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An Entropy Inequality for Quantum Measurements

G. Lindblad

Department of Theoretical Physics, Royal Institute of Technology, Stockholm, Sweden

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Abstract. It is proved that for an ideal quantum measurement the average entropy of the reduced states after the measurement is not greater than the entropy of the original state.

Consider a quantum state described by a density operator W in Hilbert space

$$W = W^* \geqq 0, \qquad \mathrm{Tr}\, W = 1 .$$

Then there exists an orthonormal set $\{|i\rangle\}$ such that

$$W = \Sigma\, w_i W_i$$

where $W_i = |i\rangle\langle i|$, $w_i > 0$, $\Sigma w_i = 1$.

Let O be an observable with eigenspaces defined by projections P_k;

$$O = \Sigma\, \omega_k P_k, \qquad \Sigma P_k = I .$$

If O is measured the value ω_k is obtained with probability $p_k = \mathrm{Tr}\, W P_k$ and W is then replaced by

$$W'_k = p_k^{-1} P_k W P_k .$$

The average over all possible outcomes gives a density operator

$$W' = \Sigma P_k W P_k = \Sigma\, p_k W'_k$$

(sum over all k such that $p_k \neq 0$).

The entropy of a state W is defined as

$$S(W) = -\,\mathrm{Tr}\, W \log W = -\,\Sigma\, w_i \log w_i .$$

It is well known that $S(W) \leqq S(W')$ [1] (the transformation $W \to W'$ is "dissipative") with equality if and only if $W = W'$. [A simple proof: use $\mathrm{Tr}\, W' \log W' = \mathrm{Tr}\, W \log W'$ and Klein's inequality $\mathrm{Tr}(W \log W - W \log W') \geqq 0$ ([2], p. 27).]

It was conjectured by Groenewold [3] that the average of the entropies of the states W'_k is not larger than $S(W)$:

$$S(W) - \Sigma\, p_k S(W'_k) \geqq 0 .$$

If entropy is identified with "missing information" the difference may be interpreted as the average information gain in the measurement process.

It is the object of this paper to prove this inequality (Theorem 2). The proof uses a subadditivity property of the entropy which is proved in Theorem 1.

Theorem 1. *Let A, B be positive operators of trace class and let*

$$\begin{aligned} S(A) &= -\operatorname{Tr} A \log A \text{ if } A \log A \text{ is of trace class} \\ &= \infty \text{ otherwise}. \end{aligned}$$

Then $S(A+B) \leqq S(A) + S(B)$.

If $S(A)$ or $S(B) = \infty$ then $S(A+B) = \infty$. If $S(A)$ and $S(B)$ are finite then the equality holds if and only if $A \cdot B = 0$.

A proof of the inequality (formulated differently) has been given by Lanford and Robinson [4]. We give an elementary proof which also gives the condition for equality.

Lemma. *If A, B are positive trace class operators such that $S(A) < \infty$ and $A \leqq B$ then*

$$\operatorname{Tr}(A \log A - A \log B) \leqq 0$$

with equality iff $B \mid \operatorname{Range} A = A \mid \operatorname{Range} A$.

Proof. Let $\{|a_i\rangle\}$, $\{|b_j\rangle\}$ be complete orthonormal sets of eigenvectors for A and B. We first show that $\operatorname{Tr}(B^{-1} A^2) \leqq \operatorname{Tr} A$.

$A \leqq B$ implies that $A + \varepsilon I \leqq B + \varepsilon I$ for every $\varepsilon > 0$. Then $A + \varepsilon I$ and $B + \varepsilon I$ have bounded inverses and ([2], p. 28)

$$(B + \varepsilon I)^{-1} \leqq (A + \varepsilon I)^{-1}.$$

Hence $\langle a_i | (A + \varepsilon I)(B + \varepsilon I)^{-1}(A + \varepsilon I) | a_i \rangle \leqq \langle a_i | (A + \varepsilon I) | a_i \rangle = a_i + \varepsilon$. Taking the limit $\varepsilon \to 0$ we obtain

$$\langle a_i | A B^{-1} A | a_i \rangle \leqq \langle a_i | A | a_i \rangle.$$

Hence $\operatorname{Tr}(A B^{-1} A) = \operatorname{Tr}(B^{-1} A^2) \leqq \operatorname{Tr} A$.

