Commun. math. Phys. 30, 69—78 (1973)

Two Equivalent Criteria for Modularity of the Lattice of All Physical Decision Effects

Günter Dähn
Mathematisches Institut der Universität Tübingen

Received November 20, 1972

Abstract. This paper answers the open question 1 of [3] in the affirmative and, conditionally, the open question 2 of [3], too. Assuming irreducibility of the orthomodular lattice G of all physical decision effects E, we shall prove in the first section that modularity of G implies symmetry of the physical probability function μ. In the second section, we shall consider the filter algebra $\mathscr{B}(B')$ being assumed to possess an involution $*$ such that $\boldsymbol{T}^*\boldsymbol{T}=\mathbf{0}$ implies $\boldsymbol{T}=\mathbf{0}$. Then this will be proved: If every atomic filter $\boldsymbol{T}_P$ is a fixpoint of $*$ and $*$ is, in a restricted manner, norm-preserving on the minimal left ideal $\mathscr{L}_P := \mathscr{B}(B')\,\boldsymbol{T}_P$, then G is modular.

I. Modularity of G

This section completes the connection between a purely lattice-theoretical property of G and a symmetry property of the physical probability function μ which induces the duality between the ensemble space B and the effect space B'.

Therefore we begin with a summary of the main results about the duality $\langle B, B'\rangle := (B, B', \mu)$:

(1) B is a real finite-dimensional base norm space having a proper generating cone B_+ for which the compact convex set K of all physical ensembles V is a compact base.

(2) The (Banach) dual B' of B is an order unit space whose order unit is denoted by $\mathbf{1}$. Its proper positive cone B'_+ is generated by the compact convex set L of all physical effects F.

(3) The extreme points E of L form an orthomodular lattice G and are called the decision effects of L. For all $E \in G$, $G(0, E)$ denotes the orthomodular lattice segment with 0 and E as zero and unit elements, respectively. The set of all atoms P of G is denoted by $A(G)$.

Further symbols, notations and definitions introduced in [2–4] will be used in the sequel without any explicit reference.

The implicit supposition for each proposition in this section is the following

Postulate. The orthomodular lattice G of all physical decision effects E is modular and irreducible. As shown by Ludwig in [6], the requirement of G being irreducible imposes no severe restriction on G. It is, above all, a mathematical convenience.

First we shall show that $B(P \vee Q)$ is even an order unit space and thereby the assertion in question.

Lemma 1. *For any different atoms* $P, Q \in A(G)$ *and every* $F \in L_{P \vee Q} \setminus \{0\}$ *there hold the unique representations: either*

(i) $F = \beta R, \beta \in \boldsymbol{R}_+^*, R \in A(G(0, P \vee Q))$ *or*

(ii) $F = \beta' P \vee Q, \beta' \in \boldsymbol{R}_+^*$ *or*

(iii) $F = \beta_1 S + \beta_2 S^\perp$, *where* $\beta_1, \beta_2 \in]0, 1[, \beta_1 > \beta_2$ *and* $S, S^\perp \in A(G(0, P \vee Q))$.

Proof. The trichotomy results from Ludwig's unique spectral decomposition of an effect F (cf. [6], Theorem 15). The fact that $S, S^\perp \in A(G(0, P \vee Q))$ is a consequence of modularity of $G(0, P \vee Q)$ whence we have the lattice-theoretical dimension statement $\dim G(0, P \vee Q) = 2$. □

Lemma 2. *For all* $P, Q \in A(G)$: $\operatorname{co} A(G(0, P \vee Q))$ *is compact.*

Proof. $A(G(0, P \vee Q))$ is compact by Ludwig's Theorem 18 of [6]. Then $\operatorname{co} A(G(0, P \vee Q))$, too, is compact (e.g. [9], Satz 3.10). □

Theorem 1. *For all* $P, Q \in A(G)$: $\operatorname{co} A(G(0, P \vee Q))$ *is a compact base of* $B'(P \vee Q)$.

Proof. Observing Lemma 2 we have only to verify that every $Y \in B'(P \vee Q)_+ \setminus \{0\}$ has a unique representation $Y = \alpha F$ where $\alpha \in \boldsymbol{R}_+^*$ and $F \in \operatorname{co} A(G(0, P \vee Q))$. By Lemma 1 such a Y has a unique representation either by

(i) $Y = \alpha R, \alpha \in \boldsymbol{R}_+^*$ or by

(ii) $Y = \alpha' P \vee Q, \alpha' \in \boldsymbol{R}_+^*$ or by

(iii) $Y = \alpha_1 S + \alpha_2 S^\perp, \alpha_1, \alpha_2 \in \boldsymbol{R}_+^*$ and $\alpha_1 \neq \alpha_2$.

