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Duality for Free Bose Fields

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Abstract. An elementary proof of Araki's duality theorem for free fields is presented. The theorem says that for a certain class of regions O in Minkowski space, the commutant of $\mathfrak{A}(O)$, the von Neumann algebra generated by all observables belonging to measurements within O, is exactly $\mathfrak{A}(O')$, where O' is the spacelike separated complement of O.

1. Introduction

In the algebraic approach to quantum field theory the principle of locality is expressed in the following way. There is a one to one correspondence between regions O in space-time and von Neumann algebras $\mathfrak{A}(O)$, the local algebras. Let $\mathfrak{A}(O)'$ denote the commuting algebra of $\mathfrak{A}(O)$ and let O' be the set of points which are spacelike separated from O. Then locality means that

$$\mathfrak{A}(O') \subset \mathfrak{A}(O)' .$$

An assumption stronger than locality is

$$\mathfrak{A}(O') = \mathfrak{A}(O)' .$$

This relation is called duality.

In recent investigations of Doplicher, Haag, and Roberts [6] [1] on the connections between gauge group, superselection sectors and irreducible representations of the observable algebra, duality plays a crucial role. The importance of the duality relation in that work makes it desirable to know whether duality holds in any quantum field theory model. Araki [1, 2] has shown that in a free field theory duality holds; his proof seems to be rather complicated. The purpose of this paper is to show that with some modifications it can be simplified.

It should be mentioned that there is an elegant proof of duality for free fields in a publication by Dell'Antonio [5]. However the methods used there – von Neumann's infinite tensor products of Hilbert spaces –

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[1] Résumés of that work can be found in Ref. [7] and [9].

are very unlikely to be of much help in a non-trivial model. Whether this same remark applies for our method as well has to be seen.

We sketch the idea of our proof. First we observe that in a free field model (we restrict ourselves to the case of one free scalar field) the local algebra $\mathfrak{A}(O)$ coincides with the time zero algebra $\mathfrak{A}(O_s)$, if O is a double cone in space-time over the time zero region O_s, and $\mathfrak{A}(O_s)$ is the von Neumann algebra generated by all bounded operators localized in the region O_s (see e.g. Ref. [8]). Thus we want to prove

$$\mathfrak{A}(O)' = \mathfrak{A}(\sim O),$$

where O is a region in $\mathbb{R}^3$ and $\sim O$ is the complement of O in $\mathbb{R}^3$. Locality gives us $\mathfrak{A}(\sim O) \subset \mathfrak{A}(O)'$, so we only have to prove

$$\mathfrak{A}(O)' \subset \mathfrak{A}(\sim O).$$

We do so by showing that $B \in \mathfrak{A}(O)'$ and $C \in \mathfrak{A}(\sim O)'$ implies $[B, C] = 0$. Then $\mathfrak{A}(O)' \subset (\mathfrak{A}(\sim O)')' = \mathfrak{A}(\sim O)'' = \mathfrak{A}(\sim O)$, as $\mathfrak{A}(\sim O)$ is a von Neumann algebra. Without loss of generality we may assume that B is self-adjoint. The main tool in our proof is a $\varphi - \pi$ *expansion* of the operator B. This means we want to write B as

$$B = \Sigma : \varphi(f_1) \ldots \varphi(f_n)\, \pi(g_1) \ldots \pi(g_m) : \tag{1}$$

and the sum on the right hand side converges and is equal to B as a bilinear form on $D \times D$. D is the dense set of those vectors in Fock space which have only a finite number of particles. In his proof of duality, Araki uses an expansion of B as a sum of Wick ordered monomials in creation and annihilation operators, the $a - a^*$ expansion, which in principle is of course equivalent to the $\varphi - \pi$ expansion. Nevertheless, recovering $B \in \mathfrak{A}(\sim O)$ from an $a - a^*$ expansion is rather troublesome, while in a $\varphi - \pi$ expansion we only have to show that all the f_i's and g_i's in (1) have support in $\sim O$ to conclude that every single term on the right hand side of (1) is affiliated with the von Neumann algebra $\mathfrak{A}(\sim O)$. This proves $[B, C] = 0$ at least for operators $C \in \mathfrak{A}(\sim O)'$ which leave the set D invariant. For a general operator $C \in \mathfrak{A}(\sim O)'$ we have to introduce a regularized version C_ε of C which can be shown to commute with B. Then because C_ε converges weakly to C, $[B, C] = 0$ follows from continuity.

In section two we introduce spaces of test functions and list some of their properties (Lemma 1). Then we introduce some Fock space notation and give a precise definition of the local algebra $\mathfrak{A}(O)$. In section three we prove duality (Theorem 1); the $\varphi - \pi$ expansion is characterized in Lemma 3. In an appendix we prove a technical lemma.

2. Spaces of Test Functions, Fock Space

We denote by L_r^2 (L^2) the space of real (complex) square integrable functions on $\mathbb{R}^3$. By μ^τ we denote the operator $(-\Delta+m^2)^{\tau/2}$, which is multiplication by $\mu^\tau(p)=(p^2+m^2)^{\tau/2}$ in Fourier transform space.

