplace transform $G(\tau)=\int e^{-\tau\alpha}\check{G}(\alpha)\,d\alpha$ of some distribution $\check{G}\in\mathscr{S}'(\overline{\mathbb{R}}_+)$.

Lemma 8.7 *Let $T(t,s)$ be as in Lemma 8.5. Then there is a distribution $\check{G}\in\mathscr{S}'(\overline{\mathbb{R}}_+)$, such that T is the Fourier-Laplace transform of $\check{G}$,*

$$T(t,s)=\int e^{-\alpha(t+is)}\check{G}(\alpha)\,d\alpha=G(\tau)\quad\textit{for}\quad t>0,\ \tau=t+is\,.$$

$G(\tau)$ is holomorphic for $\operatorname{Re}\tau>0$.

Proof. By Lemma 8.5, $T(t, s) = G(\tau)$ for some function G which is analytic in the open right half plane. We prove that $G(\tau)$ satisfies inequality (8.10) for some positive constants M, α, β. Then Lemma 8.7 follows from Lemma 8.6. Let $\tau_0 = t_0 + is_0$, $t_0 > 0$. Then for all $r \in (0, t_0)$

$$G(\tau_0) = \frac{1}{2\pi} \int_0^{2\pi} G(\tau_0 + re^{i\varphi})\, d\varphi\,.$$

Let $h(r)$ be a C^∞ function with compact support in $[\frac{1}{4}, \frac{3}{4}]$ and suppose $\int h(r) r\, dr = 1$. Then $h_{t_0}(r) = t_0^{-2} h(t_0^{-1} r)$ has its support in $\left[\frac{t_0}{4}, \frac{3t_0}{4}\right]$ and $\int h_{t_0}(r) r\, dr = 1$. Furthermore by Fubini's theorem

$$\begin{aligned} G(\tau_0) &= \frac{1}{2\pi} \int h_{t_0}(r) \left[\int_0^{2\pi} G(\tau_0 + re^{i\varphi})\, d\varphi\right] r\, dr \\ &= \frac{1}{2\pi} \int G(\tau_0 + t + is) h_{t_0}(\sqrt{t^2 + s^2})\, dt\, ds \\ &= \frac{1}{2\pi} \int T(t, s)\, h_{t_0}(\sqrt{(t - t_0)^2 + (s - s_0)^2})\, dt\, ds\,. \end{aligned}$$

Hence by the properties of T there is an $\mathscr{S}$-norm $|\cdot|_m$ and a constant c, such that

$$|G(\tau_0)| \leqq c |h_{t_0}(\sqrt{(\cdots - t_0)^2 + (\cdots - s_0)^2})|_m\,. \tag{8.11}$$

Inequality (8.10) follows from (8.11). This proves Lemma 8.7.

Lemma 8.8 *Let $S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n)$ be a distribution in $\mathscr{S}'(\mathbb{R}_+^{4\cdot n})$, and set for $m = 1, 2, \ldots n$,*

$$S_n^{(m)}(\xi_m^0, f_m) = \int S(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n)\, f_m(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_{m-1}, \underline{\xi}_m, \underset{\sim}{\xi}_{m+1} \ldots \underset{\sim}{\xi}_n)$$
$$d^4 \xi_1 \ldots d^4 \xi_{m-1}\, d^3 \xi_m\, d^4 \xi_{m+1} \ldots d^4 \xi_n\,,$$

for $f_m \in \mathscr{S}(\mathbb{R}_+^{4\cdot(m-1)}) \hat{\otimes} \mathscr{S}_{\underline{\xi}_m}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}(\mathbb{R}_+^{4\cdot(m-n)})$. Assume that for all f_m and $m = 1, 2 \ldots n$, $S_n^{(m)}(\xi_m^0, f_m)$ can be extended to a function $\hat{S}_n^{(m)}(\zeta_m^0, f_m)$ which is analytic in $\{\zeta_m^0 = \xi_m^0 + i\eta_m^0, \xi_m^0 > 0\}$ and that $\hat{S}_n^{(m)}$ is the Fourier-Laplace transform of a distribution $\check{S}_n^{(m)}(q_m^0, f_m)$ in $\mathscr{S}'(\overline{\mathbb{R}}_+)$,

$$\hat{S}_n^{(m)}(\zeta_m^0, f_m) = \int e^{-\zeta_m^0 q_m^0} \check{S}_n^{(m)}(q_m^0, f_m)\, dq_m^0\,.$$

Then $S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n)$ is the Fourier-Laplace transform of a distribution $\tilde{W}_n(\underset{\sim}{q}_1 \ldots \underset{\sim}{q}_n)$ in $\mathscr{S}'(\overline{\mathbb{R}}_+^{4\cdot n})$,

$$S_n(\underset{\sim}{\xi}_1, \ldots \underset{\sim}{\xi}_n) = \int e^{-\sum_{k=1}^{n} (\xi_k^0 q_k^0 - i \underline{\xi}_k \underline{q}_k)}\, \tilde{W}_n(\underset{\sim}{q}_1 \ldots \underset{\sim}{q}_n)\, d^{4n} q\,. \tag{8.12}$$

