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Gravitational Fields with Groups of Motions on Two-dimensional Transitivity Hypersurfaces in a Model with Matter and a Magnetic Field

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Abstract. For gravitational fields with metrics which admit of groups of motions multiply – transitive on 2-dimensional space-like invariant varieties, the exact solutions of the Einstein gravitational equations are given for the case when the sources of the gravitational field are dust-like matter and a magnetic field. A magnetic field is orientated along a direction orthogonal to transitivity hypersurface. The solutions contain arbitrary functions. In the case of transitivity hypersurface of positive curvature and in the absence of a magnetic field, the solution is reduced to the Tolman spherically symmetric solution for dust-like matter. The conditions are studied under which the solutions with a magnetic field become asymptotically isotropic and approach the flat and the open Friedmann models. The case of transitivity hypersurfaces with signature $(+ -)$ is also considered.

1. Introduction

The paper deals with exact solutions of Einstein equations in General Relativity for metrics which admit of multiply transitive groups of motions on 2-dimensional transitivity hypersurfaces V_2. Exact solutions are given for the case when the sources of gravitational field are dust (incoherent matter) and a magnetic field, the direction of which is orthogonal to V_2. These solutions contain arbitrary functions. The cases considered are those in which the signature of V_2 is $(+ +)$ and $(+ -)$ [1, 2].

In the case of space-like V_2 with positive curvature and in the absence of a magnetic field these solutions are reduced to well known Tolman-Bondi solutions for spherically symmetric gravitational fields [3, 4].

A study of gravitational fields, the sources of which are matter and a magnetic field, is important in the theory of anisotropic cosmological models with a primordial magnetic field [5–10] and also is of interest for the problem of gravitational collapse in a magnetic field. Considered solutions include a homogeneous anisotropic model with euclidian co-moving space and also contain a class of inhomogeneous solutions which asymptotically become isotropic and approach the Friedmann solutions in the flat and in the open models.

The general case of gravitational fields, whose metrics admit of 3-parameter groups of motions G_3 on V_2, is considered. Lie algebras A_3 of these groups allow extensions [11]. On the one hand, there exists a central extension of A_3 to A_4 with a space-like additional Killing vector. It leads to a group G_4 which acts on a hypersurface V_3, which is the direct topological product of V_2 and a line of a direction of a magnetic field. Such metrics have been considered previously [12, 13, 5–9]. They are consistent with the presence of a magnetic field. On the other hand, there exists noncentral extension of A_3 to A_4 which leads to a homogeneous anisotropic model with a co-moving 3-space of constant negative curvature. Also, there exist non-central extensions of A_3 to A_6, which lead to the Friedmann models. The non-central extensions, however, prove to be impossible in the case when a directed magnetic field serves as a source of gravitational field.

2. Metrics with Groups of Motions on V_2

Metrics which admit of 3-parameter groups of motions G_3, which act on space-like transitivity hypersurfaces V_2 with coordinates x^2, x^3, can be written in a synchronous system with geodesic lines of time $\tau(x^0 = c\tau)$ in the form [1, 2, 15]

$$ds^2 = (c\,d\tau)^2 - dl^2\,, \tag{2.1}$$

$$dl^2 = X^2(x^1, \tau)\,(dx^1)^2 + Y^2(x^1, \tau)\,[(dx^2)^2 + f^2(x^2)\,(dx^3)^2]\,. \tag{2.1a}$$

V_2 is a 2-space of constant curvature, which may be zero, negative and positive, and the values in (2.1a) respectively are

$$f(x^2) = 1\,, \tag{2.2a}$$

$$f(x^2) = \sinh x^2\,, \tag{2.2b}$$

$$f(x^2) = \sin x^2\,. \tag{2.2c}$$

Generators X_a of a group of motions G_3 are defined by Killing vectors $\xi^i_{(a)}$ by $X_a = \xi^i_{(a)}\partial/\partial x^i$ (Latin indices $i, k, \ldots$ run from 0 to 3). For these operators the commutation relations $[X_a, X_b] = c^c_{ab} X_c$ are valid with structure constants which may be written in the form [14] (Latin indices a, b, c run from 1 to 3; e_{abc} is a skew pseudo-tensor, $e_{123} = 1$)

$$c^c_{ab} = e_{abd} n^{dc} + \delta^c_b a_a - \delta^c_a a_b, \qquad n^{ab} a_b = 0, \qquad n^{ab} = n^{ba}\,. \tag{2.3}$$