We define $\mathscr{H}_+$ to be the set of vectors in $D(\mu^{1/2})\cap L_r^2$, equipped with the inner product

$$(g_1, g_2)_+ = (\mu^{1/2} g_1, \mu^{1/2} g_2),$$

where $D(\mu^{1/2})$ is the domain of $\mu^{1/2}$ and (,) is the inner product in L^2. As $\mu^{1/2}$ is a closed operator, $\mathscr{H}_+$ is a Hilbert space. We denote by $\mathscr{H}_-$ the closure of L_r^2 with respect to the norm $\|f\|_- = \|\mu^{-1/2} f\|_2$, equipped with the inner product $(f_1, f_2)_- = (\mu^{-1/2} f_1, \mu^{-1/2} f_2)$. ($\mathscr{H}_\pm$ are fractional Sobolev spaces of index $\pm 1/2$ and are usually denoted by $\mathscr{L}^2_{\pm 1/2}(\mathbb{R}^3)$, see e.g. [12].) Note that we have the following relations between $\mathscr{H}_+$, $\mathscr{H}_-$, and L_r^2.

a) The identity map is a bounded operator from $\mathscr{H}_+$ into L_r^2, and also from L_r^2 into $\mathscr{H}_-$.

b) The operator $\mu^{1/2}$ is an isometry of $\mathscr{H}_+$ onto L_r^2, and also of L_r^2 onto $\mathscr{H}_-$.

Next we define local Sobolev spaces. Let O be a (not necessarily bounded) region in $\mathbb{R}^3$ with sufficiently regular surface. For the rest of this paper we shall assume that the boundary ∂O of O is a piecewise many-times differentiable (two dimensional) surface, and that O has a nonempty interior $\operatorname{int} O$. We define the following subspaces of $\mathscr{H}_\pm$:

$$\mathscr{H}_+(O) = \{g \in \mathscr{H}_+, \operatorname{supp} g \subset \operatorname{int} O\}^-, \tag{2}$$

$$\mathscr{H}_-(O) = \{f \in \mathscr{H}_-, (\mu^{-1/2} f, \mu^{1/2} g) = 0 \quad \text{for all} \quad g \in \mathscr{H}_+(\sim 0)\}. \tag{3}$$

Using $\mu^{\pm 1/2}$ as an isometric operator from $\mathscr{H}_\pm$ onto L_r^2, we define subspaces of L_r^2 by

$$\mathscr{K}_+(O) = \mu^{1/2}\, \mathscr{H}_+(O), \tag{4}$$

$$\mathscr{K}_-(O) = \mu^{-1/2}\, \mathscr{H}_-(O). \tag{5}$$

Let $\mathscr{K}$ and $\mathscr{K}'$ be two subspaces of L_r^2. Then $\mathscr{K}+\mathscr{K}'$ denotes $\{f+g; f \in \mathscr{K}, g \in \mathscr{K}'\}$, and $\mathscr{K} \dot{+} \mathscr{K}'$ the closure of it. By $\{0\}$ we denote the set consisting of the null vector only. If we write $\mathscr{K}_{\substack{\alpha\\\beta}} = \mathscr{K}'_{\substack{\gamma\\\delta}}$, we mean $\mathscr{K}_\alpha = \mathscr{K}'_\gamma$ and $\mathscr{K}_\beta = \mathscr{K}'_\delta$ (but neither $\mathscr{K}_\alpha = \mathscr{K}'_\delta$ nor $\mathscr{K}_\beta = \mathscr{K}'_\gamma$), $\alpha, \beta, \gamma, \delta = +$ or $-$. The following lemma describes some relations between the subspaces $\mathscr{K}_\pm(O)$ of L_r^2.

Lemma 1. *Let O, O_1, and O_2 be regions in $\mathbb{R}^3$ as above. Then*

$$\mathscr{K}_{\pm}(O)^{\perp} = \mathscr{K}_{\mp}(\sim O)\,, \tag{6}$$

$$\mathscr{K}_{+}(O) \cap \mathscr{K}_{-}(O) = \{O\}\,, \tag{7}$$

$$\mathscr{K}_{+}(O) \dotplus \mathscr{K}_{-}(O) = L^2_r\,, \tag{8}$$

$$\mathscr{K}_{\pm}(O_1) \dotplus \mathscr{K}_{\pm}(O_2) = \mathscr{K}_{\pm}(O_1 \cup O_2)\,, \tag{9}$$

$$\mathscr{K}_{\pm}(O_1) \cap \mathscr{K}_{\pm}(O_2) = \mathscr{K}_{+}(O_1 \cap O_2)\,. \tag{10}$$

We won't give a proof of this lemma. For general local fractional Sobolev spaces $\mathscr{L}^p_\alpha(O)$ relations similar to (6–10) are sometimes quite complicated to prove, and not always true, see e.g. [3] and [13]; for the special case of Lemma 1 a complete discussion can be found in [2]. We remark that (6) follows immediately from definitions (2–5), (7) follows from the fact that μ is an "antilocal" operator, i.e. $\mu f = g$, and $f \in \mathscr{H}_+(O)$ imply that g never vanishes in the entire neighborhood of any point in $\sim O$, see [2] and [11]; (8) follows from (6) and (7). Furthermore, (10) follows from (9) and (6).

