Commun. math. Phys. 26, 271—275 (1972)

Einstein Algebras

ROBERT GEROCH
The Enrico Fermi Institute, Chicago, Illinois, USA

Received January 31, 1972

Abstract. An approach to quantization of general relatively using a reformulation of the classical theory in which the events of space-time play essentially no role is discussed.

1. Introduction

An event in physical space-time becomes, when idealized in general relativity, a point of some four-dimensional manifold. Such an idealization is natural in classical physics because, in this case, it is possible in principle to make arbitrarily precise observations whose effects on the system observed can be made arbitrarily small. The fundamental role of the events in the mathematical formalism of general relativity reflects this potential for precise measurement. In quantum theory, on the other hand, the influence of the measuring apparatus on the system being observed cannot, even in principle, be made arbitrarily small. Furthermore, in a quantum theory of the gravitational field, one would expect the metric itself to be subject to quantum fluctuations. But the metric is the primary tool for isolating individual events [1]. Thus, it is perhaps reasonable to expect that, in a quantum theory of gravitation, the mathematical formalism will, at some point, suggest a "smearing out of events".

In the various approaches to quantization of the gravitational field [2–7], one retains, at least in some form, the events of space-time. It is presumably intended that these events will lose their significance in the final theory. Although it might seem more natural to adopt an approach in which events play a secondary role from the beginning, this turns out to be difficult, for the set of events, i.e., the underlying manifold, is central in the standard treatments of classical general relativity.

The purpose of this paper is to point out that, by a judicious choice of definitions, the entire content of general relativity can be so formulated that the underlying manifold plays practically no role. This version of the theory may offer a convenient starting point for quantization. The smearing out of events, which is to be expected in the quantum theory, is already reflected (by the virtual disappearance of events) in our formulation of the classical theory.

In Section 2, we write down a set of definitions of tensor fields, the metric, the Ricci tensor, etc., on a smooth manifold M. We then observe, in Section 3, that the original manifold M is used in these definitions at just one minor point. This observation is made more explicit by the introduction of what we call an Einstein algebra.

2. Differential Geometry

In this section, we recall some facts from differential geometry [8—10].

Let M be a smooth (C^∞) manifold, and denote by $\mathscr{I}$ the collection of all smooth (real-valued) functions on M. Since (pointwise) sums and products of elements of $\mathscr{I}$ are again elements of $\mathscr{I}$, this $\mathscr{I}$ has the structure of a ring. Let $\mathscr{R}$ denote the subring of $\mathscr{I}$ consisting of the constant functions, so $\mathscr{R}$ is isomorphic with the real numbers.

A *derivation* on ($\mathscr{I}$, $\mathscr{R}$) consists of a mapping $\xi:\mathscr{I}\to\mathscr{I}$ with the following properties: i) $\xi(f+g)=\xi(f)+\xi(g)$, ii) $\xi(fg)=f\xi(g)+\xi(f)g$, and iii) if $f\in\mathscr{R}$, then $\xi(f)=0$. The collection $\mathscr{D}$ of derivations is precisely the collection of smooth contravariant vector fields on the manifold M. If ξ and η are derivations, and $g\in\mathscr{I}$, we define a new derivation, $(\xi+g\eta)$, by $(\xi+g\eta)\,(f)=\xi(f)+g\eta(f)$. Hence, $\mathscr{D}$ is a module over the ring $\mathscr{I}$. Furthermore, if ξ and η are derivations, then $(\mathscr{L}_\xi\eta)(f)=\xi(\eta(f))-\eta(\xi(f))$ defines a new derivation $\mathscr{L}_\xi\eta$, the Lie bracket of ξ and η.

Denote the dual module of $\mathscr{D}$ by $\mathscr{D}^*$. Thus, an element μ of $\mathscr{D}^*$ associates, with each $\xi\in\mathscr{D}$, an element, $\mu(\xi)$, of $\mathscr{I}$, where this action is linear: $\mu(\xi+g\eta)=\mu(\xi)+g\mu(\eta)$. The module $\mathscr{D}^*$ is precisely the collection of smooth covariant vector fields on the manifold M.