For (2.2a) the group G_3 is of type VII_0 according to Bianchi-Behr classification [1, 2, 14, 15] with the values in (2.3)

$$a_a = 0, \qquad n^{ab} = \delta^a_2 \delta^b_2 + \delta^a_3 \delta^b_3\,, \tag{2.4}$$

and with Killing vectors

$$\xi^i_{(1)} = x^3 \delta^i_2 - x^2 \delta^i_3, \quad \xi^i_{(2)} = \delta^i_2, \quad \xi^i_{(3)} = \delta^i_3, \quad X_a = \xi^i_{(a)} \partial/\partial x^i . \tag{2.4a}$$

For (2.2b) the group G_3 is of type VIII [2] with

$$a_a = 0, \quad n^{ab} = \delta^a_1 \delta^b_1 + \delta^a_2 \delta^b_2 - \delta^a_3 \delta^b_3 , \tag{2.5}$$

and Killing vectors are

$$\xi^i_{(1)} = \cos x^3 \delta^i_2 - \sin x^3 \coth x^2 \delta^i_3, \quad \xi^i_{(2)} = \partial \xi^i_{(1)}/\partial x^3, \quad \xi^i_{(3)} = \delta^i_3 . \tag{2.5a}$$

For (2.2c) the group G_3 belongs to type IX [2] with

$$a_a = 0, \quad n^{ab} = \delta^a_1 \delta^b_1 + \delta^a_2 \delta^b_2 + \delta^a_3 \delta^b_3 , \tag{2.6}$$

and Killing vectors are

$$\xi^i_{(1)} = \cos x^3 \delta^i_2 - \sin x^3 \cot x^2 \delta^i_3, \quad \xi^i_{(2)} = \partial \xi^i_{(1)}/\partial x^3, \quad \xi^i_{(3)} = \delta^i_3 . \tag{2.6a}$$

Lie algebras A_3 of generators of the groups G_3 (2.4), (2.5), (2.6) may be extended [11, 16, 17]. *Central* extension is carried out by addition of the generator X_4 which commutes with the generators of A_3, whereas for *non-central* extensions the generators, which are added, do not commute with the generators of A_3. A central extension is given by addition a Killing vector (with $\xi^2_{(4)} = \xi^3_{(4)} = 0$), space-time character of which may be arbitrary [11]. A time-like additional Killing vector would lead to static solutions which contain in particular the Reissner-Nordstrom metric. We confine ourselves to a central extension given by a space-like additional Killing vector

$$\xi^i_{(4)} = \delta^i_1, \quad X_4 = \partial/\partial x^1 , \tag{2.7}$$

which leads in (2.1a) to

$$X = b(\tau), \quad Y = a(\tau) \tag{2.8}$$

for all possible values of $f(x^2)$ (2.2a), (2.2b), (2.2c) [5–7, 9, 12, 13, 18].

For $f(x^2) = 1$ this central extension leads to the group G_4 which in virtue of (2.4a) and (2.7) contains the abelian subgroup G_3 of X_2, X_3, X_4. This subgroup G_3 is simply transitive on 3-dimensional transitivity hypersurface x^1, x^2, x^3 and belongs to Bianchi type I with $a^a = 0$, $n^{ab} = 0$ in (2.3). Such model is of interest in anisotropic cosmology [1, 19]. Lie algebra A_4 with (2.8) for $f(x^2) = 1$ is extended to A_6 by addition the Killing vectors

$$\xi^i_{(5)} = x^3 \delta^i_1 - x^1 \delta^i_3, \quad \xi^i_{(6)} = x^1 \delta^i_2 - x^2 \delta^i_1 .$$

It leads in (2.1a) to $X = Y = a(\tau)$ and yields the flat Friedmann model.

For $f(x^2) = \sinh x^2$ the central extension of (2.5a) by (2.7), (2.8) leads to the group G_4 which contains subgroup G_3 (with $2X_1, (X_2 + X_3 + X_4)/2$, $(X_2 + X_3 - X_4)/2$), which is simply transitive on 3-dimensional space

x^1, x^2, x^3 and is of Bianchi type III with $a^a = -\delta^a_1$, $n^{ab} = \delta^a_2\delta^b_2 - \delta^a_3\delta^b_3$ in (2.3) [20].

In both cases (2.2b) and (2.2c) Lie algebras A_4 of the groups G_4 with (2.8) do not allow a further extension to algebras A_6, so (2.1a), (2.8) with (2.2b) and (2.2c) can not be reduced by a further specialization to the metrics of isotropic Friedmann models.