$A \leqq B$ implies that $\langle a_i | b_j \rangle = 0$, if $a_i \neq 0$ and $b_j = 0$. For $a_i \neq 0$ we have

$$\begin{aligned} &\langle a_i | (A \log A - A \log B - B^{-1} A^2 + A) | a_i \rangle \\ &= \sum_{b_j \neq 0} |\langle a_i | b_j \rangle|^2 a_i [\log(a_i/b_j) - a_i/b_j + 1] \leqq 0 \end{aligned}$$

since $\log x \leqq x - 1$ for $x > 0$, with equality only for $x = 1$.

As $\langle a_i | (A \log A - A \log B) | a_i \rangle = 0$ for $a_i = 0$, we get

$$\operatorname{Tr}(A \log A - A \log B) \leqq \operatorname{Tr}(B^{-1} A^2 - A).$$

Equality holds iff $\langle a_i | b_j \rangle \neq 0 \Rightarrow a_i = 0$ or $a_i = b_j$.

This condition is equivalent to

$$B|a_i\rangle = \sum_j \langle b_j|a_i\rangle\, b_j|b_j\rangle = a_i|a_i\rangle = A|a_i\rangle$$

for all $a_i \neq 0$. But Range A is spanned by $\{|a_i\rangle;\ a_i \neq 0\}$ so we get

$$B|\text{Range}\, A = A|\text{Range}\, A\,.$$

If this condition is fulfilled then obviously $\mathrm{Tr}(B^{-1}A^2 - A) = 0$ and the lemma is proved.

Proof of Theorem 1. Substitute $A+B$ for B in the lemma, then interchange A and B and add the two inequalities. Then, if $S(A)$ and $S(B)$ are finite

$$S(A+B) \leqq S(A) + S(B)$$

with equality iff $(A+B)|\text{Range}\,A = A|\text{Range}\,A$ and $(A+B)|\text{Range}\,B = B|\text{Range}\,B$ which is equivalent to $A \cdot B = 0$.

If $S(A)$ or $S(B) = \infty$ then $S(A+B) = \infty$ from the concavity property of the entropy ([2], p. 28).

Corollary. *Let A, $\{A_i\}$ be positive trace class operators. If $A = \Sigma A_i$ then*

$$S(A) \leqq \Sigma S(A_i)\,.$$

If $S(A) < \infty$ then the equality holds iff $A_i \cdot A_j = 0$ for $i \neq j$.

Proof. Let the eigenvalues of $A^{(n)} = \sum_{i=1}^{n} A_i$ be $\{a_k^{(n)}\}$ arranged in decreasing order. Then by [6], Chapter 2, Lemma 1.1. $a_k^{(n)} \leqq a_k^{(n')}$ for $n \leqq n'$, all k. From $\mathrm{Tr}(A - A^{(n)}) \to 0$ and monotonity follows that the eigenvalues of A are $a_k = \lim_n a_k^{(n)}$. Let $h(x) = -x \log x$. For k sufficiently large ($a_k < e^{-1}$) and $n \leqq n'$ we have $h(a_k^{(n)}) \leqq h(a_k^{(n')})$ and the convergence $h(a_k^{(n)}) \to h(a_k)$ is monotone, hence $\lim_n S(A^{(n)}) = \lim \Sigma h(a_k^{(n)}) = \Sigma h(a_k) = S(A)$. Then $S(A^{(n)}) \leqq \sum_1^n S(A_i)$ implies that $S(A) \leqq \sum_1^\infty S(A_i)$. If $S(A) = \sum_1^\infty S(A_i) < \infty$, then equality holds in $S(A) \leqq S(A_j) + S\left(\sum_{i \neq j} A_i\right) \leqq \Sigma S(A_i)$ and $A_j \cdot \sum_{i \neq j} A_i = 0$, hence $A_i \cdot A_j = 0$ for $i \neq j$.

