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# A Laplace Transform on the Lorentz Groups

## I. Quasiregular Representations

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**Abstract.** As a first step in the generalisation of the Laplace transform to a non abelian group, we examine the representations of the groups $SO(n, 1)$ by means of transformations of (not necessarily integrable) functions defined over the hyperboloids $O(n, 1)/O(n)$. We define a regularised version of the Gel'fand-Graev transformation from the $n$-dimensional hyperboloid to its associated cone, which is valid (under certain restrictions) for polynomially bounded functions. Upon the cone we then carry out a pair of classical Laplace transforms parallel to a generator. We give inversion formulas for both these procedures, and express the Laplace transform/inversion pair directly in terms of the function on the hyperboloid.

For integrable functions our results reduce to those already known; in the non-integrable case they are new. New features include the divergence of the transform for certain discrete asymptotic behaviours; the existence of a finite dimensional kernel subspace which is annihilated; good asymptotic behaviour of both Laplace projection and inversion formulas; and the existence of discrete terms contributing to the inversion formula for even dimension. Our results are valid for all dimensions and are completely independent of the usual "Laplace transforms" involving projection by means of "second-kind representation functions"; in a final section of the paper we examine briefly the significance of that approach in the light of our own.

## I. Introduction

There has been recently [1, 2] considerable interest in possible generalisations of the Fourier transform on locally compact non-Abelian groups, with the hope of deriving expansion theorems valid for non-square-integrable functions over a non-compact group – typically one of the Lorentz groups $SO(n, 1)$. Two approaches to the problem can be distinguished: the distribution-valued-transform methods, as exemplified by (for instance) the work of Rühl [2]; and the special-function approach [1] which is often called the Laplace transform but which we shall call the Legendre transform. The former is the direct analogue of the classical one-dimensional theory; but it has the disadvantage that the function $f(g)$ with which we are concerned has to be regarded as a *distribution*, and although the Fourier transform is then certainly defined, there is no general way of evaluating it without a regularisation procedure involving $f(g)$ itself. We shall not consider this method further.

The second approach is more interesting, although the theory behind it is somewhat ad hoc. Despite any claims made by its protagonists, it still rests essentially upon the integral relation between Legendre functions

$$\int_1^\infty P_j(x)\, Q_l(x)\, dx = [(l-j)(l+j+1)]^{-1} \tag{1}$$

[valid for $\mathrm{Re}(l-j)>0$ and $\mathrm{Re}(l+j)>-1$], and its generalisations whereby the $P_l$ become the representation functions (matrix elements) of the group and the $Q_l$ the so-called "second-kind" functions [3]. There are three limitations to this approach: one (the asymptotic behaviour) we shall discuss shortly; a second is that the entire theory is cast in very strongly basis-dependent language; and the third (and most significant) is that it is restricted to $SO(2,1)$ and is incapable of generalisation [4] to higher groups like $SO(3,1)$. The last point tells us at once that if there exists a generalisation of the Laplace transform to non-Abelian groups, then this is not it.

Nonetheless, because no other approach to this important problem has so far been found, there is a considerable literature on the subject. Several authors [5] have noticed indeed that in a non-compact basis the representation functions themselves are of "second-kind" behaviour, but unfortunately this tantalising discovery merely exchanges one divergent integral for another, and we are no further advanced. Otherwise, the complete reliance upon special-function theory in a particular basis is in striking contrast to the basis-independent formalism [4, 6] developed for Fourier transforms.

In this paper we make a start on the problem of defining a true Laplace transform over a group, using the methods of group theory and integral geometry. For the sake of clarity and intuitiveness, we discuss first not the regular representation itself (that is, the "non-integrable regular representation") but rather the quasi-regular representation of $SO(n,1)$ by means of transformations of functions defined over one sheet of the hyperboloid $O(n,1)/O(n)$; we shall extend our results to the regular representations themselves in a later paper. Let us summarise the Legendre theory for $SO(2,1)$, in the especially simple case of a function $f(u)$ over the 2-dimensional hyperboloid $u \cdot u = 1$ which in a spherical coordinate system has no angular dependence: $f(u) = f(ch\,\zeta)$. We have the transform pair

$$\tilde{f}(l) = \int_1^\infty f(ch\,\zeta)\, Q_l(ch\,\zeta)\, d(ch\,\zeta)\,, \tag{2a}$$

$$f(ch\,\zeta) = \frac{1}{2\pi i} \int_C (2l+1)\, \tilde{f}(l)\, P_l(ch\,\zeta)\, dl\,, \tag{2b}$$

where $C$ is a contour running from $C-i\infty$ to $C+i\infty$; $C=\operatorname{Re} l$ is chosen to the right of the least value for which (2a) converges. If indeed $f$ is integrable we can take instead

$$\tilde{f}(l)=\int_{1}^{\infty} f(ch\zeta)\,P_l(ch\zeta)\,d(ch\zeta)\,, \tag{3a}$$

$$f(ch\zeta)=\frac{1}{2\pi i}\int_{C}(2l+1)\,\tilde{f}(l)\,Q_{-l-1}(ch\zeta)\,dl\,; \tag{3b}$$

but (3a) diverges for general $f$, whereas (2a) may converge. The pair (2) is generally called the Laplace transform; only the pair (3) can be extended [4] to higher groups.

This theory suffers however from an important defect. The classical Laplace transform on the real line converges for all exponentially bounded $f(x)$; and it gains a large measure of its usefulness from the fact that when the contour of integration in the inversion formula is pulled back to the left half-plane, we can ultimately ignore its contribution to the integral when compared with those of the singularities it has crossed. Such is not the case in (2b): the Legendre $P_l$-functions are ill-behaved as $\operatorname{Re} l\to\pm\infty$. Certainly it is true that (3b) is satisfactory as $\operatorname{Re} l\to-\infty$; but then (3a) does not usually converge. Thus the Legendre transform pair for $SO(2,1)$ can *either* be arranged to have a convergent projection formula *or* to have an attractive inversion formula – *but not both.* We shall present a theory of a Laplace transform where both these desirable features are present.

To see how our results arise, consider the classical Laplace transform. This is usually (incorrectly) stated to be an expansion in non-unitary representations of the additive group of the reals; in fact, it derives its entire usefulness from the fact that it is no such thing. Instead, it is a transform over representations of a *semigroup* (the transform integral is only from zero to infinity); the group property is possessed only by the two-sided Laplace transform, whose domain of definition is small indeed. If we want to recover the entire function $f(x)$, $-\infty<x<\infty$, we need a pair of Laplace transforms – one for each half of the real axis.

This idea tells us how to proceed: the (generalised) Laplace transform on a group $G$ is to be identified with a projection over representations of some semigroup contained in $G$. We shall elaborate these ideas in a later paper; here we concern ourselves only with the "quasi-regular" representations of $SO(n,1)$, which we know how to treat. The general procedure for decomposing an (integrable) function on the hyperboloid into its irreducible components has been given by Gel'fand and Graev [7]; see also Ref. [6], Chapters V, VI, and Refs. [8–10]. Its cardinal feature is the mapping by means of an integral transform known as the Gel'fand-Graev transform from the space of functions $f(u)$ over the hyperboloid $u \cdot u = 1$ (Lorentz metric) to a space of functions $\hat{f}(k)$ over its associated cone $k \cdot k = 0$; once on the cone, it is meaningful to expand $\hat{f}$ in homogeneous functions (i.e., carry out a Fourier transform parallel to a generator), and this decomposes $\hat{f}$ into functions transforming under a unitary irreducible representation of the associated group of motions. Hence by moving to the cone, Fourier analysing, and transforming back to the hyperboloid, we have a convenient method for Fourier analysing over the group any function defined on its homogeneous space.