Consider non-central extensions of A_3. For $f(x^2)=1$ there exists the non-central extension of algebra A_3 (2.4a) to A_4 which is given by addition the Killing vector

$$\xi^i_{(4)} = \delta^i_1 + x^2\delta^i_2 + x^3\delta^i_3, \qquad X_4 = \xi^i_{(4)}\partial/\partial x^i . \tag{2.9}$$

It leads to the group G_4 [21] of the type G_4V according to Petrov [2], which contains the subgroup G_3 of X_4, X_2, X_3. This subgroup is simply transitive on 3-space x^1, x^2, x^3 and is of Bianchi type V with $a^a = -\delta^a_1$, $n^{ab} = 0$ in (2.3). The extension (2.9) gives in (2.1a)

$$X = b(\tau), \qquad Y(x^1, \tau) = \exp(-x^1)\, a(\tau); \qquad f(x^2) = 1 . \tag{2.9a}$$

Algebra A_4 (2.4a), (2.9) is extended to A_6 by addition the Killing vectors

$$\xi^i_{(5)} = 2x^2\delta^i_1 + [(x^2)^2 - (x^3)^2 - \exp(2x^1)]\,\delta^i_2 + 2x^2x^3\delta^i_3 ,$$
$$\xi^i_{(6)} = 2x^3\delta^i_1 + 2x^2x^3\delta^i_2 + [-(x^2)^2 + (x^3)^2 - \exp(2x^1)]\,\delta^i_3 ,$$

and it leads to the open Friedmann model in (2.1a) with

$$X = a(\tau), \qquad Y = \exp(-x^1)\, a(\tau); \qquad f(x^2) = 1 . \tag{2.10}$$

For $f(x^2) = \sinh x^2$ a non-central extension of A_3 (2.5a) to A_4 is impossible. There exists the non-central extension of (2.5a) to A_6 given by addition

$$\xi^i_{(4)} = \cosh x^2\delta^i_1 - \tanh x^1 \sinh x^2\delta^i_2 ,$$
$$\xi^i_{(5)} = \sin x^3 \sinh x^2\delta^i_1 - \tanh x^1 \sin x^3 \cosh x^2\delta^i_2 - \frac{\tanh x^1 \cos x^3}{\sinh x^2}\delta^i_3 ,$$
$$\xi^i_{(6)} = \frac{\partial \xi^i_{(5)}}{\partial x^3} .$$

It leads to the open Friedmann model in (2.1a) with

$$X = a(\tau), \qquad Y = \cosh x^1\, a(\tau); \qquad f(x^2) = \sinh x^2 . \tag{2.11}$$

For $f(x^2) = \sin x^2$ there exist non-central extensions of (2.6a) to A_6 which yield the Friedmann metrics in the flat, the open and the closed models in a usual form [3] respectively with the values in (2.1a)

$$X = a(\tau), \quad Y = x^1 a(\tau), \quad \sinh x^1 a(\tau), \quad \sin x^1 a(\tau); \quad f(x^2) = \sin x^2 . \tag{2.12}$$

Formulae (2.12) are well-known while the expressions (2.10) and (2.11) for the open Friedmann model are more unusual.

Now consider Einstein's field equations (without Λ-term)

$$R_i^k - (R/2)\delta_i^k = (8\pi k/c^4)\, T_i^k \tag{2.13}$$

for the metric (2.1), (2.1a) (the notations are those of [3]). In terms of

$$\partial/\partial\tau = \dot{}\,;\ \partial/\partial x^1 = {}';\ h_1 = \dot{X}/X,\ h_2 = \dot{Y}/Y,\ \lambda_1 = X'/X,\ \lambda_2 = Y'/Y\ , \tag{2.14}$$

the non-zero components of (2.13) become

$$\begin{aligned} R_0^0 - (R/2) &= \frac{1}{c^2} h_2(h_2 + 2h_1) + \frac{1}{X^2}(-2\lambda_2' + 2\lambda_1\lambda_2 - 3\lambda_2^2) - \frac{1}{Y^2}\frac{d^2 f}{f(dx^2)^2} \\ &= \frac{8\pi k}{c^4} T_0^0\,; \end{aligned} \tag{2.15a}$$

$$R_1^1 - \frac{R}{2} = \frac{1}{c^2}(2\dot{h}_2 + 3h_2^2) - \frac{1}{X^2}\lambda_2^2 - \frac{1}{Y^2}\frac{d^2 f}{f(dx^2)^2} = \frac{8\pi k}{c^4} T_1^1\,; \tag{2.15b}$$