By $\mathscr{F}$ we denote the boson Fock space over L^2, by Φ_0 the Fock vacuum, by E_n the projection operator onto the n particle subspace of $\mathscr{F}$, by D the set of all vectors in $\mathscr{F}$ which contain only a finite number of particles. For nonnegative integers r and s we set

$$\mathscr{F}^{rs}(O) = \left\{ : \prod_{i=1}^{r} \pi(g_i) \prod_{j=1}^{s} \varphi(f_j) : \Phi_0,\ f_j \in \mathscr{H}_-(O),\ g_i \in \mathscr{H}_+(O) \right\}, \tag{11}$$

where

$$\varphi(f) = 2^{-1/2} \int \mu^{-1/2}(k)\, \tilde{f}(k)\, (a(-k) + a^*(k))\, dk$$

and

$$\pi(g) = i 2^{-1/2} \int \mu^{1/2}(k)\, \tilde{g}(k)\, (a(-k) - a^*(k))\, dk$$

are the free boson field and its time derivative at time zero, smeared over space with appropriate testfunctions f and g respectively.

The following lemma is easily proved by induction on l and using (8).

Lemma 2. *For any region $O \subset \mathbb{R}^3$ as above $\bigcup_{r+s \leq l} \mathscr{F}^{rs}(O)$ is a total set in $\bigcup_{n \leq l} E_n \mathscr{F}$. The set $\bigcup_{r,s} \mathscr{F}^{rs}(O)$ is total in $\mathscr{F}$.*

A set is called total if its linear span is dense.

Finally we define the local algebra $\mathfrak{A}(O)$ to be the von Neumann algebra generated by $\{e^{i\varphi(f)} e^{i\pi(g)},\ f \in \mathscr{H}_-(O), g \in \mathscr{H}_+(O)\}$. Note that we do not exclude unbounded regions O.

3. Duality

The following theorem states the main result of this paper (Araki [2, Theorem 4]).

Theorem. *Let O, O_1, O_2, be regions in* $\mathbb{R}^3$ *as above. Then*

$$\mathfrak{A}(O_1) \wedge \mathfrak{A}(O_2) = \mathfrak{A}(O_1 \cap O_2), \tag{12}$$

$$\mathfrak{A}(O_1) \vee \mathfrak{A}(O_2) = \mathfrak{A}(O_1 \cup O_2), \tag{13}$$

$$\mathfrak{A}(O)' = \mathfrak{A}(\sim O). \tag{14}$$

$\mathfrak{A}(O_1) \wedge \mathfrak{A}(O_2)$ denotes the intersection of the two algebras, $\mathfrak{A}(O_1) \vee \mathfrak{A}(O_2)$ stands for the von Neumann algebra generated by the set theoretical union of the two algebras. The theorem states that there is a one to one correspondence between the lattice structure in the set of regions O and the lattice structure in the set of local algebras $\mathfrak{A}(O)$.

For a proof of the theorem we note that (13) follows immediately from (9) and the continuity of $e^{i\varphi(f)}$ and $e^{i\pi(g)}$ in the test functions f and g respectively. Eq. (12) follows from (13) and (14). Thus we have only to prove (14) which is the duality relation.

According to the introduction we should take a self-adjoint operator $B \in \mathfrak{A}(O)'$ and then show that $[B, C] = 0$ for all $C \in \mathfrak{A}(\sim O)'$.

It is well known that for any bounded operator in $\mathscr{F}$ an $a - a^*$ and thus a $\varphi - \pi$ expansion exists, see e.g. Ref. [1] or [4]. What is important to us is to see that the φ's and π's occuring in the expansion (1) of B are all affiliated with $\mathfrak{A}(\sim O)$ and thus commute with C.

Lemma 3. *Suppose $B = B^* \in \mathfrak{A}(O)'$. Then there exists a sequence of (unbounded) symmetric operators $\{B_i\}_{i \in \mathbb{Z}_+}$ with common domain D, mapping D into itself, such that for $i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$*

(I) $E_k(CB_i - B_iC)E_l = 0$ *for all* $k, l \in \mathbb{Z}_+$ *and all* $C \in \mathfrak{A}(\sim O)'$.

(II) $E_k B_i E_l = 0$ *if* $k + l < i$ *or* $|k - l| > i$, *bounded otherwise.*

(III) $E_k\left(B - \sum_{m=0}^{i} B_m\right)E_l = 0$ *if* $k + l \leqq i$.

Remark. Of course the B_i's are homogeneous Wick ordered polynomials in $\varphi(f)$ and $\pi(g)$ of order i; this will yield (II). (I) says that $[B_i, C] = 0$ as bilinear forms on $D \times D$, while (III) says that $B = \sum_{m=0}^{\infty} B_m$ as a bilinear form on $D \times D$.

We prove Lemma 3 by inductive construction of the operators B_i and by verifying (I–III) at each step.

α) In order to construct B_0 we define

$$B_{00} = E_0 B E_0 .$$

As we have assumed B to be self-adjoint, it follows that $B_{00} \upharpoonright E_0 \mathscr{F}$ is multiplication by a real number b_0. We define the operator B_0 on $\mathscr{F}$ to be multiplication by b_0. Then (I–III) are obviously true for $i=0$.

β) Now we assume that we have already constructed operators $B_0, B_1, \ldots, B_{n-1}$ with all the properties stated in the theorem. In order to construct B_n we define for $k=0, 1, \ldots n$,

$$B_{k\,n-k} = E_k\left(B - \sum_{i=0}^{n-1} B_i\right) E_{n-k}\,. \tag{15}$$

We need the following three lemmas.