(i) satisfies the assertion trivially. By modularity of G we have for all $R \in A(G(0, P \vee Q))$ $P \vee Q = R + R^\perp$. So, if (ii) holds, then, by $\frac{1}{2}(P \vee Q) = \frac{1}{2}(R + R^\perp) \in \operatorname{co} A(G(0, P \vee Q))$, there exists a unique $\alpha'' \in \boldsymbol{R}_+^*$ such that $Y = \alpha'' \frac{1}{2}(P \vee Q)$. Supposing (iii) we obtain

$$Y = (\alpha_1 + \alpha_2) \left\{ \frac{\alpha_1}{\alpha_1 + \alpha_2} S + \frac{\alpha_2}{\alpha_1 + \alpha_2} S^\perp \right\},$$

where $\alpha_1 + \alpha_2 \in \boldsymbol{R}_+^*$ is the required unique $\alpha \in \boldsymbol{R}_+^*$. □

Remark 1. From Theorem 12 in [1] there follows that $(B(P \vee Q), B'(P \vee Q))$ is a duality with completely analogous properties as (B, B'). (B, B') being a finite-dimensional duality, $B(P \vee Q)$ is the Banach dual of $B'(P \vee Q)$ for which $P \vee Q$ is an order unit.

Theorem 2. *For all $P, Q \in A(G)$ there exists $V_{P \vee Q} \in K_1(P \vee Q)$ such that*

(i) *$B(P \vee Q)$ is an order unit space with an order unit $2V_{P \vee Q}$*

(ii) *$V_{P \vee Q} = \frac{1}{2}(V_R + V_{R^\perp})$ for all $R \in A(G(0, P \vee Q))$.*

Proof. (i) Theorem 1 says that $B'(P \vee Q)$ is a base norm space. Thus we can apply Theorem 5 of Ellis' paper [4] which states that, for the dual partial ordering, $B(P \vee Q) = B''(P \vee Q)$ has an order unit norm. The order unit $X_{P \vee Q}$ in $B(P \vee Q)$ is strictly positive satisfying $\langle X_{P \vee Q}, F \rangle = 1$ for all F in the order base $\operatorname{co} A(G(0, P \vee Q))$ of $B'(P \vee Q)$ (cf. [4], Lemma 2 and Theorem 5). $B(P \vee Q)$ itself being a base norm space, $X_{P \vee Q}$ has a unique representation by $X_{P \vee Q} = \beta V_{P \vee Q}$ with $\beta \in \boldsymbol{R}_+^*$ and $V_{P \vee Q} \in K_1(P \vee Q)$ being a compact base of $B(P \vee Q)$.

By Theorem 15 of Stolz [8] $\partial K_1(P \vee Q)$ coincides with the extreme boundary $\partial_e K_1(P \vee Q)$ of $K_1(P \vee Q)$. To prove that $V_{P \vee Q}$ is an internal point of $K_1(P \vee Q)$ let us assume $V_{P \vee Q} \in \partial_e K_1(P \vee Q)$. Then, as proved in [1], Theorem 4, there exists only one $S \in A(G(0, P \vee Q))$ such that $\langle V_{P \vee Q}, S \rangle = 1$. $\langle X_{P \vee Q}, R \rangle = 1$ for all $R \in A(G(0, P \vee Q))$, however, implies the contradiction $1 = \langle X_{P \vee Q}, S^\perp \rangle = \beta \langle V_{P \vee Q}, S^\perp \rangle = 0$. Therefore, $V_{P \vee Q}$ must be a proper convex combination $V_{P \vee Q} = \lambda V_S + (1 - \lambda) V_T$ with $V_S, V_T \in \partial_e K_1(P \vee Q)$. From $\langle V_{P \vee Q}, P \vee Q \rangle = 1$ and $\langle X_{P \vee Q}, \frac{1}{2}(P \vee Q) \rangle = 1$ we conclude that $\beta = 2$ and so $X_{P \vee Q} = 2V_{P \vee Q}$.