$$\begin{aligned} R_2^2 - (R/2) = R_3^3 - (R/2) &= \frac{1}{c^2}(\dot{h}_1 + \dot{h}_2 + h_1^2 + h_2^2 + h_1 h_2) + \frac{1}{X^2}(-\lambda_2' + \lambda_1\lambda_2 - \lambda_2^2) \\ &= \frac{8\pi k}{c^4} T_2^2\,; \end{aligned} \tag{2.15c}$$

$$R_1^0 = \frac{2}{c}[\lambda_2(h_1 - h_2) - h_2'] = \frac{8\pi k}{c^4} T_1^0; \qquad R_2^0 = R_3^0 = 0\,. \tag{2.15d}$$

The non-zero components of Rucci 3-tensor in the space (2.1a) are

$$\begin{aligned} P_1^1 &= \frac{2}{X^2}[-\lambda_2' + \lambda_2(\lambda_1 - \lambda_2)]\,, \\ P_2^2 = P_3^3 &= \frac{1}{X^2}[-\lambda_2' + \lambda_2(\lambda_1 - 2\lambda_2)] - \frac{d^2 f}{Y^2 f(dx^2)^2}\,. \end{aligned} \tag{2.16}$$

Weyl tensor for the metric (2.1), (2.1a) in a general case is of type D; in special cases (2.10), (2.11), (2.12) it is reduced to zero.
In the cases when the signature of transitivity hypersurfaces V_2 is $(+\,-)$, two forms for an interval are possible

$$\begin{aligned} ds^2 &= Y^2(x^2, x^3)\,[(dx^0)^2 - f^2(x^0)\,(dx^1)^2] \\ &\quad - X^2(x^2, x^3)\,(dx^2)^2 - (dx^3)^2\,, \end{aligned} \tag{2.17a}$$

$$\begin{aligned} ds^2 &= Y^2(x^2, x^3)\,[f^2(x^1)\,(dx^0)^2 - (dx^1)^2] \\ &\quad - X^2(x^2, x^3)\,(dx^2)^2 - (dx^3)^2 \end{aligned} \tag{2.17b}$$

with the functions (2.2a), (2.2b), (2.2c) of the respective arguments.

For $f=1$ in (2.17a), (2.17b) the group G_3 [2] is of type VI_0 of Bianchi-Behr classification with the values in (2.3) and with Killing vectors given by

$$a_a=0,\ n^{ab}=\delta_1^a\delta_1^b-\delta_2^a\delta_2^b;\ \xi^i_{(1)}=\delta^i_0,\ \xi^i_{(2)}=\delta^i_1,\ \xi^i_{(3)}=x^1\delta^i_0+x^0\delta^i_1. \quad (2.18)$$

In the case $f(y)=\sinh y$ in (2.17a), (2.17b) G_3 is of type VIII with (2.5) and Killing vectors for (2.17a) are given by

$$\xi^i_{(1)}=\cosh x^1\delta^i_0-\sinh x^1\coth x^0\delta^i_1,\quad \xi^i_{(2)}=\delta^i_1,\quad \xi^i_{(3)}=\partial\xi^i_{(1)}/\partial x^1. \quad (2.19)$$

For $f(y)=\sin y$ in (2.17a), (2.17b) G_3 is also of type VIII and for (2.17a)

$$\xi^i_{(1)}=\cosh x^1\delta^i_0-\sinh x^1\cot x^0\delta^i_1,\quad \xi^i_{(2)}=\delta^i_1,\quad \xi^i_{(3)}=\partial\xi^i_{(1)}/\partial x^1. \quad (2.20)$$

Killing vectors for (2.17b) are given by (2.19), (2.20) with the permutation of x^0 with x^1 and of ξ^0 with ξ^1.

The central extension of A_3 for (2.17a), (2.17b) is given by addition the Killing vector $\xi^i_{(4)}=\delta^i_2$. It leads in (2.17a), (2.17b) to

$$X=X(x^3),\qquad Y=Y(x^3). \quad (2.21)$$

Weyl tensor C_{iklm} for the metrics (2.17a), (2.17b) is also of type D with the equal eigenvalues of $C_{0\alpha 0\beta}$ ($C_{0\alpha\beta\gamma}=0$) in the orthonormal tetrad in x^2, x^3 directions.