Our task is now clear. First, we must define a version of the Gel'fand-Graev transform which holds for non-integrable functions; second, once on the cone we carry out a pair of Laplace transforms. These will define a corresponding transform on the hyperboloid itself, which will be a true Laplace transform on the group.

In Section II then we devote ourselves to defining the Gel'fand-Graev transform for polynomially bounded functions. The result is a two-stage regularisation process using the method of "analytic continuation in the coordinates", which reduces to the usual result for integrable functions and (in general) converges satisfactorily. The caveat is because the transform may indeed not exist for a (discrete) set of functions – we should not expect it to. For consider the function $f(u) = (ch\zeta)^{\mu}$; this is analytic in $\mu$, and under a natural transformation should remain so. Therefore $\hat{f}(k)$ will have singularities in the complex $\mu$-plane; and these turn out to lie at certain integers or half-integers, with an interesting difference between the even and odd-dimensional cases. We also find that the transform has a non-empty kernel: it annihilates all those functions which transform under a finite-dimensional representation of the group. Provided the transform exists, we derive an inversion formula (modulo the kernel); this depends explicitly on the dimensionality of the space.

In Section III we carry out the Laplace transform on the cone and express this in terms of the function on the hyperboloid. We obtain a pair of formulas which have all the desirable properties of the classical Laplace transform pair, with the single exception of diverging for the special cases we mentioned above. The inversion formula we find to depend critically on the dimensionality: if $n = 2m+1$ is odd, only a contour integral is required in the complex $l$-plane; while if $n = 2m$ is even, we need also a sum of discrete contributions from the positive integers. We note that the poles and zeros of the measures in the inversion formulas dovetail exactly with the zeros and divergences of the transforms.

In Section IV we descend from our general $n$-dimensional basis-independent formalism and look briefly at the quasi-regular representation of $SO(2,1)$, in order to compare our results with the Legendre transform. We find that the essential difference springs from the fact that the scalar product on the hyperboloid induces one on the cone which has a *two-point* measure. In the case of the Fourier transform, the difference this makes turns out to be quite trivial; but for a Laplace or Legendre transform the consequences are important, and lead to a distinction between the two approaches.

## II.1. The Gel'Fand-Graev Transform

We parametrise the upper sheet $H$ of the $n$-dimensional two-sheeted hyperboloid $O(n, 1)/O(n)$ by a vector $u = (u_0, \boldsymbol{u})$ with $u \cdot u \equiv u_0^2 - \boldsymbol{u} \cdot \boldsymbol{u} = 1$. The associated cone we shall call $K$ and parametrise by $k$ with $k \cdot k = 0$. Consider any $f(u)$ which is $C^\infty$ and of compact support; then the Gel'fand-Graev transform $\Gamma$ is defined by [11]

$$\Gamma : f \to \hat{f} = \int_\omega f(u)\, d\omega \tag{4}$$

where the integration is over the horosphere $\omega$. Given any $k \in K$, this defines a horosphere by the points $u$ satisfying $u \cdot k = 1$; hence (4) becomes

$$\Gamma : f(u) \to \hat{f}(k) = \int f(u)\, \delta(u \cdot k - 1)\, du\,. \tag{5}$$

The measure $du$ is the usual invariant measure upon $H$. This converges for all $f(u)$ of compact support, and defines a function $\hat{f}(k)$ which is indefinitely differentiable and which vanishes [11] both at infinity and in a neighbourhood of the vertex of $K$. We wish to extend the definition of the transform $\Gamma$ to functions $f$ which belong to the space $P$ of polynomially bounded $C^\infty$ functions on $H$. It is natural to attempt this by taking

$$\Gamma : f \to \hat{f} = \frac{1}{2\Gamma(\varrho)} \int f(u)\, |u \cdot k - 1|^\varrho\, du\big|_{\varrho = -1} \tag{6}$$

where the integral is evaluated at $\operatorname{Re} \varrho \ll 0$ and then analytically continued to $\varrho = -1$; however, this fails because (a) at $u \cdot k = 1$ we then obtain a large divergence factor, and (b) we can have $|u|^2 \equiv u_0^2 + \boldsymbol{u} \cdot \boldsymbol{u} \to \infty$ without also $u \cdot k \to \infty$.

To turn the potential convergence factor into an actual one we therefore regularize this integral in its turn, by the method of "analytic continuation in the coordinates" (see Ref. [6], Chapter V, Section 3.2). Let $\eta$ be any vector inside the positive cone: i.e., $\eta \cdot \eta > 0$, $\eta_0 > 0$. Then the functional $(u \cdot k + iu \cdot \eta - 1)^\varrho$ is readily shown to be analytic in the coordinates of the set of all $k + i\eta$ in this "upper half-plane", as well as analytic in $\varrho$. Making use of the result

$$|x|^\lambda = \frac{e^{-i\pi\lambda/2}(x+i0)^\lambda + e^{i\pi\lambda/2}(x-i0)^\lambda}{2\cos\pi\lambda/2}, \tag{7}$$

which is true for generalised functions of a single variable [12], we *define* the generalised transform $\Gamma$ to be

$$\Gamma : f(u) \to \hat{f}(k) = \operatorname*{Lim}_{\eta\to 0} \{ \int f(u)\, J_\varrho(u,\eta,k)\, du|_{\varrho=-1} \} \tag{8}$$

where

$$J_\varrho(u,\eta,k) = \frac{e^{-i\pi\varrho/2}(u\cdot k + iu\cdot\eta - 1)^\varrho + e^{i\pi\varrho/2}(u\cdot k - iu\cdot\eta - 1)^\varrho}{4\Gamma(\varrho)\cos\pi\varrho/2}. \tag{9}$$

The limits and integrations are to be performed in the following order: first integrate, with $\operatorname{Re}\varrho$ sufficiently negative to ensure convergence, and $\eta$ in the positive cone; then analytically continue to $\varrho = -1$; finally let $\eta \to 0$. Provided it exists, we shall call (8) too the Gel'fand-Graev transform of $f$.

To investigate the convergence, we split $f(u)$ into two parts. Let $F_1$ be the subspace of $P$ of all functions $f_1(u)$ of compact support; and let $F_\infty$ be the subspace of $P$ of all functions $f_\infty(u)$ which as $|u| \to \infty$ grow like $|u|^{1-n}$ or faster. We see that $F_\infty$ is the subspace of functions which are not integrable. Then upon $F_1$ we can take the limits in (8) before integrating, and we recover exactly the classical transform (5). Upon $F_\infty$, however, the generalised transform may have singularities – i.e., not exist. The conditions for this are given in the following lemma.