Lemma 4. *Let $A=\varphi(f)$ or $\pi(g)$ with $f \in \mathscr{H}_-(O)$, $g \in \mathscr{H}_+(O)$. Then for any Φ, $\Psi \in D$*

$$\left(A\Phi, \left(B - \sum_{i=0}^{n-1} B_i\right)\Psi\right) = \left(\Phi, \left(B - \sum_{i=0}^{n-1} B_i\right) A\Psi\right).$$

Proof. We have

$$\left(e^{-i\lambda A}\Phi, \left(B - \sum_{i=0}^{n-1} B_i\right)\Psi\right) = \left(\Phi, \left(B - \sum_{i=0}^{n-1} B_i\right) e^{i\lambda A}\Psi\right), \tag{16}$$

because $e^{i\lambda A} \in \mathfrak{A}(O) \subset \mathfrak{A}(\sim O)'$ and thus commutes with $B \in \mathfrak{A}(O)'$, and with $\sum_{i=0}^{n-1} B_i$ by (I). Since A is defined on D and maps D into itself, the lemma follows if we take weak derivatives with respect to λ on both sides of (16) and set $\lambda=0$.

Lemma 5. *For $k=0, 1, \ldots n$, the operators $B_{k\,n-k}$ are uniquely determined by B_{0n}.*

Proof. Let $f_i \in \mathscr{H}_-(O)$, $g_i \in \mathscr{H}_+(O)$, $\Psi \in \mathscr{F}$. Then for $0 \leqq r \leqq k \leqq n$

$$\begin{aligned}
&\left(:\prod_{i=1}^{r} \pi(g_i) \prod_{i=r+1}^{k} \varphi(f_i): \Phi_0, B_{k\,n-k}\Psi\right) \\
&\quad = \left(\prod_{i=1}^{r} \pi(g_i) \prod_{i=r+1}^{k} \varphi(f_i)\Phi_0, E_k\left(B - \sum_{i=0}^{n-1} B_i\right) E_{n-k}\Psi\right) \\
&\quad = \left(\prod_{i=1}^{r} \pi(g_i) \prod_{i=r+1}^{k} \varphi(f_i)\Phi_0, \left(B - \sum_{i=0}^{n-1} B_i\right) E_{n-k}\Psi\right)
\end{aligned}$$

using (III),

$$= \left(\Phi_0, E_0\left(B - \sum_{i=0}^{n-1} B_i\right) \prod_{i=r+1}^{k} \varphi(f_i) \prod_{i=1}^{r} \pi(g_i) E_{n-k}\Psi\right)$$

by Lemma 4,

$$=\left(\Phi_0, E_0\left(B-\sum_{i=0}^{n-1} B_i\right) E_n \prod_{i=r+1}^{k} \varphi(f_i) \prod_{i=1}^{r} \pi(g_i) E_{n-k} \Psi\right).$$

again using (III),

$$=(\Phi_0, B_{0n} \Psi'), \tag{17}$$

where

$$\Psi'=:\prod_{i=r+1}^{k} \varphi(f_i) \prod_{i=1}^{r} \pi(g_i): E_{n-k} \Psi .$$

Thus we get the matrix elements of $B_{k\,n-k}$ between a total set of vectors (Lemma 2) from matrix elements of B_{0n}. As $B_{k\,n-k}$ is a bounded operator, Lemma 5 follows.

Lemma 6. *Let $\Phi \in \mathscr{F}^{r,n-r}(O)$ for some $r \in \{0, 1, \dots n\}$. Then $(\Phi_0, B_{0n}\Phi)$ is real.*

Proof. In Eq. (17) we set $k=n$, $\Psi=\Phi_0$, and

$$\Phi=:\prod_{i=1}^{r} \pi(g_i) \prod_{i=r+1}^{n} \varphi(f_i): \Phi_0 .$$

Then we get

$$\begin{aligned}(\Phi, B_{n0}\Phi_0)&=(\Phi_0, B_{0n}\Phi)\\ &=(B_{0n}\Phi, \Phi_0),\end{aligned}$$

which proves Lemma 6.

Now we are prepared to construct B_n out of B_{0n} and to finish the proof of Lemma 3. The restriction of B_{0n} to $E_n\mathscr{F}$ is a bounded operator mapping $E_n\mathscr{F}=(L^2)^{\otimes_s n}$, the symmetrized n-fold tensor product of L^2, into the complex numbers $\mathbb{C}$. By the Riesz representation theorem it is given by a kernel $B_{0n}(x_1, \dots x_n) \in (L^2)^{\otimes_s n}$. Lemma 6 says that

$$\begin{aligned}0&=\operatorname{Im}(\Phi_0, B_{0n}\Psi)\\ &=\operatorname{Im}\int B_{0n}(x_1, \dots x_n) \prod_{i=1}^{r} (i\mu^{1/2} g_i)(x_i) \prod_{i=r+1}^{n} (\mu^{-1/2} f_i)(x_i)\, dx_1 \dots dx_n,\end{aligned}$$

for all $f_i \in \mathscr{H}_-(O)$, $g_i \in \mathscr{H}_+(O)$ and all $r=0, 1, \dots n$. This implies that for all $r=0, 1 \dots n$,

$$\operatorname{Im} i^r B_{0n}(x_1, \dots x_n) \in [(\mathscr{H}_+(O))^{\otimes r} \otimes (\mathscr{H}_-(O))^{\otimes n-r}]^{\perp} \cap (L_r^2)^{\otimes_s n},$$