3. Gravitational Fields in a Model with Dust and a Magnetic Field

We shall consider solutions of the Einstein equations under assumption that the energy-momentum tensor in (2.13) is the superposition of that of dust-like matter and that of electromagnetic field:

$$T_i^k=(T_i^k)^{\mathrm{I}}+(T_i^k)^{\mathrm{II}}. \quad (3.1)$$

The energy-momentum tensor for dust (pressure-free matter) is

$$(T_i^k)^{\mathrm{I}}=eu_iu^k,\qquad u_iu^i=1, \quad (3.2)$$

where u^i is the 4-velocity and $e=\varrho c^2$ is the energy density of matter. The invariance of the metric leads to the invariance of the 4-velocity in (2.13), (3.2) under transformations of the groups G_3 (2.4), (2.5), (2.6), so the operator $u^i\partial/\partial x^i$ must commute with the generators of the groups. It gives for (2.1), (2.1a)

$$u^i=u^0(x^1,\tau)\delta^i_0+u^1(x^1,\tau)\delta^i_1, \quad (3.3)$$

and in the case of the extensions with (2.8) and (2.9a)

$$u^0 = u^0(\tau), \qquad u^1 = u^1(\tau). \tag{3.4}$$

The conservation laws are

$$(T_i^k)_{;k} = 0. \tag{3.5}$$

We shall assume for the electromagnetic field that the Lorentz 4-force is reduced to zero i.e.

$$(T_i^k)^{\mathrm{II}}_{;k} = 0. \tag{3.5a}$$

For dust Eqs. (3.5) in virtue of (3.5a) give the continuity equation

$$u^i (T_i^k)^{\mathrm{I}}_{;k} = (e u^k)_{;k} = 0, \tag{3.6}$$

and the equations of momentum

$$(T_i^k)^{\mathrm{I}}_{;k} = e u^k (u_i)_{;k} = 0, \tag{3.7}$$

which mean that the world lines of dust are geodesic.

Eq. (3.7) together with $u^i u_{i;k} = 0$ lead to the equation for vorticity

$$u^i \omega_{ik} = 0, \qquad \omega_{ik} = u_{i;k} - u_{k;i},$$

which is reduced under condition (3.3) to

$$\omega_{ik} = 0, \qquad u_i = \partial \varphi(x^0, x^1)/\partial x^i.$$

Thus, in the case under consideration a motion of matter is irrotational. On account of this the system of coordinates (2.1), (2.1a) can be transformed into the synchronous and comoving system by means of the transformation

$$(x^0)' = \varphi(x^0, x^1), \qquad u^m \partial (x^\alpha)'/\partial x^m = 0, \qquad \alpha = 1, 2, 3.$$

In such a system the geodesic lines τ are world lines of matter.

In the further consideration with groups G_3 we imply that the system (2.1), (2.1a) is co-moving, so in (3.3), (3.2)

$$u^0 = u_0 = 1, \qquad u^1 = 0; \qquad (T_i^k)^{\mathrm{I}} = e \delta_i^0 \delta_0^k. \tag{3.3a}$$

Electromagnetic field can be described in terms of space-like 4-vectors h^i and e^i which are defined through the electromagnetic field tensor F_{ik} and the 4-velocity u^i by [22, 23]

$$\begin{gathered} h_k = \tfrac{1}{2} E_{iklm} u^i F^{lm}, \qquad e_k = u^i F_{ik}, \qquad h^i u_i = e^i u_i = 0, \\ h^i h_i = -|h|^2, \qquad e^i e_i = -|e|^2; \qquad F_{ik} = u_i e_k - u_k e_i - E_{iklm} u^l h^m, \end{gathered} \tag{3.8}$$

where $E_{iklm} = e_{iklm}(-g)^{1/2}$, e_{iklm} is a skew pseudo-tensor with $e^{0123} = 1$. The energy-momentum tensor of electromagnetic field is written in terms of h^i, e^i and u^i by

$$(T_i^k)^{\text{II}} = \frac{1}{4\pi}\left[\left(u_i u^k - \frac{1}{2}\delta_i^k\right)(|h|^2 + |e|^2) - h_i h^k - e_i e^k - u_i v^k - u^k v_i\right], \tag{3.8a}$$

$$v_i = E_{iklm} e^k h^l u^m .$$

Maxwell equations with aid of (3.8) are written

$$(u^i h^k - u^k h^i + E^{iklm} u_l e_m)_{;k} = 0\,, \tag{3.9}$$

$$F^{ik}_{;k} \equiv (u^i e^k - u^k e^i - E^{iklm} u_l h_m)_{;k} = -4\pi j^i/c\,. \tag{3.10}$$