(ii) From $1 = \langle X_{P \vee Q}, S^\perp \rangle = \langle 2V_{P \vee Q}, S^\perp \rangle = 2(1 - \lambda) \langle V_T, S^\perp \rangle$ and $1 = \langle X_{P \vee Q}, T^\perp \rangle = \langle 2V_{P \vee Q}, T^\perp \rangle = 2\lambda \langle V_S, T^\perp \rangle$ we infer $\lambda = \frac{1}{2}$, hence $V_T = V_{S^\perp}$ and $V_{P \vee Q} = \frac{1}{2}(V_S + V_{S^\perp})$. Consider any $V_R \in \partial_e K_1(P \vee Q)$. $V_{P \vee Q}$ being internal, the unique line through V_R and $V_{P \vee Q}$ intersects $\partial_e K_1(P \vee Q)$ in V_R and $V_{R'}$ such that $V_{P \vee Q}$ is a proper convex combination $V_{P \vee Q} = \alpha V_R + (1 - \alpha) V_{R'}$. Then the same argumentation as above shows that $\alpha = \frac{1}{2}$ and $V_{R'} = V_{R^\perp}$. □

Corollary 1. *For all $P, Q \in A(G)$: any $V \in K_1(P \vee Q) \setminus \{V_{P \vee Q}\}$ has a unique convex ortho-decomposition*

$V = \beta V_R + (1 - \beta) V_{R^\perp}$ with $R, R^\perp \in A(G(0, P \vee Q))$ and $R \perp R^\perp$.

Proof. (i) For any $V \in \partial_e K_1(P \vee Q)$ the assertion is trivially valid with $\beta = 1$.

(ii) For any internal point $V \in K_1(P \vee Q) \setminus \{V_{P \vee Q}\}$ consider the unique line through V and $V_{P \vee Q}$ intersecting $\partial_e K_1(P \vee Q)$ in V_R and $V_{R'}$ such that V and $V_{P \vee Q}$ are proper convex combinations of V_R and $V_{R'}$. Then Theorem 2 implies $V_{R'} = V_{R^\perp}$. □

Corollary 2. *Relative to the supremum norm in $B(P \vee Q)$ there holds for all $R \in A(G(0, P \vee Q))$ $\|V_R - V_{P \vee Q}\| = \frac{1}{2}$.*

Proof. For all $R \in A(G(0, P \vee Q))$ we have $\|V_R - V_{R^\perp}\| = 1$ and $V_{P \vee Q} = \frac{1}{2}(V_R + V_{R^\perp})$. □

Lemma 3. *For all $P, Q \in A(G)$: $B(P \vee Q)_+$ is the convex hull of its extreme rays each of which being generated by an extreme point of $K_1(P \vee Q)$.*

Proof. Since $\partial K_1(P \vee Q) = \partial_e K_1(P \vee Q)$ (cf. [8], Theorem 15) and $B(P \vee Q)_+$ is locally compact with compact base $K_1(P \vee Q)$, the assertion follows from a theorem of Klee [5]. □

Theorem 3. *For all $P, Q \in A(G)$: any*

$$X \in B(P \vee Q) \setminus (B(P \vee Q)_+ \cup -B(P \vee Q)_+)$$

has a unique representation $X = \beta_1 V_R - \beta_2 V_{R^\perp}$ with $\beta_1, \beta_2 \in \boldsymbol{R}_+^*$, $R, R^\perp \in A(G(0, P \vee Q))$ and $R \perp R^\perp$.

Proof. Since $B(P \vee Q)_+$ is a convex body with $V_{P \vee Q}$ in its interior, the unique line through X and $V_{P \vee Q}$ intersects $\partial B(P \vee Q)_+$ in a unique point λV_R such that $\lambda \in \boldsymbol{R}_+^*$, $V_R \in \partial_e K_1(P \vee Q)$ and λV_R lies in the open line segment $]V_{P \vee Q}, X[$. This follows from Lemma 3. Decomposing $V_{P \vee Q}$ by V_R and $V_{R^\perp}$, we obtain $\lambda V_R = \beta_1' \frac{1}{2}(V_R + V_{R^\perp}) + \beta_2' X$ with $\beta_1', \beta_2' \in \boldsymbol{R}_+^*$ and $\beta_1' + \beta_2' = 1$. Hence $X = \beta_1 V_R - \beta_2 V_{R^\perp}$ where $\beta_1 := \dfrac{2\lambda - \beta_1'}{2\beta_2'}, \beta_2 := \dfrac{\beta_1'}{2\beta_2'} \in \boldsymbol{R}_+^*$ and $\dfrac{2\lambda - \beta_1'}{2\beta_2'} > 0$, too, by hypothesis. □

Remark 2. By Corollary 1 of Theorem 2 an analogous statement relative to Theorem 3 holds for

$$X \in B(P \vee Q)_+ \setminus \{\lambda V_{P \vee Q} \mid \lambda \in \boldsymbol{R}_+\} \quad \text{with} \quad \beta_1, -\beta_2 \in \boldsymbol{R}_+ .$$

The corollary of Theorem 5 in [3] states the equivalence of the following postulates:

(1) $\sum_{i \in \boldsymbol{N}_n} \beta_i P_i = 0$ iff $\sum_{i \in \boldsymbol{N}_n} \beta_i V_{P_i} = 0$ with $\beta_i \in \boldsymbol{R}$ for every $i \in \boldsymbol{N}_n$ and any $n \in \boldsymbol{N}$.