where we have used (4) and (5). In other words

$$\operatorname{Re} B_{0n}(x_i, \dots x_n) \in \operatorname{Sym} \bigcap_{r\ \mathrm{odd}} [(\mathscr{K}_+(O))^{\otimes r} \otimes (\mathscr{K}_-(O))^{\otimes n-r}]^{\perp},$$
$$\operatorname{Im} B_{0n}(x_i, \dots x_n) \in \operatorname{Sym} \bigcap_{r\ \mathrm{even}} [(\mathscr{K}_+(O))^{\otimes r} \otimes (\mathscr{K}_-(O))^{\otimes n-r}]^{\perp}, \tag{18}$$

where Sym is the symmetrizer from $(L_r^2)^{\otimes n}$ to $(L_r^2)^{\otimes_s n}$.

The proof of the following lemma is based on Lemma 1 and is given in the appendix.

Lemma 7. $\operatorname{Sym} \bigcap_{\substack{r\ \mathrm{odd} \\ (\mathrm{even})}} [(\mathscr{K}_+(O))^{\otimes r} \otimes (\mathscr{K}_-(O))^{\otimes n-r}]^{\perp}$

$$= \operatorname{Sym} \dot{\sum_{\substack{s\ \mathrm{even} \\ (\mathrm{odd})}}} (\mathscr{K}_+(\sim O))^{\otimes s} \otimes (\mathscr{K}_-(\sim O))^{\otimes n-s}, \tag{19}$$

where $\dot{\sum}_{\nu \in N} \mathscr{K}_\nu$ denotes the closure in L_r^2 of the set of all linear combinations of vectors in $\mathscr{K}_\nu$, $\nu \in N$. Let $\{\mu^{-1/2} f_\nu\}$ and $\{\mu^{1/2} g_\nu\}$ be complete orthonormal sets of vectors in $\mathscr{K}_-(\sim O)$ and in $\mathscr{K}_+(\sim O)$ respectively. Then $f_\nu \in \mathscr{H}_-(\sim O)$, $g_\nu \in \mathscr{H}_+(\sim O)$, and (18) and (19) allow us to write

$$\operatorname{Re} B_{0n}(x_1, \dots x_n)$$
$$= \operatorname{Sym} \lim_{\varrho \to \infty} \sum_{s\ \mathrm{even}} \sum_{\underset{\sim}{\nu}} c_{s \underset{\sim}{\nu} \varrho} \prod_{i=1}^{s} (\mu^{1/2} g_{\nu_i})(x_i) \prod_{i=s+1}^{n} (\mu^{-1/2} f_{\nu_i})(x_i),$$
$$\operatorname{Im} B_{0n}(x_1, \dots x_n)$$
$$= \operatorname{Sym} \lim_{\varrho \to \infty} \sum_{s\ \mathrm{odd}} \sum_{\underset{\sim}{\nu}} c_{s \underset{\sim}{\nu} \varrho} \prod_{i=1}^{s} (\mu^{1/2} g_{\nu_i})(x_i) \prod_{i=s+1}^{n} (\mu^{-1/2} f_{\nu_i})(x_i), \tag{20}$$

where $\underset{\sim}{\nu} = (\nu_1, \dots \nu_n)$, and the $c_{s \underset{\sim}{\nu} \varrho}$ are real coefficients. Note that the infinite sums and $\lim_{\varrho \to \infty}$ converge in $(L_r^2)^{\otimes n}$ norm. The ϱ limit is necessary because the subspaces of $(L_r^2)^{\otimes n}$ which occur in $\dot{\Sigma}$, Eq. (19), are not mutually orthogonal.

Now we define the operator B_n. For $\Phi \in D$ we set

$$B_n \Phi = \lim_{\varrho \to \infty} \sum_{s} \sum_{\underset{\sim}{\nu}} \left(\frac{2^n}{n!} \right)^{1/2} (-1)^{[s/2]} c_{s \underset{\sim}{\nu} \varrho} : \prod_{i=1}^{s} \pi(g_{\nu_i}) \prod_{i=s+1}^{n} \varphi(f_{\nu_i}) : \Phi, \tag{21}$$

where $[s/2]$ is the entire part of $\frac{s}{2}$. The coefficient $\left(\frac{2^n}{n!} \right)^{1/2} (-1)^{[s/2]}$ is chosen such that $B_{0n} \Phi = E_0 B_n E_n \Phi$. The limit $\varrho \to \infty$ and the infinite sum over $\underset{\sim}{\nu}$ in (21) converge strongly. The operator B_n, defined by (21) is certainly a symmetric operator, defined on D and mapping D into itself. Property (I) of Lemma 3 holds for $i=n$, because on D, B_n is the strong limit of a sum of operators which are all affiliated with $\mathfrak{A}(\sim O)$ and thus commute with $C\in\mathfrak{A}(\sim O)'$ in the sense of (I). Property (II) follows immediately from the explicit formula (21) for B_n. Finally (III) holds for $k+l<n$ by the induction assumption; for $k+l=n$ and $k=0$ the relation follows from our construction of B_n, namely

$$E_0 B_n E_n = B_{0n} = E_0 \left(B - \sum_{i=1}^{n-1} B_i\right) E_n .$$

For $k \neq 0$ we use Lemma 5 to go back to the case $k=0$.