The invariance of the metric (2.1), (2.1a) under the groups of motions G_3 leads to the invariance of the vectors h^i and e^i in (3.8a) under these groups. It yields similarly to (3.3)

$$h^i = h^0(x^1, \tau)\delta_0^i + h^1(x^1, \tau)\delta_1^i, \qquad e^i = e^0(x^1, \tau)\delta_0^i + e^1(x^1, \tau)\delta_1^i\,.$$

In the case of the co-moving system (2.1), (2.1a) with (3.3a) the orthogonality conditions (3.8) lead to

$$h^i = h^1(x^1, \tau)\delta_1^i, \qquad e^i = e^1(x^1, \tau)\delta_1^i\,. \tag{3.11}$$

Consideration of F_{ik} with (3.11) in an orthonormal frame yields

$$|h|^2 = X^2(h^1)^2 = H^2, \qquad |e|^2 = X^2(e^1)^2 = E^2\,.$$

It shows that in the co-moving system (2.1), (2.1a) there exist the collinear magnetic and electric fields, which are directed along x^1 and have intensities H and E.

Maxwell equations (3.10) show that the 4-current j^i is equal to zero in accordance with (3.5a) whereas (3.9) for (3.11) gives

$$\begin{gathered} \frac{h_1}{X} = H = \frac{K_1}{Y^2}, \qquad \frac{e_1}{X} = E = \frac{K_2}{Y^2}, \qquad K_1 = \text{const}\,, \\ K_2 = \text{const}; \qquad e^i = \frac{K_2}{K_1} h^i\,. \end{gathered} \tag{3.12}$$

Non-zero components of (3.8a) in virtue of (3.12) are given by

$$(T_0^0)^{\text{II}} = (T_1^1)^{\text{II}} = -(T_2^2)^{\text{II}} = -(T_3^3)^{\text{II}} = W = \frac{H^2 + E^2}{8\pi} = \frac{K^2}{8\pi Y^4}, \tag{3.12a}$$

$$K^2 = K_1^2 + K_2^2 = \text{const}\,.$$

In the case of the central extension of A_3 to A_4 with (2.8) the Eq. (2.15d) yields $R_{01} = 0$. It shows due to (3.8a), (3.2) that in this case also $u^1 = 0$, so (3.3a) and (3.12), (3.12a) with $H = H(\tau)$, $E = E(\tau)$ remain valid.

For the metric (2.1), (2.1a) with (3.3a) one obtains

$$c u^i_{;k} = h_1 \delta^i_1 \delta^1_k + h_2(\delta^i_2 \delta^2_k + \delta^i_3 \delta^3_k),$$

so a motion of matter is described by expansion with the scalar $\Theta = (h_1 + 2h_2)/3$ and by shear for $h_1 \neq h_2$ while rotation is absent.

Now consider the integration of the Einstein gravitational equations (2.15), (3.1) in the co-moving synchronous system (2.1), (2.1a) for dust with (3.3a) and electromagnetic field with (3.12), (3.12a). The case when both a magnetic and an electric field are present and the case when only a magnetic field is present are distinguished in (3.12a) merely by the value of a constant K. For the sake of simplicity we shall speak only of the presence of a magnetic field, which is physically reasonable for many problems.

The continuity Eq. (3.6) gives

$$e = \Psi(x^1)/X Y^2 . \tag{3.13}$$

In the case

$$\lambda_2 = 0, \qquad Y = Y(\tau), \tag{3.14}$$

the Eq. (2.15d) (with $T_1^0 = 0$) is satisfied identically. In this case Eqs (2.15a) and (2.15c) with (3.12a) lead to the relation $X = \varphi_1(x^1)\, X(\tau)$ which can be written with the aid of possible transformations of x^1 in the form

$$X = X(\tau) . \tag{3.14a}$$

Thus, $\lambda_2 = 0$ corresponds to the group G_4 with (2.7). The result of the integration of the Einstein field equations for dust and a magnetic field under conditions (3.14), (3.14a) is given in [7] (also [6, 18]). It must be pointed out that this solution becomes isotropic asymptotically as $|\tau| \to \infty$ only for the value $f(x^2) = 1$, this being in agreement with the possibility in this case of the extension of A_4 to A_6, which corresponds to the flat Friedmann model. The solution of the Einstein equations for (3.14), (3.14a) in the case when only matter is present is considered in [12, 13, 24].