(2) $\langle V_P, Q \rangle = \langle V_Q, P \rangle$ for all $P, Q \in A(G)$.

(2) can be interpreted as a symmetry postulate of the physical probability function $\mu: K \times L \to [0, 1]$ inducing the duality $\langle B, B' \rangle$.

Our further procedure will consist in verifying (1) relative to $G(0, P \vee Q)$ provided G is modular and irreducible.

Theorem 4. *For all $P, Q \in A(G)$: (1) is valid with $P_i \in A(G(0, P \vee Q))$ for all $i \in \boldsymbol{N}_n$.*

Proof. (i) Consider the non-trivial case where not all β_i vanish, and suppose $\sum_{i \in \boldsymbol{N}_n} \beta_i P_i = 0$ with $P_i \in A(G(0, P \vee Q))$ for all $i \in \boldsymbol{N}_n$. Theorem 2 implies $0 = \sum_{i \in \boldsymbol{N}_n} \beta_i \langle 2 V_{P \vee Q}, P_i \rangle = \sum_{i \in \boldsymbol{N}_n} \beta_i \langle V_{P_i} + V_{P_i^\perp}, P_i \rangle = \sum_{i \in \boldsymbol{N}_n} \beta_i$.

Assume $\sum_{i \in N_n} \beta_i V_{P_i} = : X \neq 0$. There holds $\langle X, P \vee Q \rangle = \sum_{i \in N_n} \beta_i = 0$:

1) $X \in B(P \vee Q)_+$ is impossible because $X = \lambda V$ with $\lambda \in \boldsymbol{R}_+^*$ and $V \in K_1(P \vee Q)$ would imply $\langle X, P \vee Q \rangle = \lambda > 0$. An analogous contradiction can be derived from assuming $X \in -B(P \vee Q)_+$.

2) $X \notin B(P \vee Q)_+ \cup -B(P \vee Q)_+$ admits, by Theorem 3, a representation $X = \beta' V_R - \beta'' V_{R^\perp}$ with $\beta', \beta'' \in \boldsymbol{R}_+^*$; $R, R^\perp \in A(G(0, P \vee Q))$. Using $0 = \langle X, P \vee Q \rangle = \beta' - \beta''$, we obtain $\beta' = \beta'' = : \beta$. Being the dual space of the partially ordered Banach space B' having an order unit norm, B has, by Theorem 4 of [4], the minimal decomposition property, i.e. every $X_0 \in B$ can be decomposed into $X_0 = X_1 - X_2$ such that $X_1, X_2 \in B_+$ and $\|X\| = \|X_1\| + \|X_2\|$. Therefore X has a representation by $X = \alpha_1 V_1 - \alpha_2 V_2$ such that $\alpha_1, \alpha_2 \in \boldsymbol{R}_+^*$; $V_1, V_2 \in K_1(P \vee Q)$ and $\|X\| = \alpha_1 + \alpha_2$. Again from $\langle X, P \vee Q \rangle = 0$, there follows $\alpha_1 = \alpha_2 = : \alpha$ and thus $X = \beta(V_R - V_{R^\perp}) = \alpha(V_1 - V_2)$. From $\|V_R - V_{R^\perp}\| = 1$ we infer that $\|X\| = \beta = 2\alpha$, i.e. $\alpha = \frac{\beta}{2}$. Hence we obtain

$$\langle X, R \rangle = \beta = \frac{\beta}{2} (\langle V_1, R \rangle - \langle V_2, R \rangle),$$

whence $2 = \langle V_1, R \rangle - \langle V_2, R \rangle$, contrary to $|\langle V_1, R \rangle - \langle V_2, R \rangle| \leqq 1$.

(ii) Again, consider the non-trivial case $\sum_{i \in N_n} \beta_i V_{P_i} = 0$. By Theorem 2, $B(P \vee Q)$ is an order unit space with an order unit $X_{P \vee Q}$, and $\operatorname{co} A(G(0, P \vee Q))$ is a base of $B'(P \vee Q)$. Thus $\sum_{i \in N_n} \beta_i P_i = 0$ follows in a manner completely analogous to (i). □

Corollary 1. *For all $P, Q \in A(G)$ there holds*

$$\langle V_P, Q \rangle = \langle V_Q, P \rangle.$$

Proof. The assertion is a direct consequence of remark 2 and the last Theorem. □

Corollary 2. *B' becomes a real Hilbert-space.*

Proof. Theorem 6 in [3]. □

Combining these corollaries with Theorem 14 of [3] we can state the first main theorem:

Theorem 5. *Suppose that G is irreducible. Then there holds: Symmetry of the physical probability function μ is equivalent with modularity of the lattice G of all physical decision effects E.*

Remark 3. Notice that no dimension requirement of G is necessary (except $\dim G > 1$ to exclude triviality).