This completes our analysis of the $\varphi - \pi$ expansion and the proof of Lemma 3.

Obviously for a $C\in\mathfrak{A}(\sim O)'$ which maps D into itself, Lemma 3 shows that $[B, C]=0$, whenever $B\in\mathfrak{A}(O)'$. For a general $C\in\mathfrak{A}(\sim O)'$ we follow the idea of Araki [1, Section 9]. For $k=1,2$, let $\Psi_k = A_k \Phi_0$, where $A_k = \prod_{i=1}^{r_k} \pi(g_{ki}) \prod_{j=1}^{s_k} \varphi(f_{kj})$ and

$$g_{ki}\in\mathscr{H}_-(O), \quad f_{kj}\in\mathscr{H}_+(O) .$$

Let $\xi_\varepsilon(\lambda)$ be a positive C_0^∞ function with support in $|\lambda|<\varepsilon$ and with $\int \xi_\varepsilon(\lambda)\, d\lambda = 1$. Then we define

$$W_{k\varepsilon} = \prod_{i=1}^{r_k} \left(\int e^{i\lambda\pi(g_{ki})} \xi_\varepsilon(\lambda)\, d\lambda\right) \prod_{j=1}^{s_k} \left(\int e^{i\lambda\varphi(f_{jk})} \xi_\varepsilon(\lambda)\, d\lambda\right) .$$

It is easy to show that for an arbitrary vector θ, $W_{k\varepsilon}\theta$ is in the domain of A_k as well as in the domain of A_k^*, and W_k converges strongly to 1 as $\varepsilon \downarrow 0$. Obviously $W_{k\varepsilon}\in\mathfrak{A}(O)\subset\mathfrak{A}(\sim O)'$ and thus $C\in\mathfrak{A}(\sim O)'$ implies $W_{1\varepsilon} C W_{2\varepsilon}\in\mathfrak{A}(\sim O)'$. Lemma 3, property I gives us for all $i\in\mathbb{Z}_+$

$$(\Psi_1, C B_i \Psi_2) = (\Psi_1, B_i C \Psi_2) ,$$

and we want to sum both sides over i and to replace ΣB_i by B. We first replace C by $C_\varepsilon = W_{1\varepsilon} C W_{2\varepsilon}$. We get

$$\begin{aligned}(A_1\Phi_0, C_\varepsilon B_i A_2 \Phi_0) &= (A_1\Phi_0, B_i C_\varepsilon A_2\Phi_0) \\ &= (A_2^* C_\varepsilon^* A_1 \Phi_0, B_i\Phi_0) = (B_i\Phi_0, A_1^* C_\varepsilon A_2 \Phi_0) .\end{aligned} \tag{22}$$

But in (22) summation over i and substituting B for ΣB_i is allowed, because for any $\Phi\in\mathscr{F}$

$$\sum_{i=0}^{N} (B_i\Phi_0, \Phi) = \sum_{j=0}^{N} \left(E_j\left(\sum_{i=0}^{N} B_i\right) E_0\Phi_0, \Phi\right)$$

using (II) of Lemma 3,

$$=\sum_{j=0}^{N}(E_j B E_0 \Phi_0, \Phi)$$

using (III) of Lemma 3,

$$=\left(\sum_{j=0}^{N} E_j B \Phi_0, \Phi\right)$$

$$\rightarrow (B\Phi_0, \Phi) \quad \text{as} \quad N \rightarrow \infty .$$

Thus we get from (22)

$$(\Psi_1, C_\varepsilon B \Psi_2) = (\Psi_1, B C_\varepsilon \Psi_2) . \tag{23}$$

Due to the strong convergence of C_ε to C in the limit $\varepsilon \downarrow 0$, and because the set of all Ψ_k's is total in $\mathscr{F}$ (Lemma 2), it follows that B and C commute. This completes the proof of duality and thus of our theorem.

Note added in proof. As was pointed out by Prof. E. H. Wichmann the last argument of our proof (regularization of C) can be avoided. Namely Lemma 3 already implies that $(\Phi_0, [B, C]\Phi_0) = O$ for $B \in \mathfrak{A}(O)'$ and $C \in \mathfrak{A}(\sim O)'$. It is then immediately clear that $(A_1\Phi_0, [B, C]A_2\Phi_0) = 0$ for all $A_1, A_2 \in \mathfrak{A}(O)$, which proves $[B, C] = 0$, as $\{A\Phi_0, A \in \mathfrak{A}(O)\}$ is dense in $\mathscr{F}$ by a theorem of Reeh and Schlieder.

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Appendix

Proof of Lemma 7.

We denote $\mathscr{K}_\pm(O)$ by $K_\pm$ and L_r^2 by K. Then it follows from Lemma 1 that

$$K_+ \dotplus K_- = K_+ \dotplus K_-^\perp = K_+^\perp \dotplus K_- = K_+^\perp \dotplus K_-^\perp = K, \text{ or equivalently}$$

$$K_+^\perp \cap K_-^\perp = K_+^\perp \cap K_- = K_+ \cap K_-^\perp = K_+ \cap K_- = \{O\} . \tag{A 1}$$

Let P_+ be the projection operator onto K_+.