**Lemma 1.** *If $n = 2m$ is even, the Gel'fand-Graev transform* (8), (9) *does not exist for functions $f_\infty(u) \in F_\infty$ which contain components whose behaviour is of the form*

$$f(|u| \to \infty) \sim |u|^{r-m+\frac{1}{2}} \qquad r \geqq 0\,;$$

*if $n = 2m+1$ is odd, it does not exist if*

$$f(|u| \to \infty) \sim |u|^{r-m} \qquad r \geqq 0\,,$$

*where in both cases $r$ is an integer. In either case, the kernel of the transformation is the "polynomial subspace" $Q$ whose $|u|-$ dependence is given in a spherical coordinate system by linear combinations of the functions*

$$(sh\,\zeta)^k\, C_{l-k}^{k+\frac{1}{2}n-\frac{1}{2}}(ch\,\zeta)$$

*where $C_m^\nu$ is a Gegenbauer polynomial and $l, k$ are integers.*

*Sketch of Proof.* Fix $\boldsymbol{k}$ and choose a spherical coordinate system with $\boldsymbol{k}$ as the polar axis. Then (8) is the regularization (if it exists) of the integral (6), which can be written

$$\frac{1}{2\Gamma(\varrho)}\int F(\zeta,\theta)\,[k_0(ch\zeta - sh\zeta\cos\theta)-1]^{\varrho}\,(sh\zeta)^{n-1} \cdot (\sin\theta)^{n-2}\,d\theta\,d\zeta|_{\varrho=-1}\,, \tag{10}$$

$$F(\zeta,\theta)=\int f(\zeta,\theta,\Omega_{n-2})\,d\Omega_{n-2}\,. \tag{11}$$

[The label $\Omega$ incorporates all the angular dependence of $f$ upon $S^{n-2}$, which is irrelevant to (10).] For a given $\theta$-independent asymptotic behaviour of $F$, no remaining $\theta$-dependence can worsen the behaviour of $\hat{f}$, and so to find the singularities of (10) it suffices to examine the case when $F(\zeta,\theta)=F(ch\zeta)$ alone.

In this case we find from (5) that we need to consider the regularisation of the integral

$$\int_{ch\alpha}^{\infty} F(ch\zeta)\,(ch\zeta - ch\alpha)^{\frac{n-3}{2}}\,d(ch\zeta)\,, \tag{12}$$

where we have set $k_0=e^{\alpha}$. Make the substitution $ch\zeta=y^{-1}$, and set $F_1(y)=F(y^{-1})$. Because $F(y)$ is polynomially bounded, we can write $F_1(y)$ as a sum of terms of the form[1] $y^{-p}(\ln y)^{\nu}\,F_2(y)$ where $F_2(y)=c(1+0(y))$ near $y=0$; then (12) becomes

$$\int_0^{\operatorname{sech}\alpha} F_2(y)\,(\ln y)^{\nu}\,(1-y\,ch\alpha)^{\frac{n-3}{2}}\,y^{-p-\frac{1}{2}n-\frac{1}{2}}\,dy\,. \tag{13}$$

This is just

$$\int_0^{\operatorname{sech}\alpha} F_2(y)\,(\ln y)^{\nu}\,(1-y\,ch\alpha)^{\frac{n-3}{2}}\,y^{\lambda}\,dy$$

evaluated for $\operatorname{Re}\lambda>0$ and analytically continued to $\lambda=-p-\frac{1}{2}n-\frac{1}{2}$. Since $F_2$ is finite and continuous at $y=0$, we can apply standard theory [12] and conclude that the integral can have singularities only at $\lambda=-r<0$; that is, at $p=r-\frac{1}{2}n+\frac{1}{2}$ with $r\geqq 0$.

With this knowledge, we can evaluate (12) with $F(x)=x^p$ to obtain further details. We find that

$$\widehat{ch^p}\,\zeta(k)=(2\pi)^{\frac{1}{2}(n-1)}\,e^{-(n-1)\alpha/2}(ch\alpha)^{p+\frac{1}{2}(n-1)}\,\frac{\Gamma(\frac{1}{2}-p-n/2)}{\Gamma(-p)} \tag{14}$$

[1] The inclusion of more slowly varying functions, like $(\ln\ln y)^{\mu}$, does not alter the argument further: the position of the singularities remains what it would be if the factor were just a logarithm.

when we take into account all the factors involved. The analytic continuation is to be applied to the parameter $n$; $p$ is regarded as fixed, so that (14) has a zero for positive integer $p$ for all values of $n$. This becomes clearer by working directly from (8).

If now we include logarithmic factors, then because we can no longer expand $y^p F_1(y)$ in a Taylor series about $y=0$ we no longer obtain the automatic zero at positive integer $p$. Together with the expression (14), this proves the first part of the lemma. It is of interest to remark that the insertion of a factor $[\ln(ch\,\zeta)]^q$ worsens the behaviour at the integer points by a pole of order $q$; notice the analogy with the classical Mellin transform.

To investigate the kernel of $\Gamma$ we start from the fact just proved, that for trivial dependence of $f$ on $S^{n-1}$ it consists of all polynomials in $(ch\,\zeta)$. Consider the representation of $O(n, 1)$ by $T_g : f(u) = f(ug)$, where $g \in O(n, 1)$. Under the transform $\Gamma$ this becomes just

$$\hat{T}_g : \hat{f}(k) = \hat{f}(kg)$$

(since both $H$ and $K$ are homogeneous spaces of $O(n, 1)$, this is meaningful); see Ref. [6], Chapter VI, Section 3.2. Hence if $f$ lies in the kernel of $\Gamma$, then so does $T_g f$. But the polynomials just found are merely the angular-independent part of a function on $H$ transforming under a *finite-dimensional* representation of $SO(n, 1)$; all such functions can be expressed as a finite linear combination of functions of the form [13]

$$(sh\,\zeta)^{k_1}\, C_{l-k_1}^{\frac{1}{2}n-\frac{1}{2}+k_1}(ch\,\zeta) \cdot \prod_{j=1}^{n-2} (\sin\theta_j)^{k_{j+1}}\, C_{k_j-k_{j+1}}^{\frac{1}{2}(n-j-1)+k_{j+1}}(\cos\theta_j)\, e^{ik_{n-1}\phi} \tag{15}$$

and these functions span a closed invariant subspace $Q$ of $P$. This proves the second part of the lemma.

## II.2. The Inversion Formula

We must first examine the behaviour of $\hat{f}(e^\alpha k)$ as $\alpha \to \pm\infty$. A simple and obvious argument, which we do not give here, yields the following result.

**Lemma 2.** *Let $f(u) \in P$ be a function satisfying the conditions of Lemma 1 (so that its Gel'Fand-Graev transform exists), with asymptotic behaviour $f(u) \sim |u|^p (\ln|u|)^q$ as $|u| \to \infty$ for all fixed $\mathbf{u}/|u|$; then as $\alpha \to \infty$, $\hat{f}(e^\alpha k)$ has the behaviour*

$$\hat{f}(e^\alpha k) \sim \alpha^q e^{\alpha p}$$
$$\hat{f}(e^{-\alpha} k) \sim \alpha^q e^{\alpha(p+n-1)}$$

*to leading order in $e^{\alpha}$ and $\alpha^{1}$. Furthermore, $\hat{f}(k)$ is indefinitely differentiable in the coordinates of k, except possibly at the vertex.*

We now introduce some technical lemmas on the singularities of certain linear functionals on spaces of polynomially bounded functions over $H$ and $K$.