II. Separating Involutions on $\mathscr{B}(B')$

This section investigates the connexion between the symmetry condition in [3] and a separating involution on the filter algebra $\mathscr{B}(B')$ which leaves fixed all atomic filters $\boldsymbol{T}_P$ of the orthomodular filter lattice $\mathscr{T}(G)$. Henceforth we call an involution * on $\mathscr{B}(B')$ *separating* iff $\boldsymbol{T}^*\boldsymbol{T}=\boldsymbol{0}$ implies $\boldsymbol{T}=\boldsymbol{0}$.

For the other terminology see [2]. There we proved in Theorem 13, its corollary and Theorem 14 that for every $\boldsymbol{T}_P$

(i) $\mathscr{R}_P:=\{X\otimes P \mid X\in B\}=\boldsymbol{T}_P\mathscr{B}(B')$ is a minimal right ideal being linearly order isomorphic to B.

(ii) $\mathscr{L}_P:=\{V_P\otimes Y \mid Y\in B'\}=\mathscr{B}(B')\,\boldsymbol{T}_P$ is a minimal left ideal being linearly isomorphic to B'.

Provided G is irreducible we gather from the Remarks 4 and 5 in [2] that $\mathscr{B}(B')$ is generated by the orthomodular filter lattice $\mathscr{T}(G)=\{\boldsymbol{T}_E \mid E\in G\}$. Thus it is plausible that the operation * is determined on the whole of $\mathscr{B}(B')$ by the way it operates on $\bigcup_{m\in\boldsymbol{N}}\mathscr{T}(G)^m$ with $\mathscr{T}(G)^m=\{\boldsymbol{T}_{E_{\iota_1}}\boldsymbol{T}_{E_{\iota_2}}\ldots\boldsymbol{T}_{E_{\iota_m}} \mid \boldsymbol{T}_{E_{\iota_k}}\in\mathscr{T}(G) \text{ and } k\in\boldsymbol{N}_m\}$ for any $m\in\boldsymbol{N}$.

Suppose that the relation $*\colon \bigcup_{m\in\boldsymbol{N}}\mathscr{T}(G)^m\to\bigcup_{m\in\boldsymbol{N}}\mathscr{T}(G)^m$ defined by $(\boldsymbol{T}_{E_{\iota_1}}\boldsymbol{T}_{E_{\iota_2}}\ldots\boldsymbol{T}_{E_{\iota_{m-1}}}\boldsymbol{T}_{E_{\iota_m}})^*=\boldsymbol{T}_{E_{\iota_m}}\boldsymbol{T}_{E_{\iota_{m-1}}}\ldots\boldsymbol{T}_{E_{\iota_2}}\boldsymbol{T}_{E_{\iota_1}}$ is a mapping. Then this mapping has all the multiplicative properties of an involution on $\mathscr{B}(B')$ and every filter $\boldsymbol{T}_E\in\mathscr{T}(G)$, being idempotent, remains fixed under*. To guarantee a unique linear bijective extension of * to the whole of $\mathscr{B}(B')$ we additionally assume the validity of:

"For every $\boldsymbol{T}\in\bigcup_{m\in\boldsymbol{N}}\mathscr{T}(G)^m$: $\boldsymbol{T}^*=\boldsymbol{0}$ implies $\boldsymbol{T}=\boldsymbol{0}$."

(This extension condition holds always for $\bigcup_{m\in\boldsymbol{N}}A\mathscr{T}(G)^m$; cf. [2], Theorem 18).

* to be separating can be equivalently substituted by "For every $\boldsymbol{T}\in\mathscr{B}(B')$: $\boldsymbol{T}^*\boldsymbol{T}=\boldsymbol{0}$ iff $\boldsymbol{T}\boldsymbol{T}^*=\boldsymbol{0}$."

For, if this equivalence is valid, then the right ideal $\mathscr{R}:=\{\boldsymbol{T} \mid \boldsymbol{T}\in\mathscr{B}(B')$ and $\boldsymbol{T}^*\boldsymbol{T}=\boldsymbol{0}\}$ of $\mathscr{B}(B')$ is even two-sided. Simplicity of $\mathscr{B}(B')$ then implies $\mathscr{R}=(\boldsymbol{0})$, hence * is separating. The reverse direction of the equivalence asserted is trivial.