Lemma A 1 (Araki [1, Lemma 4.1]). *There exists a unique, closed operator φ from K_+ into $K_+^\perp$ with domain $D_\varphi = P_+ K_-$, range $\Delta_\varphi = (1 - P_+)K_-$ and graph $G_\varphi = K_-$. D_φ and Δ_φ are dense in K_+ and $K_+^\perp$ respectively.*

Proof. We define φ on $P_+ K_-$ by $\varphi P_+ x = (1 - P_+)x$, $x \in K_-$. Assume $D_\varphi = P_+ K_-$ is not dense in K_+. Then there is an $x \in K_+$ such that

$O=(x, P_+ y)=(P_+ x, y)=(x, y)$ for all $y \in K_-$ and thus $x \in K_-^\perp$. But $K_+ \cap K_-^\perp = \{0\}$ by assumption, therefore $x=0$, and D_φ is dense in K_+. The proof for Δ_φ is similar, and the rest of the lemma is obvious.

The operator φ has a polar decomposition [10, p. 334]; we write it as $\varphi = U\alpha^{1/2}$, where $\alpha=(\varphi^* \varphi)^{1/2}$ is a nonnegative, self-adjoint operator in K_+ and U is an isometry from K_+ onto $K_+^\perp$. In fact α is positive definite since $\varphi x=0$ implies $x \in K_+ \cap K_-$ and thus $x=0$. Nevertheless the spectrum of α is in general not bounded away from zero, which means geometrically, that for any $\varepsilon > 0$ there are elements $x \in K_+$, $y \in K_-$, such that the angle between x and y is smaller than ε. To handle this difficulty, which of course only occurs in infinite dimensional Hilbert spaces, we use the spectral decomposition $\alpha = \int_0^\infty \lambda \, dE(\lambda)$ of α to decompose α. We define projection operators $E^{(n)}$ in K_+ by

$$E^{(n)} = \int_{(n+1)^{-1}}^{n^{-1}} dE(\lambda) + \int_n^{n+1} dE(\lambda), \qquad n \in \mathbb{Z}'_+ = \{1, 2, \ldots\}.$$

Obviously $\sum_1^\infty E^{(n)}=1$ and $E^{(n)}E^{(m)}=0$ if $n \neq m$, on K_+. Using the isometry $U: K_+ \to K_+^\perp$, we define projection operators $F^{(n)}$ in K by

$$F^{(n)} = E^{(n)} P_+ + U E^{(n)} U^{-1}(1-P_+).$$

Again $\sum_1^\infty F^{(n)}=1$ and $F^{(n)}F^{(m)}=0$ if $n \neq m$, on K.

We define

$$K_\pm^{(n)} = F^{(n)} K_\pm, \quad K_+^{\perp(n)} = F^{(n)} K_+^\perp, \quad K^{(n)} = F^{(n)} K.$$

Hence $K_+ = \bigoplus_{n=1}^\infty K_+^{(n)}$, etc.

By $\varphi^{(n)}$ we denote the restriction of φ to $K_+^{(n)}$, i.e. on K_+

$$\varphi^{(n)} = (U E^{(n)} U^{-1}) \varphi E^{(n)} = U E^{(n)} \alpha^{1/2} E^{(n)} = U \alpha^{1/2} E^{(n)} = \varphi E^{(n)},$$

where we have used the fact that α commutes with $E^{(n)}$. By construction $\varphi^{(n)}$ maps $K_+^{(n)}$ onto $K_+^{\perp(n)}$ and has a bounded inverse. The graph of $\varphi^{(n)}$ is $K_-^{(n)}$. Let $P_+^{(n)}$ be the projection (in $K^{(n)}$) onto $K_+^{(n)}$. We define

$$Q_+^{(n)} = P_+^{(n)} - P_+^{(n)}(\varphi^{(n)})^{-1}(1-P_+^{(n)}),$$

$$Q_-^{(n)} = (1-P_+^{(n)}) + P_+^{(n)}(\varphi^{(n)})^{-1}(1-P_+^{(n)}).$$

Obviously for each $n \in \mathbb{Z}'_+$, $Q_\pm^{(n)}$ is a bounded idempotent operator in $K^{(n)}$, $Q_+^{(n)} + Q_-^{(n)} = 1$, $Q_+^{(n)} Q_-^{(n)} = Q_-^{(n)} Q_+^{(n)} = 0$ and $Q_\pm^{(n)} K^{(n)} = K_\pm^{(n)}$. Note that, had we defined $Q_\pm$ in the big space K, it would have become an unbounded operator, as φ has not a bounded inverse.

Geometrically $Q_\pm^{(n)}$ is the (nonorthogonal) projection onto $K_\pm^{(n)}$ along $K_\mp^{(n)}$.

Now we introduce tensor products. We define inductively

$$H_{(1)} = K, \quad H_{(1)\pm} = K_\pm, \quad \hat{H}_{(1)\pm} = K_\pm^\perp,$$

$$H_{(r)} = H_{(r-1)} \otimes K,$$

$$H_{(r)\pm} = H_{(r-1)\pm} \otimes K_- \dotplus H_{(r-1)\mp} \otimes K_+,$$

$$\hat{H}_{(r)\pm} = \hat{H}_{(r-1)\pm} \otimes K_+^\perp \dotplus \hat{H}_{(r-1)\mp} \otimes K_-^\perp.$$

Note that

$$\operatorname{Sym} H_{(r)\pm} = \operatorname{Sym} \sum_{\substack{s\,\text{odd} \\ (\text{even})}} (K_+)^{\otimes s} \otimes (K_-)^{\otimes r-s},$$

$$\operatorname{Sym} \hat{H}_{(r)\pm} = \operatorname{Sym} \sum_{\substack{s\,\text{even} \\ (\text{odd})}} (K_-^\perp)^{\otimes s} \otimes (K_+^\perp)^{\otimes r-s},$$

and Lemma 7 follows from

Lemma A2. *For all* $r \in \mathbb{Z}_+$, $\hat{H}_{(r)\pm}^\perp = H_{(r)\pm}$.