**Lemma 3.1.** *Let $f \in P$ vanish in a neighbourhood of $|u|=0$. Consider the linear functional $\Phi$ which in a spherical coordinate system is given by*

$$\Phi = (sh\,\zeta)^{\mu}\,(ch\,\zeta/2)^{2-n-\mu}\,.$$

*Then as a function of $\mu$, $(\Phi, f)$ is singular at $\mu = -n$ only if, as $|u| \to \infty$, $f$ has asymptotic behaviour $|u|^{r}\,(\ln|u|)^{s}$ where r is a positive integer or zero[1]. When acting upon the entire space P, the kernel of $\Phi$ is the "polynomial subspace" Q of Lemma 1.*

**Lemma 3.2.** *Let $g_{\infty}(k)$ be a polynomially-bounded $C^{\infty}$ function over K which vanishes in a neighbourhood of the vertex. Consider the linear functional $\Psi$:*

$$\Psi = (a \cdot k)^{\frac{1}{2}(\mu-n)}$$

*where $a \cdot a = 1$; then as a function of $\mu$, $(\Psi, g_{\infty})$ is singular at $\mu = -n$ only if the asymptotic behaviour of $g$ is $g \sim k_0^{r}\,(\ln k_0)^{s}$ where r is a strictly positive integer[1].*

**Lemma 3.3.** *Let $g_0(k)$ be a function on K of compact support which is indefinitely differentiable at all points except possibly the vertex. Consider the linear functional $X$:*

$$X = (a \cdot k)^{-\frac{1}{2}(\mu+n)}$$

*where $a \cdot a = 1$; then as a function of $\mu$, $(X, g_0)$ is singular at $\mu = -n$ only if, as $k_0 \to 0$, $g_0$ behaves like $k_0^{1-r-n}(\ln k_0)^{s}$ where r is a strictly positive integer[1].*

The proofs of these lemmas are similar to that of Lemma 1. We remark that the presence of logarithmic terms in the asymptotics worsens the singularities but does not alter their positions.

We are now ready to derive the inversion formula for the transform $\Gamma$, under certain restrictions. We shall henceforth always assume that $f \in P$ has no components satisfying the conditions of Lemma 1 – i.e., that the transform $\Gamma f(k)$ exists; and we shall *also* require that $f(u)$ has no components of the asymptotic behaviour[1] $|u|^{r}(\ln|u|)^{s}$ where $r$ is a positive integer or zero. We shall discuss this restriction later. Under these assumptions, then, consider the following linear functional $\Psi$ operating on the space of transforms $\hat{f}(k)$:

$$(\Psi, \hat{f}) = \int \hat{f}(k)\, |a \cdot k - 1|^{\mu} (a \cdot k)^{-\frac{1}{2}(\mu+n)}\, dk \tag{16}$$

where $a$ lies on $H$ and the measure $dk$ is the usual invariant measure on the cone. The regularisation of the integral is performed for fixed $a$ by setting

$$\hat{f}(k) = \hat{f}_0(k) + \hat{f}_1(k) + \hat{f}_\infty(k),$$

where $\hat{f}_0$ vanishes outside a compact set which contains the vertex of the cone but none of the points $a \cdot k = 1$; $\hat{f}_1$ vanishes outside a compact set which contains these points but not the vertex; and $\hat{f}_\infty$ vanishes *within* a compact set containing both $k = 0$ and $a \cdot k = 1$. Upon $\hat{f}_1$ the integral converges classically for $\mathrm{Re}\,\mu > 0$; on $\hat{f}_0$ and $\hat{f}_\infty$ it converges for $\mathrm{Re}\,\mu$ sufficiently negative, as follows from the asymptotic behaviour of $\hat{f}$ given in Lemma 2. The value of (16) is then found by analytic continuation in $\mu$.

Now if $\Gamma$ exists, $\Psi\Gamma \equiv \Phi$ defines a linear functional on $P$. By (5), this functional is given explicitly by

$$\Phi(u, a; \mu) = \int |a \cdot k - 1|^{\mu} (a \cdot k)^{-\frac{1}{2}(\mu+n)}\, \delta(u \cdot k - 1)\, du\,. \tag{17}$$

The integral can be performed [11], and gives

$$\Phi(u, a; \mu) = \frac{2\pi^{\frac{1}{2}(n-1)}\, \Gamma(\frac{1}{2}\mu + \frac{1}{2})}{\Gamma(\frac{1}{2}\mu + \frac{1}{2}n)} \; \frac{sh^{\mu} r}{ch^{\mu+n-2} r/2} \tag{18}$$

where $ch\, r = a \cdot u$; so we find

$$(\Psi, \hat{f}(k))(a) = \int \Phi(u, a; \mu)\, f(u)\, du\,. \tag{19}$$

The discussion now depends upon the parity of $n$. Suppose first that $n = 2m + 1$ is odd; and consider Eq. (19). Then upon the space $H_0$ of functions $f \in P$ of compact support, $\Phi$ has a simple pole at $\mu = -n$, with residue

$$\frac{2^{n+1}\, \pi^{n-1} (-1)^m}{(n-1)!}\, \delta(\boldsymbol{u} - \boldsymbol{a})\,. \tag{20}$$

By Lemma 3.1 the kernel of $\Phi$ is the polynomial subspace $Q$; and if $f(u)$ has no behaviour[1] of the form $|u|^r (\ln |u|)^s$, where $r$ is an integer and $s \neq 0$, then (20) are the *only* singularities of $\Phi$ on the entire space $P/Q$. Under these assumptions (19) becomes

$$(\Psi, \hat{f})(a) = \frac{1}{(\mu+n)} \cdot \frac{2^{n+1}\, \pi^{n-1}}{(n-1)!} (-1)^m f'(a) + \text{regular terms} \tag{21}$$

where $f' \in P/Q$ is just $f$ *modulo* any components in $Q$. The remaining terms above are regular at $\mu = -n$.

Now consider $\Psi$ acting by (16) upon the space of functions $\hat{f}_1(k)$ of compact support on $K$. The generalised function $|t|^\mu$ of a single variable has a simple pole at $\mu = -n$ if $n = 2m+1$ is odd, with residue

$$\frac{2}{(n-1)!}\,\delta^{(n-1)}(t)\,. \tag{22}$$

By Lemma 3.2, at the point $\mu = -n$, $\Psi$ is regular upon $\hat{f}_\infty(k)$ unless $\hat{f}_\infty$ has an asymptotic component of $k_0^r(\ln k_0)^s$ with $r$ an integer; and by Lemma 3.3 $\Psi$ is regular upon $\hat{f}_0(k)$ unless $\hat{f}_0$ has a component behaving like $k_0^{1-n-r}(\ln k_0)^s$ as $k_0 \to 0$. Hence by Lemma 2, the only singularities of $\Psi$ acting on $\hat{f}_\infty$ or $\hat{f}_0$ arise from components of $f(u)$ growing exactly [1] as $|u|^r(\ln|u|)^s$. By our assumptions, there are no such components with $s \neq 0$; and if $s = 0$, then by Lemma 1, such components of $f$ are in the kernel of $\Gamma$ and hence cannot appear in $\hat{f}$. Therefore if $\hat{f}(k)$ is the transform of a function $f(u) \in P$ satisfying our assumptions, the *only* singularities of the functional $\Psi(a, k; -n)$ are those found when it acts upon $\hat{f}_1(k)$ and given by (22).