A *metric* on M consists of an isomorphism g from the module $\mathscr{D}$ to the module $\mathscr{D}^*$ which is symmetric, i.e., which satisfies $g(\xi,\eta)=g(\eta,\xi)$ for all $\xi,\eta\in\mathscr{D}$, where we have set $g(\xi,\eta)=[g(\xi)]\,(\eta)$.

A *tensor field* of rank n is a multilinear mapping $\alpha:\mathscr{D}\times\cdots\times\mathscr{D}\to\mathscr{I}$ (n factors). (These are the covariant tensor fields. Since g gives an isomorphism between $\mathscr{D}$ and $\mathscr{D}^*$, there is no need to consider contravariant tensor fields.) For example, a metric defines a tensor field of rank two. The tensor field of rank n clearly form a module over $\mathscr{I}$. Furthermore, if α and β are tensor fields of rank n and m, respectively, then $\alpha(\xi,\ldots,\tau)\cdot\beta(\lambda,\ldots,\eta)$ is multilinear in $\xi,\ldots,\eta$, and so represents a tensor field, $\alpha\times\beta$, of rank $(n+m)$. This is the outer product operation on tensor fields.

We next define the (covariant) derivative. Let $\xi \in \mathscr{D}$, $\mu \in \mathscr{D}^*$. Then the right side of

$$[\nabla_\xi \mu](\eta) = \tfrac{1}{2}\{[g^{-1}(\mu)](g(\xi,\eta)) + \xi(\mu(\eta)) - \eta(\mu(\xi)) + g(\xi, \mathscr{L}_\eta g^{-1}(\mu)) + g(\eta, \mathscr{L}_\xi g^{-1}(\mu)) - \mu(\mathscr{L}_\xi \eta)\} \tag{1}$$

is linear in $\eta \in \mathscr{D}$. Hence, Eq. (1) defines a tensor field, $\nabla_\xi \mu$, of rank one. More generally, if α is a tensor field of rank n, we define $\nabla_\xi \alpha$ by

$$[\nabla_\xi \alpha](\eta, \ldots, \tau) = \xi[\alpha(\eta, \ldots, \tau)] - \alpha(g^{-1}(\nabla_\xi g(\eta)), \ldots, \tau) - \cdots - \alpha(\eta, \ldots, g^{-1}(\nabla_\xi g(\tau))). \tag{2}$$

It follows immediately from the definition that the derivative satisfies all the usual rules: i) additivity ($\nabla_\xi(\alpha+\gamma) = \nabla_\xi \alpha + \nabla_\xi \gamma$), ii) Liebnitz rule ($\nabla_\xi(\alpha \times \beta) = \alpha \times \nabla_\xi \beta + (\nabla_\xi \alpha) \times \beta$), iii) linearity in $\xi(\nabla_{(\xi+g\eta)}\alpha = \nabla_\xi \alpha + g\nabla_\eta \alpha)$, and iv) the vanishing of the derivative of the metric tensor field.

To define the Riemann tensor, note that, since the right side of

$$R(\xi, \eta, \tau, \lambda) = [\nabla_\xi \nabla_\eta g(\lambda) - \nabla_\eta \nabla_\xi g(\lambda)](\tau) - [\nabla_{\mathscr{L}_\xi \eta} g(\lambda)](\tau) \tag{3}$$

is linear in $\xi, \eta, \tau, \lambda \in \mathscr{D}$, this equation defines a tensor field $R(\xi, \eta, \tau, \lambda)$. The algebraic identities on the Riemann tensor ($R(\xi,\eta,\tau,\lambda) = -R(\eta,\xi,\tau,\lambda) = -R(\xi,\eta,\lambda,\tau) = R(\tau,\lambda,\xi,\eta), R(\xi,\eta,\tau,\lambda) + R(\eta,\tau,\xi,\lambda) + R(\tau,\xi,\eta,\lambda) = 0$), as well as the Bianchi identity, follow directly from (1), (2), and (3).

Finally, in order to obtain the Ricci tensor from the Riemann tensor, we must introduce the contraction operation on tensor fields. It suffices to define contraction on tensor fields of rank two. The contraction, $C(\alpha)$, of an α of rank two is to be an element of $\mathscr{I}$, where C has the properties: i) linearity ($C(\alpha + f\beta) = C(\alpha) + fC(\beta)$), and ii) outer product rule (If $\mu, \nu \in \mathscr{D}^*$, then $C(\mu \times \nu) = \mu(g^{-1}(\nu))$.) Since our tensor fields are on a manifold, we have

Contraction Property. There is precisely one contraction operation C satisfying i) and ii) above.