Proof. Obviously the lemma holds for $r=1$. Now we assume it has been proved for $r = 1, 2, \ldots s-1$. Before proving it for $r = s$, we use the projection operators $F^{(n)}$ to define (for $n_i \in \mathbb{Z}_+$, $i = 1, \ldots r$)

$$H(n_1, \ldots n_r) = (F^{(n_1)} \otimes \cdots \otimes F^{(n_r)}) H_{(r)},$$

$$H_\pm(n_1, \ldots n_r) = (F^{(n_1)} \otimes \cdots \otimes F^{(n_r)}) H_{(r)\pm}.$$

Of course $H_{(r)} = \bigoplus_{n_1, \ldots n_r} H(n_1, \ldots n_r)$ and $H_{(r)\pm} = \bigoplus_{n_1, \ldots n_r} H_\pm(n_1, \ldots n_r)$.

We also need tensor products of the operators $Q_\pm^{(n)}$. We set $Q_\pm(n_1) = Q_\pm^{(n_1)}$ and define inductively

$$Q_\pm(n_1, \ldots n_s) = Q_\pm(n_1, \ldots n_{s-1}) \otimes Q_-^{(n_s)} + Q_\mp(n_1, \ldots n_{s-1}) \otimes Q_+^{(n_s)}.$$

It follows by induction that $Q_\pm(n_1, \ldots n_s)$ is a bounded, idempotent operator in $H(n_1, \ldots n_s)$ with

$$Q_+(n_1, \ldots n_s)\, Q_-(n_1 \ldots n_s) = Q_-(n_1, \ldots n_s)\, Q_+(n_1, \ldots n_s) = O$$

and $Q_+(n_1 \ldots n_s) + Q_-(n_1 \ldots n_s) = 1$ on $H(n_1, \ldots n_s)$. Geometrically $Q_\pm(n_1, \ldots n_s)$ projects onto $H_\pm(n_1, \ldots n_s)$ along $H_\mp(n_1, \ldots n_s)$.

Now we compute

$$\begin{aligned}\hat{H}^{\perp}_{(s)\pm} &= (\hat{H}^{\perp}_{(s-1)\pm} \otimes K \dotplus H_{(s-1)} \otimes K_+) \cap (\hat{H}^{\perp}_{(s-1)\mp} \otimes K \dotplus H_{(s-1)} \otimes K_-) \\ &= (H_{(s-1)\pm} \otimes K \dotplus H_{(s-1)} \otimes K_+) \cap (H_{(s-1)\mp} \otimes K \dotplus H_{(s-1)} \otimes K_-) \\ &= \bigoplus_{n_1, \dots n_s} L_\pm(n_1, \dots n_s)\end{aligned}$$

where

$$\begin{aligned}L_\pm(n_1, \dots n_s) = (H_\pm(n_1, \dots n_{s-1}) \otimes K^{(n_s)} \dotplus H(n_1, \dots n_{s-1}) \otimes K^{(n_s)}_+) \\ \cap (H_\mp(n_1, \dots n_{s-1}) \otimes K^{(n_s)} \dotplus H(n_1, \dots n_{s-1}) \otimes K^{(n_s)}_-) .\end{aligned}$$

Obviously $H_\pm(n_1, \dots n_s) \subset L_\pm(n_1, \dots n_s)$. Let now $x \in L_\pm(n_1, \dots n_s)$. Then

$$\begin{aligned}x &= (Q_\pm(n_1, \dots n_{s-1}) \otimes 1_{n_s} + 1_{n_1, \dots n_{s-1}} \otimes Q^{(n_s)}_+) \\ &\quad \cdot (Q_\mp(n_1, \dots n_{s-1}) \otimes 1_{n_s} + 1_{n_1, \dots n_{s-1}} \otimes Q^{(n_s)}_-) x \\ &= (Q_\pm(n_1, \dots n_{s-1}) \otimes Q^{(n_s)}_- + Q_\mp(n_1 \dots n_{s-1}) \otimes Q^{(n_s)}_+) x , \\ &= Q_\pm(n_1, \dots n_s) x \in H_\pm(n_1, \dots n_s) ,\end{aligned}$$

where 1_{n_s} and $1_{n_1 \dots n_{s-1}}$ are the identity operators on $K^{(n_s)}$ and $H(n_1, \dots n_{s-1})$ resp. Hence $L_\pm(n_1 \dots n_s) \subset H_\pm(n_1 \dots n_s)$. We conclude that $L_\pm(n_1, \dots n_s) = H_\pm(n_1 \dots n_s)$, which proves

$$\hat{H}^{\perp}_{(s)\pm} = \bigoplus_{n_1, \dots n_s} H_\pm(n_1 \dots n_s) = H_{(s)\pm} .$$

This proves the lemma for $r = s$ and thus for all $r \in \mathbb{Z}_+$.

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