Equating the residues of the poles at $\mu = -n$ of Eqs. (21) and (16) with the aid of (22), we find then

$$f'(a) = \frac{(-1)^m}{2(2\pi)^{2m}} \int \hat{f}(k)\,\delta^{(2m)}(a\cdot k - 1)\,dk\,. \tag{23}$$

The integral converges because for fixed $a$ this is a continuous closed contour upon the cone.

We now turn our attention to the case of even $n = 2m$. (18) is now regular at $\mu = -n$; so too is the generalised function $|t|^\mu$ of a single variable [12], so that arguing just as before we obtain the result

$$f'(a) = \frac{(-1)^m(2m-1)!}{(2\pi)^{2m}} \int \hat{f}(k)\,|a\cdot k - 1|^{-2m}\,dk\,. \tag{24}$$

The function $f'(a)$ is defined as before; and the regularisation of (24) is to be understood in the sense described after (16). We now summarise the results of this section in the form of a theorem.

**Theorem 1.** *Let $f(u)$ be a $C^\infty$ polynomially-bounded function over the upper sheet $H$ of the hyperboloid $O(n,1)/O(n)$ whose Gel'fand-Graev transform $\hat{f}(k)$ exists [and is given by (8) and (9)]. Let $f'(u)$ be just $f(u)$ modulo any polynomial terms (in the sense of Lemma 1). Then if $n = 2m+1$ is odd, we have the inversion formula*

$$f'(u) = \frac{(-1)^m}{2(2\pi)^{2m}} \int \hat{f}(k)\,\delta^{(2m)}(a\cdot k - 1)\,dk\,.$$

*If $n=2m$ is even, the inversion formula is valid only for transforms of functions $f(u)$ with no components which behave asymptotically as $|u|^r(\ln|u|)^s$ where $r$ is a positive integer*[1]. *If these conditions are met, we have the result*

$$f'(u)=\frac{(-1)^m(2m-1)!}{(2\pi)^{2m}}\,\mathrm{Reg.}\int|a\cdot k-1|^{-2m}\hat{f}(k)\,dk\,.$$

These results are formally just those of Ref. [6], Chapter V, Section 2.3; and indeed we should expect them to be, since our results must reduce to the classical ones if $f(u)$ is of compact support. What we have done is to define a regularisation of the Gel'fand-Graev transform which exists for a large class of polynomially bounded functions, and show that the old inversion formulas hold in the sense of Theorem 1. The new features are the non-existence of the transform for certain discrete asymptotic behaviours; the precise definition of the regularisation procedures to be adopted; and the annihilation of all functions on $H$ which transform under a finite-dimensional representation of $O(n,1)$.

The divergence of the inversion formula in the even-dimensional case for certain discrete asymptotic behaviours is unpleasant but apparently unavoidable; we should perhaps point out that the reason this does not occur in the odd-dimensional case is that by Lemma 1, the transform itself does not exist for these asymptotic behaviours. Finally, although we have assumed in all our work that the asymptotics can be represented by powers of $|u|$ and $\ln|u|$, it is clear that functions growing more slowly at infinity (such as $\ln\ln|u|$) introduce no new features.

### III.1. The Laplace Transform

The canonical method of decomposing the quasiregular representation of $O(n,1)$ into its irreducible components has been given by Gel'fand and Graev [6, 7]. It consists of passing to the transform space of functions $\hat{f}(k)$ defined by (5) and then carrying out a Fourier transform parallel to the generator of the cone:

$$\tilde{f}(k,l)=\int_{-\infty}^{\infty}\hat{f}(e^{\alpha}k)e^{-l\alpha}\,d\alpha\,. \tag{25}$$

If however $f(u)$ is not integrable, neither is $\hat{f}(k)$, and the Fourier transform diverges. It is then natural to introduce the *Laplace* transform on the cone by the pair of transforms

$$\tilde{f}(k_+,l)=\int_{0}^{\infty}\hat{f}(e^{\alpha}k)e^{-l\alpha}\,d\alpha \tag{26}$$

$$\tilde{f}(k_-,l)=\int_{0}^{\infty}\hat{f}(e^{-\alpha}k)e^{-(l+n-1)\alpha}\,d\alpha\,. \tag{27}$$

Because of Lemma 2, if the generalised transform $\Gamma f = \hat{f}$ exists, these integrals both converge for Re$l$ sufficiently positive, and indeed define analytic functions of $l$.

Now (26) defines a linear functional upon the space $P(u)$ itself. To find an explicit expression for this, we combine (26) with (9) and consider the integral

$$\Phi^{(+)}(p, p'; \varrho, l) = \int_0^\infty J_\varrho(u, \eta, e^\alpha k) e^{-l\alpha} \, d\alpha \tag{28}$$

where $p \equiv u \cdot k$, $p' \equiv u \cdot \eta$; so that

$$\tilde{f}(k_+, l) = \lim_{\eta \to 0} \{\int f(u) \, \Phi^{(+)}(p, p'; \varrho, l) \, du|_{\varrho = -1}\} . \tag{29}$$

The integral (28) can be performed, and yields

$$\begin{aligned} \Phi^{(+)}(p, p'; \varrho, l) &= [4\Gamma(\varrho) \cos \pi\varrho/2]^{-1} \\ &\left\{ e^{-i\pi\varrho/2} \frac{p^\varrho}{(l-\varrho)} F\left(-\varrho, l-\varrho; l-\varrho+1; \frac{1-ip'}{p}\right) \right. \\ &\left. + e^{i\pi\varrho/2} \frac{p^\varrho}{(l-\varrho)} F\left(-\varrho, l-\varrho; l-\varrho+1; \frac{1+ip'}{p}\right) \right\} . \end{aligned} \tag{30}$$

Now $F(a, b; c; z)$ has a cut along the positive real $z$-axis from $z = 1$ to $\infty$, but is analytic elsewhere in the complex $z$-plane. Hence it is clear that (29) has vanishing contribution from the region $p > 1$; for $p < 1$ however the limit $\eta \to 0$ (i.e., $p' \to 0$) in (29) takes the two hypergeometric functions to opposite sides of the cut, and a non-zero limit exists. To determine this limit, we transform [14] the first term in the braces above to

$$\begin{aligned} &e^{i\pi\varrho/2} l^{-1} (1 - ip')^\varrho F\left(-\varrho, -l; 1-l; \frac{p}{1-ip'}\right) \\ &\qquad + e^{-i\pi(l-\varrho/2)} \frac{\Gamma(l-\varrho)\Gamma(-l)}{\Gamma(-\varrho)} p^l (1-ip')^{\varrho-l} . \end{aligned} \tag{31}$$