The condition $\lambda_2 \neq 0$ corresponds to a group G_3 for (2.1), (2.1a) (and to the non-central extensions). In this case Eq. (2.15d) gives

$$X = \psi(x^1) Y' . \tag{3.15}$$

We shall use an arbitrary function $\varphi(x^1)$ defined by

$$1/\psi^2(x^1) = \pm \alpha^2 + \varphi(x^1), \tag{3.15a}$$

where

$$\mp \alpha^2 = d^2 f(x^2)/f (d x^2)^2 \tag{3.16}$$

with the functions (2.2a), (2.2b), (2.2c), so $\alpha^2 = 0$ for (2.2a), $\alpha^2 = 1$ with the lower sign for (2.2b) and $\alpha^2 = 1$ with the upper sign for (2.2c). Eqs. (3.15a) and (3.16) lead to the inequalities

$$f(x^2)=1: \qquad \varphi(x^1)>0\,, \tag{3.17a}$$

$$f(x^2)=\sinh x^2: \quad \varphi(x^1)>1\,, \tag{3.17b}$$

$$f(x^2)=\sin x^2: \qquad \varphi(x^1)+1>0\,. \tag{3.17c}$$

Eq. (3.15) by virtue of (3.15a) becomes

$$X=\pm Y'/[\varphi(x^1)\pm\alpha^2]^{1/2}\,. \tag{3.15b}$$

The combination $R_2^2-R_1^1$ of the Einstein equations is reduced (for $T_1^0=0$) due to (2.15c) and (2.15b) to the relation

$$2\lambda_2(T_1^1-T_2^2)+\partial T_1^1/\partial x^1=0\,,$$

which is the component of the conservation laws (3.5) with $i=1$ and is satisfied by (3.3a), (3.13) and (3.12a). The combination $(R_0^0-R_1^1)h_1+2(R_0^0-R_2^2)h_2$ of the Einstein equations is reduced (for $T_1^0=0$) to the relation

$$h_1(T_0^0-T_1^1)+2h_2(T_0^0-T_2^2)+\partial T_0^0/\partial\tau=0\,,$$

which is the component of (3.5) with $i=0$ and is also satisfied by (3.3a), (3.13) and (3.12a).

The Eq. (2.15b) together with (3.15b), (3.3a) and (3.12a) yields after the first integration

$$\frac{1}{c^2}\dot{Y}^2=\varphi(x^1)-\frac{B^2}{Y^2}+\frac{F_1(x^1)}{Y}, \qquad B^2=\frac{kK^2}{c^4}=\text{const}\,, \tag{3.18}$$

with an arbitrary function $F_1(x^1)$. The further integration of (3.18) depends upon the sign of function $\varphi(x^1)$.

In the cases $f(x^2)=1$ and $f(x^2)=\sinh x^2$ with the conditions (3.17a) and (3.17b) the result of the integration of (3.18) can be given in a parametric form with a parameter η which depends upon τ and x^1:

$$\begin{aligned} c\tau&=c\tau_0(x^1)+F(x^1)\,\{\pm[1+B^2\chi(x^1)]^{1/2}\sinh\eta-\eta\}\,,\\ Y&=[\varphi(x^1)]^{1/2}\,F(x^1)\,\{\pm[1+B^2\chi(x^1)]^{1/2}\cosh\eta-1\}\,,\\ F(x^1)&\equiv F_1(x^1)/2[\varphi(x^1)]^{3/2}, \qquad \chi(x^1)=1/F^2(x^1)\,\varphi^2(x^1)\,, \end{aligned} \tag{3.19}$$

where $\tau_0(x^1)$ is an arbitrary function. Two signs in (3.19) corresponds to two types of the solutions [7, 13]. Formulae (3.19) give

$$\begin{aligned} Y'=\frac{1}{2}\frac{\varphi'}{\varphi}Y-2\varphi^{1/2}F'-\frac{\varphi^{1/2}FB^2\chi'}{2(1+B^2\chi)}+\frac{\varphi F}{Y}\\ \cdot\{F'[B^2\chi\pm(1+B^2\chi)^{1/2}\eta\sinh\eta]\mp(1+B^2\chi)^{1/2}c\tau_0'\sinh\eta\\ +FB^4\chi'\chi[2(1+B^2\chi)]^{-1}\}\,. \end{aligned} \tag{3.19a}$$

For X the formulae (3.15b) and (3.19a) are valid with the values (3.16), (3.17a) and (3.17b). The substitution of (3.19), (3.15b) into the equation $R_0^0-(R/2)$ gives the expression for e (3.13) in the form

$$e=\frac{c^4 F|\varphi|^{3/2}}{4\pi k Y^2 Y'}\left(\frac{3}{2}\frac{\varphi'}{\varphi}+\frac{F'}{F}\right)=\frac{c^4}{8\pi k}\frac{F_1'}{Y^2 Y'}\,. \tag{3.20}$$

(3.19), (3.15b), (3.19a), (3.20) and (3.12) give the complete solution for the values (2.2a) and (2.2b) in (2.1), (2.1a). The presence of a magnetic field manifests itself in a nonzero value of a constant B. The solution contains arbitrary functions $\tau_0(x^1)$, $F(x^1)$ and also $\varphi(x^1)$ with the conditions (3.17a), (3.17b).