Thus, fixing ξ and τ, $S(\eta, \lambda) = R(\xi, \eta, \tau, \lambda)$ is bilinear in η, λ. Then $C(S)$ is bilinear in ξ, τ, and so defines the (rank two) Ricci tensor.

We now have the machinery for writing any equation, and for carrying out any calculation, involving smooth tensor fields on the manifold M.

3. Einsteins Algebras

In the previous section, we stated a sequence of definitions, leading from a smooth manifold M to a metric on M and finally to all the tensor operations on M. We now make the following observation: the manifold

M entered the discussion of Section 2 at only one point: in the construction of the ring $\mathscr{I}$ of smooth functions on M and the subring $\mathscr{R}$ of the constant functions. All constructions and definitions thereafter were purely algebraic ones on $(\mathscr{I}, \mathscr{R})$.

We are thus led to the following definition. An *Einstein algebra* consists of i) a commutative ring $\mathscr{I}$, ii) a subring $\mathscr{R}$ of $\mathscr{I}$, isomorphic with the real numbers (and such that the identity 1 of $\mathscr{R}$ is the identity of $\mathscr{I}$), and iii) a metric g, such that the contraction property is satisfied and the Ricci tensor vanishes. More generally, one could introduce Einstein algebras with sources (e.g., electromagnetic fields, fluids, etc.) by introducing smooth tensor fields to represent the sources, and suitably modifying Einstein's equation. Thus, an Einstein algebra is a purely algebraic object, making no direct reference to a manifold. Of course, every space-time which is a solution of Einstein's equation defines an Einstein algebra. One could just as well take, as the underlying mathemathical framework of general relativity, an Einstein algebra rather than the usual smooth manifold with smooth metric tensor field.

In fact, one could even regard Einstein algebras as representing a theory of gravitation, a theory which includes general relativity as a special case. Such a point of view would be difficult to maintain, however, unless it can be demonstrated that this "theory of gravitation" is capable of making experimental predictions. It appears that such predictions can indeed be made: it is only necessary to formulate the prediction in terms of tensorial properties of smooth tensor fields. For example, to describe the perihelion shift of Mercury, one would require that the Einstein algebra have a smooth fluid stress-energy (representing that of the Sun), and that the solution have the usual properties (spherical symmetry, et.) associated with the solar system. The planet Mercury would be described by a second smooth stress-energy field, which is conserved, but which is not inserted into Einstein's equation. The conservation equation would guarantee "geodesic motion". The perihelion shift would then be interpreted in terms of the behavior of the stress-energy field of Mercury relative to the Killing fields.

References

1. Geroch, R.: Ann. Phys. **48**, 526 (1968).
2. Bergmann, P.: Helv. Phys. Acta Suppl. **4**, 79 (1956).
3. — Komar, A.: In: Recent developments in general relatively. New York: Pergamon Press 1962.
4. DeWitt, B.: Phys. Rev. **160**, 1113 (1967); **162**, 1195 (1967); **162**, 1239 (1967).
5. Kundt, W.: In: Springer Tracts in Modern Physics, Vol. **40**. Berlin-Heidelberg-New York: Springer 1966.

6. Wheeler, J. A.: In: DeWitt, C., Wheeler, J. A. (Eds.); Battelle recontres. New York: Benjamin 1968.
7. Arnowitt, R., Deser, S., Misner, C. W.: In: Witten, L. (Ed.); Gravitation. New York: Wiley and Sons 1962.
8. Nomizu, K.: Lie groups and differential geometry. Math. Soc. of Japan, Tokyo, 1956.
9. Hicks, N. J.: Notes on differential geometry. Princeton: D. Van Nostrand 1965.
10. Helgason, S.: Differential geometry and symmetric spaces. New York: Academic Press 1962.

R. Geroch
The Enrico Fermi Institute
933 E. 56th Street
Chicago, Ill. 60637, USA