Theorem 6. *Suppose that G is irreducible. Then*

(i) *modularity of G implies the existence of a separating involution * on $\mathscr{B}(B')$;*

(ii) *every filter $\boldsymbol{T}_E$ is a fixed element under this involution;*

(iii) *for all $P, Q\in A(G)$: * preserves the L-norm of $V_P\otimes Q\in\mathscr{L}_P$.*

Proof. (i) Corollary 2 of Theorem 4 states that B' becomes a real Hilbert space. Hence $\mathscr{B}(B')$ becomes a C^*-algebra (Rickart's terminology, cf. [7]).

(ii) is the statement of Theorem 15 in [3].

(iii) For every $P \in A(G)$ there holds $V_P = P$ (cf. [2]). Therefore, $\|P \otimes Q\|_L = \sup\{\|(P \otimes Q)F\| \mid F \in L\} = \sup\{\langle P \mid F\rangle \, \|Q\| \mid F \in L\} = \langle P \mid \mathbf{1}\rangle = 1$ and $(P \otimes Q)^* = Q \otimes P$ imply $\|Q \otimes P\|_L = \sup\{\|(Q \otimes P)F\| \mid F \in L\} = 1 = \|P \otimes Q\|_L$. □

The converse will be verified by two steps.

Lemma 4. *If $\mathscr{B}(B')$ possesses a separating involution * such that every $T_P \in A\mathscr{T}(G)$ is a fixpoint of *, then there exists a linear isomorphism $J_P: B' \to B$ being positive in both directions.*

Proof. Every T_P belongs to $\mathscr{L}_P \cap \mathscr{R}_P = \mathscr{B}(B')\, T_P \cap T_P \mathscr{B}(B')$ and, being a fixpoint of *, there holds $\mathscr{L}_P^* = \mathscr{R}_P$ and $\mathscr{R}_P^* = \mathscr{L}_P$. Hence * induces a canonical linear isomorphism $J_P: B' \to B$ defined by $V_P \otimes Y \mapsto J_P Y \otimes P = (V_P \otimes Y)^*$. From T_P being a fixpoint of * we conclude that for every $P \in A(G)$: $J_P P = V_P$. J_P will be shown to be positive in both directions: From the Theorems 4.10.3 and 4.10.7 of [7] there follows that $\langle\cdot|\cdot\rangle_P : B' \times B' \to \mathbf{R}$ is an inner product on B' which is defined by $\langle Y_1 | Y_2\rangle_P \, T_P = (V_P \otimes Y_2)^* \circ V_P \otimes Y_1 = \langle J_P Y_2, Y_1\rangle \, T_P$. We infer from its symmetry that J_P equals the transposed isomorphism J_P^t because B is finite-dimensional. Thus for all $Y_1, Y_2 \in B'$: $\langle Y_1 | Y_2\rangle_P = \langle J_P Y_2, Y_1\rangle = \langle J_P Y_1, Y_2\rangle$. Since $(V_P \otimes \mathbf{1})^2 = \langle V_P, \mathbf{1}\rangle \, V_P \otimes \mathbf{1} = V_P \otimes \mathbf{1}$, so $(V_P \otimes \mathbf{1})^{2*} = (J_P \mathbf{1} \otimes P)^2 = \langle J_P \mathbf{1}, P\rangle \, J_P \mathbf{1} \otimes P = J_P \mathbf{1} \otimes P$, whence $\langle J_P \mathbf{1}, P\rangle = 1$. Hence for all $P, Q \in A(G)$: $T_Q \circ J_P \mathbf{1} \otimes \mathbf{1} \circ T_P = V_Q \otimes Q \circ J_P \mathbf{1} \otimes \mathbf{1} \circ V_P \otimes P = \langle V_Q, \mathbf{1}\rangle \langle J_P \mathbf{1}, P\rangle \, V_P \otimes Q = V_P \otimes Q$ and therefore

$$J_P Q \otimes P = (V_P \otimes Q)^* = T_P \circ J_P \mathbf{1} \otimes \mathbf{1} \circ T_Q = \langle J_P \mathbf{1}, Q\rangle \, V_Q \otimes P$$
$$= \langle J_P Q, \mathbf{1}\rangle \, V_Q \otimes P \, .$$

To ensure $\langle J_P Q, \mathbf{1}\rangle \in \mathbf{R}_+^*$ we observe that $0 < \langle Q | Q\rangle_P = \langle J_P Q, Q\rangle$ and $(J_P Q \otimes P)\, Q = \langle J_P Q, Q\rangle \, P = \langle J_P Q, \mathbf{1}\rangle \, P$. Thus for all

$$P, Q \in A(G): J_P Q = \langle J_P Q, \mathbf{1}\rangle \, V_Q = \langle J_P \mathbf{1}, Q\rangle \, V_Q \in B_+ \setminus \{0\}.$$