Proof. We prove the lemma for $n=2$; the proof for arbitrary n is then straightforward. Let $f_1(\underline{\xi}_1, \xi_2) \in \mathscr{S}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}(\mathbb{R}_+^4)$ and $h(\xi_1^0) \in \mathscr{S}(\mathbb{R}_+)$. Then

$$\begin{aligned} S_2(h \times f_1) &= \int S_2(\xi_1, \xi_2)\, h(\xi_1^0)\, f_1(\underline{\xi}_1, \xi_2)\, d^4\xi_1\, d^4\xi_2 \\ &= \int h(\xi_1^0) \left(\int e^{-\xi_1^0 q_1^0} \check{S}_2^{(1)}(q_1^0, f_1)\, dq_1^0\right) d\xi_1^0 , \end{aligned} \tag{8.13}$$

for some distribution $\check{S}_2^{(1)}(q_1^0, f_1) \in \mathscr{S}'(\overline{\mathbb{R}}_+)$, according to the assumption of the lemma. By Lemma 8.4 we can change the order of integration in (8.13) and obtain

$$S_2(h \times f_1) = \int \check{h}(q_1^0)\, \check{S}_2^{(1)}(q_1^0, f_1)\, dq_1^0 , \tag{8.14}$$

where $\check{h}(q_1^0) = \int e^{-q_1^0 \xi_1^0} h(\xi_1^0)\, d\xi_1^0 \upharpoonright \{q_1^0 \geqq 0\}$ is an element in $\mathscr{S}(\overline{\mathbb{R}}_+)$. By Lemma 8.2 the right hand side of (8.14) defines a bilinear continuous functional on $\mathscr{S}(\overline{\mathbb{R}}_+) \times (\mathscr{S}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}(\mathbb{R}_+^4))$ and thus by the nuclear theorem a unique distribution $\check{W}_2^1(q_1^0, \underline{\xi}_1, \xi_2)$ in the dual space of $\mathscr{S}(\overline{\mathbb{R}}_+) \hat{\otimes} \mathscr{S}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}(\mathbb{R}_+^4)$. Taking the Fourier transform with respect to $\underline{\xi}$, we obtain

$$S_2(\xi_1, \xi_2) = \int e^{-q_1^0 \xi_1^0 + i \underline{q}_1 \underline{\xi}_1}\, \tilde{W}_2^1(q_1, \xi_2)\, d^4 q$$

where $\tilde{W}_2^1(q_1, \xi_2)$ is now a distribution in the dual space of $\mathscr{S}(\overline{\mathbb{R}}_+^4) \hat{\otimes} \mathscr{S}(\mathbb{R}_+^4)$. Now take $f(\xi_1) \in \mathscr{S}(\mathbb{R}_+^4)$, $g(\xi_2^0) \in \mathscr{S}(\mathbb{R}_+)$, $h(\underline{\xi}_2) \in \mathscr{S}(\mathbb{R}^3)$. Then $\check{f}(q_1) = \int e^{-q_1^0 \xi_1^0 - i \underline{q}_1 \underline{\xi}_1} f(\xi_1)\, d^4\xi_1 \upharpoonright \{q_1^0 \geqq 0\}$ is in $\mathscr{S}(\overline{\mathbb{R}}_+^4)$ and $S_2(f \times g \times h) = \tilde{W}_2^1(\check{f} \times g \times h)$. According to the assumption of the lemma,

$$\tilde{W}_2^1(\check{f}, \xi_2^0, h) \equiv \int \tilde{W}_2^1(q_1, \xi_2^0, \underline{\xi}_2)\, \check{f}(q_1)\, h(\underline{\xi}_2)\, d^4 q_1\, d^3\xi_2$$

is a real analytic function in $\xi_2^0 > 0$ and it can be written as $\tilde{W}_2^1(\check{f}, \xi_2^0, h) = \int e^{-\xi_2^0 q_2^0}\, \check{W}_2^2(\check{f}, q_2^0, h)\, dq_2^0$ for some distribution $\check{W}_2^2(\check{f}, q_2^0, h)$ in $\mathscr{S}'(\overline{\mathbb{R}}_+)$. As before we obtain

$$S_2(f \times g \times h) = \int \check{g}(q_2^0)\, \check{W}_2^2(\check{f}, q_2^0, h)\, dq_2^0 , \tag{8.15}$$

and the right hand side of (8.15) defines a bilinear continuous functional on $\mathscr{S}(\overline{\mathbb{R}}_+^4) \times \mathscr{S}(\overline{\mathbb{R}}_+) \times \mathscr{S}(\mathbb{R}^3)$ and thus a distribution $\check{W}_2^2(q_1, q_2^0, \underline{\xi}_2)$ in the dual space of $\mathscr{S}(\overline{\mathbb{R}}_+^4) \hat{\otimes} \mathscr{S}(\overline{\mathbb{R}}_+) \hat{\otimes} \mathscr{S}(\mathbb{R}^3)$. Taking the Fourier transform with respect to $\underline{\xi}_2$ we finally obtain

$$S_2(f \times g \times h) = \int \check{f}(q_1)\, \check{g}(q_2^0)\, \tilde{h}(\underline{q}_2)\, \tilde{W}_2(q_1, q_2)\, d^4 q_1\, d^4 q_2 , \tag{8.16}$$

where $\tilde{W}(q_1, q_2)$ is now a distribution in $\mathscr{S}(\overline{\mathbb{R}}_+^{4 \cdot 2})$, and $\tilde{h}$ is the Fourier transform of h. Again by Lemma 8.4 we obtain (8.12) from (8.16).

Acknowledgements. We are extremely grateful to Prof. A. Jaffe for proposing this work and for calling our attention to the many advantages of a Euclidean approach to constructive quantum field theory. We also thank him for his permanent interest in our work and for many fruitful conversations. It is a pleasure to thank Prof. L. Ehrenpreis for a very helpful conversation.

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