As we take the limits in (29) the contribution of the pair of terms like the first one above vanishes; the second pair gives us

$$\begin{aligned} &\Phi^{(+)}(p, p'; \varrho, l) \\ &\quad = \frac{-i}{2 \sin \pi l} p^l \{e^{il\pi}(1+ip')^{\varrho-l} - e^{-il\pi}(1-ip')^{\varrho-l}\} \, \theta(1-p) . \end{aligned} \tag{32}$$

In the same way as (29), $\tilde{f}(k_-, l)$ is defined by a functional $\Phi^{(-)}$, which is given by

$$\begin{aligned}\Phi^{(-)}(p, p'; \varrho, l) = \tfrac{1}{2} i \operatorname{cosec} \pi l \, p^{1-l-n} \\ \cdot \{e^{-il\pi}(1+ip')^{\varrho+l+n-1} - e^{il\pi}(1-ip')^{\varrho+l+n-1}\}\, \theta(p-1).\end{aligned} \tag{33}$$

The transforms (26), (27), given in terms of $f(u)$ by the linear functionals (32), (33), we shall call the Laplace transforms[2] on $H$. Notice that in (32) we can take the limit $\varrho \to -1$ before integrating, provided that Re $l$ is sufficiently large; in (33) however this is impossible. If $f(u) \in L^1(u)$ we can indeed transfer both the limits from (29) to (32), (33), which then become

$$\begin{aligned}\Phi^{(+)}(p, l) &= p^l \theta(1-p) \\ \Phi^{(-)}(p, l) &= p^{1-l-n}\theta(p-1).\end{aligned} \tag{34}$$

These results are more easily obtained from the "classical" transform, which is valid for $f \in L^1(u)$. In such a case, the Fourier transform (25) exists, and is defined [8, 9] by a functional

$$\Phi(p, l) = \Phi^{(+)}(p, l) + \Phi^{(-)}(p, 1-l-n) = p^l.$$

### III.2. The Laplace Inversion Formula

The inversion formulas for even and odd dimension show considerable differences. We start by examining the case of even $n = 2m$, assuming initially that $f(u)$ satisfies the conditions of Theorem 1.

Choose some (fixed) cross-section $\mathscr{K}$ of the cone K, and write $k = \not{k} e^{\alpha}$, with $\not{k} \in \mathscr{K}$. The measure $d\not{k}$ is then defined by

$$dk = e^{(n-1)\alpha} d\alpha \, d\not{k}.$$

From the classical Laplace inversion formula, we have

$$\hat{f}(\not{k} e^{\alpha}) = \frac{1}{2\pi i} \int_C \tilde{f}(\not{k}_+, l) e^{l\alpha} dl \qquad (\alpha > 0)$$

where $C$ is the usual contour $C - i\infty$ to $C + i\infty$ to the right of all singularities of $\tilde{f}$ in $l$. Hence by (24) the contribution of $\tilde{f}(\not{k}_+, l)$ (that is, of

[2] We note in passing that the generalised Paley-Wiener theorem will impose restrictions upon these transforms, so they are not entirely independent.

the part of the cone lying above $\mathscr{K}$) to $f(u)$ is given by

$$f'_+(u) = \frac{(-1)^m (2m-1)!}{(2\pi)^{2m+1} i} \cdot \int |e^\alpha u \cdot k - 1|^{-2m} e^{(2m-1)\alpha} \, d\alpha \, dk \int_C \tilde{f}(k_+, l) e^{l\alpha} \, dl \,. \tag{35}$$

Interchange the orders of integration and look at the $\alpha$-integral; writing as before $p \equiv u \cdot k$ this is

$$\text{Reg.} \int_0^\infty |e^\alpha p - 1|^{-2m} e^{\alpha(l+2m-1)} \, d\alpha \equiv A(p) \,. \tag{36}$$

The integral can be performed (in the sense of its regularisation) using standard formulas [14]. For $p > 1$ we obtain

$$A(p > 1) = \frac{p^{-2m}}{(1-l)} F(2m, 1-l; 2-l; p^{-1}) \,; \tag{37}$$

while for $p < 1$ we find

$$A(p<1) = \lim_{\mu \to m} \left\{ p^{1-l-m-\mu} \frac{\Gamma(1-2\mu)\,\Gamma(\mu-l-m+1)}{\Gamma(2-\mu-l-m)} [1 - e^{-2i\pi\mu}] + \frac{(1-p)^{1-2\mu}}{p} (l-\mu+m-1)^{-1} F(1, 2-\mu-l-m; 2+\mu-l-m; p^{-1}+i0) \right\}. \tag{38}$$

The first term above is finite; using [14] HTF 2.9.34 on the second, we can write the whole of (38) as

$$A(p<1) = \frac{\Gamma(1-l)}{(2m-1)!\,\Gamma(2-l-2m)} \pi \cot l\pi \, p^{-l-2m+1} + \frac{(1-p)^{-2m}}{(1-2m-l)} F\left(1, 2m; l+2m; \frac{p}{p-1}\right). \tag{39}$$

Hence (35) becomes

$$f'_+(u) = B \int_C dl \int \tilde{f}(k_+, l) \, A(u \cdot k, l) \, dk \tag{40}$$

where $B$ is a numerical factor, and $A(u \cdot k, l)$ is given by (37) and (39). Now $\tilde{f}(k_+, l)$ is analytic in $l$ to the right of the contour $C$; so too is the second term in (39), if $\operatorname{Re} C > 1 - 2m$. Hence the contribution to (40) of this term vanishes. Consider the contribution of (37). $A(p > 1)$ tends to zero at infinity in the $l$-plane like $|l|^{-1}$ in all directions except along the positive real axis; where it has simple poles at the integers, with residue

$$\frac{-1}{(l-1)!} \frac{(2m+l-2)!}{(2m-1)!} p^{1-l-2m} \qquad (l = 1, 2, 3, \ldots) \,. \tag{41}$$

Hence the total contribution of $\tilde{f}(\not{k}_+, l)$ to the inversion formula (24) becomes

$$\frac{(-1)^m}{2i(2\pi)^{2m}} \int_C dl \frac{\Gamma(1-l)}{\Gamma(2-l-2m)} \cot \pi l \int_{u \cdot \not{k} < 1} \tilde{f}(\not{k}_+, l)(u \cdot \not{k})^{1-l-2m} d\not{k} \\ + \frac{(-1)^m}{(2\pi)^{2m}} \sum_l \frac{(l+2m-2)!}{(l-1)!} \int_{u \cdot \not{k} > 1} \tilde{f}(\not{k}_+, l)(u \cdot \not{k})^{1-l-2m} d\not{k}, \tag{42}$$

where the discrete summation runs over all positive or zero integers to the right of the contour $C$. The contribution to $f(u)$ of $\tilde{f}(\not{k}_-, l)$ is derived similarly, and the resulting complete inversion formula we shall give in Theorem 2.

Now consider the odd-dimensional case, when $n = 2m + 1$. By (16) and (22), instead of (36) we now need to find

$$\operatorname*{Res.}_{\mu=-n} \int_0^\infty |e^\alpha u \cdot \not{k} - 1|^\mu (e^\alpha u \cdot \not{k})^{-\frac{1}{2}(\mu+n)} e^{(l+n-1)\alpha} d\alpha. \tag{43}$$

By methods similar to those above, we can show that if $u \cdot \not{k} > 1$ this vanishes; while if $u \cdot \not{k} < 1$ it is proportional to

$$\frac{\Gamma(l+2m)}{\Gamma(l)} (u \cdot \not{k})^{-l-2m}. \tag{44}$$

We summarise these results in the form of a Theorem.