Theorem 2. *Let W be a state with $S(W) < \infty$, W' and W_k' as defined above. Then*

$$\Sigma p_k S(W_k') \leqq S(W)\,.$$

Equality holds if and only if $S(W_k') = S(W)$ for all k with $p_k \neq 0$.

Proof. The statement is trivial if W is pure: $S(W) = S(W_k') = 0$ all k. Let $W = \Sigma w_i W_i$, W_i pure, $W_i \cdot W_j = 0$ for $i \neq j$ and put $s\{w_i\} = -\Sigma w_i \log w_i$.

Then $S(W) = s\{w_i\}$ and

$$W'_k = p_k^{-1} P_k W P_k = \sum_i w_i p_{i,k} W'_{i,k}$$

where $p_{i,k} = p_k^{-1} \operatorname{Tr} P_k W_i$ and $W'_{i,k} = (\operatorname{Tr} P_k W_i)^{-1} P_k W_i P_k$. By the corollary of Theorem 1, noting that all $W'_{i,k}$ are pure

$$S(W'_k) \leqq \sum_i S(w_i p_{i,k} W'_{i,k}) = \sum_i w_i p_{i,k} S(W'_{i,k}) + s\{w_i p_{i,k}\} = s\{w_i p_{i,k}\} \tag{1}$$

$$\text{i.e.} \qquad \sum_k p_k S(W'_k) \leqq \sum_k p_k s\{w_i p_{i,k}\}.$$

Use the fact that $s\{\ \}$ is a concave function and $\Sigma p_k = 1$, $\Sigma P_k = I$:

$$\sum_k p_k s\{w_i p_{i,k}\} \leqq s\left\{\sum_k p_k w_i p_{i,k}\right\} = s\left\{w_i \sum_k \operatorname{Tr}(P_k W_i)\right\} = s\{w_i\} = S(W). \tag{2}$$

If $S(W'_k) = S(W)$ for all k such that $p_k \neq 0$ then the equality obviously holds. Conversely, if $\Sigma p_k S(W'_k) = S(W)$, then equality holds in (2). This implies that for each i, $w_i p_{i,k}$ are equal for all k such that $p_k \neq 0$ ([5], § 3.8) i.e. $w_i p_{i,k} = \alpha_i$, hence

$$w_i \operatorname{Tr}(P_k W_i) = \alpha_i p_k.$$

Summation over k gives $\alpha_i = w_i$, i.e. $p_{i,k} = 1$ and

$$\operatorname{Tr}(P_k W_i) = \langle i | P_k | i \rangle = p_k.$$

Furthermore we must have equality in (1):

$$S(W'_k) = \sum_i S(w_i p_{i,k} W'_{i,k}) = \Sigma\, S(w_i W'_{i,k}) = s\{w_i\} = S(W)$$

and the theorem is proved.

Remark. Equality in (1) implies by Theorem 1

$$W'_{i,k} W'_{j,k} = 0 \quad \text{for} \quad i \neq j$$

hence $\operatorname{Tr}(P_k W_i P_k W_j) = |\langle j | P_k | i \rangle|^2 = 0$ for $i \neq j$.

This and the condition $p_{i,k} = 1$ can be summarized by

$$\langle i | P_k | j \rangle = p_k \delta_{ij} \quad \text{or} \quad W_i P_k W_j = p_k \delta_{ij} W_i.$$

Introducing $P_W = \Sigma W_i$ (the support of W) this condition reads

$$P_W P_k P_W = p_k P_W \quad \text{for all} \quad k,$$

which is an equivalent condition for equality in Theorem 2.

This relation is obviously satisfied if W is pure. If $P_W = I$ the condition implies that $P_k = I$ and $p_k = 1$ for one k. If the observable commutes with W then $P_k P_W = p_k P_W$, hence $P_W \leqq P_k$ and $p_k = 1$ for one k.

There are also nontrivial cases provided the dimension of P_W is not larger than the dimension of $I - P_W$.

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G. Lindblad
The Royal Institute of Technology
Department of Theoretical Physics
Lindstedtsvägen 15
S-10044 Stockholm 70, Sweden