Since all V_Q generate B_+ and all Q generate B'_+, J_P^{-1} is also positive. □

Lemma 5. *In addition to Lemma 4, suppose G to be irreducible. Then for every $P \in A(G)$ the positive isomorphism J_P is unique up to a positive multiplicative constant.*

Proof. For all $P, Q \in A(G)$ and all $Y_1, Y_2 \in B'$: there exists a strictly positive (self-adjoint) linear operator A on $(B', \langle\cdot|\cdot\rangle_P)$ such that $\langle Y_1 | Y_2\rangle_Q = \langle Y_1 | A\, Y_2\rangle_P$. This is a well-known fact from Hilbert space theory. Utilizing Lemma 4, we obtain for all

$$R, S \in A(G): \langle J_Q R, S\rangle = \langle R | S\rangle_Q = \langle R | A\, S\rangle_P = \langle A\, R | S\rangle_P = \langle J_P A\, R, S\rangle \, ,$$

and so $J_P A R = J_Q R = \langle J_Q \mathbf{1}, R\rangle V_R$. This implies $A R = \langle J_Q \mathbf{1}, R\rangle J_P^{-1} V_R = \frac{\langle J_Q \mathbf{1}, R\rangle}{\langle J_P \mathbf{1}, R\rangle} R = \beta(Q, P) R$ with $\beta(Q,P) := \frac{\langle J_Q \mathbf{1}, R\rangle}{\langle J_P \mathbf{1}, R\rangle} \in \boldsymbol{R}_+^*$. Thus every $R \in A(G)$ is a proper vector of A and A commutes with every atomic filter $\boldsymbol{T}_R$. G being irreducible, we may apply Theorem 20 of [1] to obtain that A is a scalar operator. Therefore, $J_P A R = J_Q R$ implies $\beta(Q, P) J_P = J_Q$, the desired result. □

Lemma 6. *Given the hypothesis of Lemma 5 and, additionally, $J_P \mathbf{1} = J_Q \mathbf{1}$ for all $P, Q \in A(G)$. Then the symmetry condition $\langle V_P, Q\rangle = \langle V_Q, P\rangle$ holds.*

Proof. $J_P \mathbf{1} = J_Q \mathbf{1}$ implies $\beta(Q, P) = 1$, thus $J_P = J_Q$. Then, using $J_P Q = V_Q$, we have $\langle V_P, Q\rangle V_Q \otimes P = \boldsymbol{T}_P \boldsymbol{T}_Q = (\boldsymbol{T}_Q \boldsymbol{T}_P)^* = \langle V_Q, P\rangle (V_P \otimes Q)^* = \langle V_Q, P\rangle V_Q \otimes P$. □

Corollary. *For all $P \in A(G)$, J_P equals the canonical order isomorphism J of* [3].

Proof. [3], Theorem 5 and its corollary. □

Theorem 7. *Suppose G to be irreducible and $\mathscr{B}(B')$ to have a separating involution * such that for all $P \in A(G)$: $\boldsymbol{T}_P^* = \boldsymbol{T}_P$. Then, for all $P, Q \in A(G)$, these propositions are equivalent:*

(i) $\|V_P \otimes Q\|_L = \|J_P Q \otimes P\|_L$.

(ii) $J_P \mathbf{1} = J_Q \mathbf{1}$.

(iii) $\langle V_P, Q\rangle = \langle V_Q, P\rangle$.

Proof. (i)⇒(ii): By Lemma 3 of [2] there holds

$$\|V_P \otimes Q\|_L = \sup\{\|(V_P \otimes Q) F\| \mid F \in L\} = \sup\{\langle V_P, F\rangle \mid F \in L\} = 1 .$$

Using (i) and Lemma 5, we obtain $\|J_P Q \otimes P\|_L = \beta(Q, P)^{-1} \cdot \|J_Q Q \otimes P\|_L = \beta(Q, P)^{-1} \circ \|V_Q \otimes P\|_L = \beta(Q, P)^{-1} = 1$; hence for all $R \in A(G)$: $\langle J_P \mathbf{1}, R\rangle = \langle J_Q \mathbf{1}, R\rangle$, whence $J_P \mathbf{1} = J_Q \mathbf{1}$.

(ii)⇒(iii): Lemma 6.