**Theorem 2.** *Suppose that $f(u)$ satisfies the conditions of Theorem 1. Then the Laplace transforms $\tilde{f}(k_\pm, l)$ of $f$ are given by*

$$\tilde{f}(k_\pm, l) = \lim_{\eta \to 0} \{\int f(u)\, \Phi^{(\pm)}(k \cdot u, k \cdot \eta; \varrho, l)\, du|_{\varrho=-1}\}$$

*where $\Phi^{(\pm)}$ are defined by* (32), (33). *If the dimension of the Lobachevskii space is odd, $n = 2m+1$, the inversion formula is*

$$f'(u) = \frac{(-1)^m}{(2\pi)^{2m+1} 2i} \int_C dl \frac{\Gamma(l+2m)}{\Gamma(l)} \\ \cdot \left\{ \int_{u \cdot \not{k} < 1} \tilde{f}(\not{k}_+, l)(u \cdot \not{k})^{-l-2m} d\not{k} + \int_{u \cdot \not{k} > 1} \tilde{f}(\not{k}_-, l)(u \cdot \not{k})^l d\not{k} \right\}$$

*where $\not{k}$ lies on any cross-section $\mathscr{K}$ of the cone (that is, any "contour" on K that intersects every generator once) such that the $\not{k}$-integrals converge, and $C$ is a contour from $C - i\infty$ to $C + i\infty$ to the right of all* *singularities of* $\tilde{f}(\ell_{\pm}, l)$. *If the dimension is even,* $n=2m$, *we have instead*

$$f'(u)=\frac{(-1)^m}{2i(2\pi)^{2m}}\int_C dl\,\frac{\Gamma(1-l)}{\Gamma(2-l-2m)}\cot\pi l\int_{u\cdot\ell<1}\tilde{f}(\ell_+, l)\,(u\cdot\ell)^{1-l-2m}\,d\ell$$

$$+\frac{(-1)^{m+1}}{2i(2\pi)^{2m}}\int_C dl\,\frac{\Gamma(l+2m)}{\Gamma(l+1)}\cot\pi l\int_{u\cdot\ell>1}\tilde{f}(\ell_-, l)\,(u\cdot\ell)^{l}\,d\ell$$

$$+\frac{(-1)^m}{(2\pi)^{2m}}\sum_r\frac{(r+2m-2)!}{(r-1)!}\int_{u\cdot\ell>1}\tilde{f}(\ell_+, r)\,(u\cdot\ell)^{1-r-2m}\,d\ell$$

$$+\frac{(-1)^{m+1}}{(2\pi)^{2m}}\sum_r\frac{(r+2m-1)!}{r!}\int_{u\cdot\ell<1}\tilde{f}(\ell_-, r)\,(u\cdot\ell)^{r}\,d\ell$$

*where the summations run over all positive (or zero) integers* $r$ *to the right of the contour* $C$.

We remark that the measures appearing in the $l$-integrations here are classical, as given for example by Vilenkin [8]. Our numerical factors differ because we have not normalised to unity the measure of $\mathscr{K}$.

Let us now discuss these results briefly. First notice the difference between the even and the odd-dimensional cases: the former have a discrete sum appearing in the inversion formula, whereas the latter do not. This is very reminiscent of the appearance of a discrete series of representations in the Fourier inversion formula for the regular representation of $SO(2, 1)$, but not for $SO(3, 1)$; but the precise relationship is unclear. We know that the discrete series cannot be realised upon $H$ because of covariance constraints, while the polynomial representations $Q$ are annihilated by the Gel'fand-Graev transform. They may perhaps correspond to the second set of finite-dimensional representations, which are realised not on a polynomial subspace of $P$ but rather upon a factor space (see, e.g., Ref. [6], Chapter VII, Section 4.3; Chapter III, Section 5.3), which we have not investigated.

Now consider the asymptotics of the inversion formula. Since $u\cdot\ell$ appears with exponent $(1-l-n)$ if it is less than unity, and with exponent $l$ if it is greater, it is clear that we can indeed move our contour of $l$-integration to the left, and eventually ignore its contribution compared with those of the singularities it has crossed. This is a consequence of our taking a Laplace transform in (26) instead of the usual Fourier transform (25) (which in general is of course divergent); it is this highly desirable feature which is absent from the Legendre transform approach which is discussed in the introduction. Part of the price paid for this is the existence of a *pair* of functions instead of just one; but this is true too for the classical Laplace transform.

It is possible to derive a much simpler inversion formula by taking the lower limit of the $\alpha$-integration in (36) and (43) down to $-\infty$: the properties of the Laplace transform will ensure that the contribution of the extra range is annihilated by the $l$-integral. If we do this, we recover exactly the classical inversion formulas for each of the $\tilde{f}(k_{\pm}, l)$: in Theorem 2, the limits on the $k$-integrals are removed, and the discrete sums disappear. While the results are of course equivalent, the attractive asymptotic behaviour of our formulas is completely obscured.

The cross-section $\mathcal{K}$ of $K$ apparently plays an important role in Theorem 2. This is illusory: *any* contour on the cone will do, provided only that the relevant integrals converge. In practice, unless $f$ is integrable this will restrict $\mathcal{K}$ to be a closed contour; and if we make the usual requirement that the coordinate system allow separation of the Laplace operator, this in turn restricts our coordinates to be either spherical or of ellipsoidal type. Such an ellipsoidal system has been briefly discussed for $SO(2, 1)$ in Appendix $A$ of Ref. [15]; for $SO(3, 1)$ see Olevskii [16]. Therefore the number of useful new special-function results derivable from Theorem 2 is limited.

Finally, consider the measures of the $l$-integrations. Comparing them with the results of Lemma 1, we see that their zeroes lie exactly at the positions corresponding to functions with such asymptotic behaviour that their Gel'fand-Graev transforms do not exist (i.e., the Laplace transform (29) diverges). The poles possessed by the measure in the even-dimensional case correspond to those *extra* functions that we were forced to exclude from Theorem 1 because the inversion formula did not converge. We are now able to make use of the analyticity in $l$ to remove these extra restrictions.

**Lemma 4.** *If $n = 2m$ is even, the Laplace inversion formula in Theorem 2 converges to $f'(u)$ even for functions with asymptotic behaviours $|u|^r(\ln|u|)^s$ with integer $r$, which were excluded from the Gel'fand-Graev inversion formula in Theorem 1*[1].

## IV.1. Relation to the Legendre Transform. The Group $O(2, 1)$

Having derived a general theory of Laplace transforms, we shall now show how the Legendre transforms fit into our scheme. We shall in this section consider the two-dimensional hyperboloid, which has the group of motions $SO(2, 1)$; for it is this which has been most extensively investigated.

To clarify the relationship of the transforms we shall need some theorems of the Plancherel or inner-product type. Suppose that $f$ and $g$ are integrable over $H$. Let

$$(f, g)_H = \int \overline{f(u)}\, g(u)\, du\,.$$

Then we know that [8] also

$$(f, g)_H = (\tilde{f}, \tilde{g}) \\ \equiv \frac{1}{8\pi^2 i} \int_C dl.\, l \cot \pi l \int \overline{\tilde{f}(k, -\bar{l}-1)}\, \tilde{g}(k, l)\, dk\,. \tag{45}$$

On the cone, however, we can only derive an inner product which has a two-point measure. By inserting (25) (with $n=2$) into (45) we obtain in the integrable situation

$$(f, g)_H = (\hat{f}, \hat{g})_K \\ \equiv \frac{-1}{16\pi^2} \int \overline{\hat{f}(k e^{\alpha})}\, \hat{g}(k e^{\beta}) \operatorname{cosech}^2\left(\frac{\beta-\alpha}{2}\right) e^{\alpha}\, d\alpha\, d\beta\, dk\,. \tag{46}$$

Similar theorems (subject to the provisos of Theorem 1) can be obtained when $fg$ is integrable while $f$ is not. We shall not need them here.

We are now ready to study the Legendre transform. Choose a spherical coordinate system and for simplicity suppose that the dependence of $f$ and $g$ upon the azimuthal angle $\phi$ is trivial. We can therefore write $f(u) = f(ch\zeta)$ and $\hat{f}(k) = \hat{f}(\alpha)$. Let the cross-section $\mathscr{K}$ of $K$ be $x_0 = 1$; then for $\tilde{f}(\not{k}, l)$ we shall write $\tilde{f}(l)$.

Now consider the Fourier projection formula (3a); we ask what this becomes on the cone. It is simple to verify using (5) that $P_l(ch\zeta)$ is transformed into

$$[\Gamma : P_l(ch\zeta)]\,(\alpha) = \frac{\tan \pi l}{(2l+1)} \{e^{\alpha l} + e^{-\alpha(l+1)}\}\,, \tag{47}$$

and indeed we can show by (8) that this is true for all values of $l$ (we exclude the case $l = n + \frac{1}{2}$, when the transform diverges). Now in (46) set $\hat{g}(\not{k} e^{\beta}) = e^{l\beta}$; the equation becomes

$$(\hat{f}, \hat{g})_K = \frac{-1}{8\pi} \int \overline{\hat{f}(\alpha)} e^{\alpha}\, d\alpha \int_{-\infty}^{\infty} e^{l\beta} \operatorname{cosech}^2\left(\frac{\beta-\alpha}{2}\right) d\beta\,. \tag{48}$$

The $\beta$-integral can be performed (in the sense of the regularisation) and yields

$$-4\pi l \cdot \cot \pi l \cdot e^{l\alpha} \tag{49}$$

so that from (46)–(49) we find

$$\int \overline{f(ch\zeta)}\, P_l(ch\zeta)\, d(ch\zeta) \\ = \frac{1}{2(2l+1)} \int \overline{\hat{f}(\alpha)}\, [l e^{l\alpha} + (l+1) e^{-\alpha(l+1)}]\, e^{\alpha}\, d\alpha\,. \tag{50}$$

Therefore the result (49) has the very important consequence that the projection formula on the cone, (50), is of simple Fourier form, so that the usual projection formula (3a) does indeed have the significance we ascribed to it. (We do not enter here into a discussion of the simultaneous appearance of terms in $l$ and $-l-1$; the equivalence of these representations is a consequence of the Paley-Wiener theorem for the Gel'fand-Graev transform, which has its roots in the fact that upon $H$ the hyperbolic angle $\zeta$ takes only positive values.)

Now turn to the Legendre transform (2a). We find that

$$[\Gamma : Q_l(ch\zeta)]\,(\alpha) = \begin{cases} \dfrac{2\pi}{(2l+1)}\, e^{l\alpha} & \alpha < 0 \\[2ex] \dfrac{2\pi}{(2l+1)}\, e^{-\alpha(l+1)} & \alpha > 0\,. \end{cases} \tag{51}$$

At first sight then we might expect (2a) to be just the sum of our two Laplace transforms $\tilde{f}(l_+)$ and $\tilde{f}(l_-)$; but this is not so. For our Laplace transforms are *exactly* the classical Laplace transforms on the cone, whereas (51) has to be used in conjunction with a two-point measure; and unlike the Fourier transform just treated, we no longer obtain an integral over the cone whose integrand is essentially the product of $\hat{f}(\alpha)$ and (51).

The principal distinction between the Laplace and Legendre transforms therefore lies in the two-point measure on the cone[3] given by (46). Our "Laplace" transform takes a genuine Laplace transform on the cone, which corresponds to projection with some more complicated function on the hyperboloid; the Legendre transform chooses a simple function on $H$, but this corresponds to a more complicated one on $K$. The advantage of our approach is that it enables us to derive new formulas with "good" asymptotics; its disadvantage is its non-convergence at "integer-points" and the existence of the kernel subspace $Q$.

## IV.2. The Groups $SO(n, 1)$

The existence of the two-dimensional Legendre transform discussed above rests on a cancellation of singularities. Near $\zeta = 0$, the projection function behaves like

$$Q_l^m(ch\zeta) \sim (sh\zeta/2)^{-|m|}$$

so that in general it is neither continuous nor locally integrable on the hyperboloid. (On the cone this manifests itself by a discontinuity in the analogue of (51) at $\alpha = 0$.) By virtue of the inversion theorem (2b),

[3] For $f \notin L^1(u)$ the distinction between the Laplace and Legendre transforms is much greater; for the latter may converge (or be non-zero) where the former diverges (or vanishes).

however, the covariant part $f_m$ of $f(u)$ itself behaves near this point like

$$P_l^m(ch\zeta) \sim (sh\zeta)^{+|m|} \tag{52}$$

so that the product of $f_m(u)$ and $Q_l^m(ch\zeta)$ is indeed both continuous and locally integrable, and the transform (2a) exists.

Similar cancellations occur for the quasi-regular representations of the higher groups $SO(n, 1)$; but not for their regular representations [4]. Only the group $SO(2, 1)$ has a Legendre transform similar to (2a) over the group itself: for $SO(3, 1)$ and higher groups we can use Eqs. (3) but not Eqs. (2).

Finally, we mention another related approach to the definition of a Laplace transform on $SO(2, 1)$. In a recent paper (ICTP preprent IC/72/41) Cronström has acheived a truly Laplacian asymptotic behavious by retaining the projection formula (2a) but altering the inversion formula (2b) so that the kernel functions are no longer just the $P_l(ch\zeta)$, but are instead given by

$$\begin{aligned} R_l(ch\zeta) = & -\frac{1}{\pi}\tan\pi l\, Q_{-l-1}(ch\zeta) \\ & -\frac{1}{\pi^2}\sum_{n=-\infty}^{\infty}(l+n+\tfrac{1}{2})^{-1}\, Q_{n-\frac{1}{2}}(ch\zeta)\,. \end{aligned}$$

These functions arise by considering an integral representation of the Legendre function, but with the "wrong limits", and are closely related to those which arise from our theory when we specialise to a spherical coordinate system. We shall consider them further in a subsequent paper, when we shall show how our formalism can be extended to all the Lorentz groups.

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