(iii)⇒(i): By the corollary to Lemma 6 there holds for all $P \in A(G)$ $J_P = J$. This implies $J_P Q = J Q = V_Q$ and so we arrive at $\|V_P \otimes Q\|_L = 1 = \|V_Q \otimes P\|_L$. □

Corollary 1. *If any of the equivalent propositions of Theorem 7 holds, then G is modular.*

Proof. Theorem 14 of [3]. □

Although, unless G Boolean, no filter of $\mathscr{T}(G)$ can be additively decomposed into atomic filters, we can state the

Corollary 2. *Every filter $\boldsymbol{T}_E \in \mathscr{T}(G)$ is a fixpoint of the involution provided any proposition of Theorem 7 is valid.*

Proof. From Theorem 4.10.7 and the Corollary 4.10.8 in [7] we gather that every $T^* \in \mathscr{B}(B')$ is the adjoint operator relative to the inner product $\langle\cdot|\cdot\rangle_P$ on B' which was defined in the proof of Lemma 4. By Theorem 7, this inner product coincides with that which is induced on B' by the symmetry condition (cf. [3], Theorems 6 and 7). The assertion then follows from Theorem 15 in [3]. □

Lemma 7. *A separating involution * on $\mathscr{B}(B')$ such that for all $P, Q \in A(G)$: $T_P^* = T_P$ and $\|V_P \otimes Q\|_L = \|J_P Q \otimes P\|_L$ is unique.*

Proof. Since $J_P = J$, the adjoint operator of any $\boldsymbol{T} \in \mathscr{B}(B')$ relative to the inner product on B' by J equals $\boldsymbol{T}^*$ for every involution satisfying the hypothesis. □

Remark 4. At this stage of our deduction we think some motivating remarks on the preceding lemmata and theorems to be necessary. As represented in Rickart's monograph [7], for instance, (and already utilized extensively) a separating involution on $\mathscr{B}(B')$ having minimal idempotents induces a Hilbert space structure on B'. Of course, this property then induces a canonical isomorphism between B' and B. However, it is not obvious that this isomorphism preserves order in both directions (not even in one). This is the reason why we additionally postulated $A\mathscr{T}(G)$ to be a fixpoint subset of the separating involution given. As to the postulate concerning the L-norm we conjecture that there exists a separating involution on $\mathscr{B}(B')$ leaving fixed every atomic filter but not possessing the L-norm property of Theorem 7. In fact, a perusal of the proof of self-adjointness of $\boldsymbol{T}_E$ reveals no restriction of J_P except for its positivity already ensured by the $\boldsymbol{T}_P$-postulate. Therefore, only to require all filters of $\mathscr{T}(G)$ to be fixpoints of a separating involution seems to impose no additional structure on the filter lattice $\mathscr{T}(G)$. We conjecture, but failed to verify that such an involution would necessarily pertain to a non-modular filter lattice.

A concluding theorem will summarize this paper and [3]:

Theorem 8. *For an irreducible lattice G of all physical decision effects, the following postulates are equivalent:*

i) *The physical probability function μ satisfies the symmetry condition: for all atomic decision effects P, Q:*

$$\mu(V_P, Q) = \mu(V_Q, P).$$

(ii) *The filter algebra $\mathscr{B}(B')$ has a (unique) separating involution leaving fixed every atomic filter $\boldsymbol{T}_P$ and preserving the L-norm of $V_P \otimes Q$ for all $P, Q \in A(G)$.*

(iii) *G is modular.*

Proof. (i)⇒(ii): Theorem 15 of [3], Theorems 6 and 7 and Lemma 7. (ii)⇒(iii): Corollary 1 of Theorem 7. (iii)⇒(i): Theorem 5. □

References

1. Dähn, G.: Attempt of an axiomatic foundation of quantum mechanics and more general theories. IV. Commun. math. Phys. **9**, 192—211 (1968).
2. Dähn, G.: The algebra generated by physical filters. Commun. math. Phys. **28**, 109—122 (1972).
3. Dähn, G.: Symmetry of the physical probability function implies modularity of the lattice of decision effects. Commun. math. Phys. **28**, 123—132 (1972).
4. Ellis, A. J.: The duality of partially ordered normed linear spaces. J. London Math. Soc. **39**, 730—744 (1964).
5. Klee, V. L.: Convexity. Princeton.
6. Ludwig, G.: Attempt of an axiomatic foundation of quantum mechanics and more general theories. III. Commun. math. Phys. **9**, 1—12 (1968).
7. Rickart, C. E.: General theory of Banach algebras. Princeton: Van Nostrand 1960.
8. Stolz, P.: Attempt of an axiomatic foundation of quantum mechanics and more general theories. V. Commun. math. Phys. **11**, 303—313 (1969).
9. Valentine, F. A.: Konvexe Mengen, Bd. 402/402a. Mannheim: Bibliographisches Institut A.G. 1968.

Günter Dähn
Mathematisches Institut der Universität
D-7400 Tübingen
Brunnenstraße 27
Federal Republic of Germany