In the case $f(x^2)=\sin x^2$ a function $\varphi(x^1)$ in (3.18) may be positive, zero or negative. For $\varphi(x^1)>0$ Eqs. (3.19), (3.19a) and (3.15b), (3.20) remain valid with (3.16), (3.15b). For values of $\varphi(x^1)$ in (3.17c) in the interval

$$-1<\varphi(x^1)<0 \tag{3.21}$$

the result of the integration of (3.18) is given in a parametric form

$$\begin{aligned} c\tau &= c\tau_0(x^1)+F(x^1)\,\{\eta-[1-B^2\chi(x^1)]^{1/2}\sin\eta\}\,,\\ Y &= F(x^1)\,|\varphi(x^1)|^{1/2}\{1-[1-B^2\chi(x^1)]^{1/2}\cos\eta\}\,,\\ F(x^1) &\equiv F_1(x^1)/2[-\varphi(x^1)]^{3/2}, \qquad \chi(x^1)=1/F^2(x^1)\,\varphi^2(x^1)\,, \end{aligned} \tag{3.22}$$

with an arbitrary function $\tau_0(x^1)$. Formulae (3.22) give

$$\begin{aligned} Y' &= \frac{1}{2}\frac{\varphi'}{\varphi}\,Y+2|\varphi|^{1/2}F'-\frac{|\varphi|^{1/2}FB^2\chi'}{2(1-B^2\chi)}+\frac{\varphi F}{Y}\\ &\cdot\{F'[B^2\chi+(1-B^2\chi)^{1/2}\eta\,\sin\eta]+(1-B^2\chi)^{1/2}c\tau_0'\sin\eta\\ &-FB^4\chi'\chi[2(1-B^2\chi)]^{-1}\}\,. \end{aligned} \tag{3.22a}$$

X is given by (3.15b) with the values (3.16), (3.21) and with (3.22a). For e in the case (3.21) the formula (3.20) remains valid. In virtue of (3.22), (3.22a) an arbitrary function $\chi(x^1)$ for (3.21) must satisfy the inequality $\chi(x^1)\leqq 1/B^2$, i.e. $F^2\varphi^2\geqq B^2$.

Finally, in the case $f(x^2)=\sin x^2$ for

$$\varphi(x^1)=0 \tag{3.23}$$

the result of the integration of (3.18) is given by

$$\begin{aligned} c\tau &= c\tau_0(x^1)+\frac{1}{12}F_1(x^1)\lambda^3+\frac{1}{F_1(x^1)}B^2\lambda\,,\\ Y &= \frac{1}{4}F_1(x^1)\lambda^2+\frac{1}{F_1(x^1)}B^2\,. \end{aligned} \tag{3.24}$$

Eqs. (3.24) and (3.15b), (3.16), (3.23) give also

$$X = \pm Y',$$

$$Y' = \frac{F_1'}{3F_1^2}(YF_1 + 4B^2) - \frac{1}{2Y}\left[\frac{16}{3}B^4\frac{F_1'}{F_1^3} + \lambda F_1 c\tau_0'(x^1)\right], \quad (3.24a)$$

$$e = \frac{c^4}{8\pi k}\frac{F_1'}{Y^2 Y'}.$$

The energy density of a magnetic field W is given by (3.12a). If a magnetic field is present ($B \neq 0$), then in all possible cases due to (3.19), (3.22) and (3.24) a value of Y never is reduced to zero. So W remains finite for all moments of time. The moment τ^* when X becomes zero, being obtained from (3.15b), (3.19a), (3.22a) and (3.24a), corresponds due to (3.20), (3.15b) to a singularity with $e = \infty$. In a general case of a metric (2.1a) with groups G_3 a singular state is achieved for various points x^1 at various moments τ^*. Near a singular state $\eta = \eta^*(x^1)$, $\tau = \tau^*(x^1)$ the dependence of X and Y upon τ has a structure

$$X = [\tau - \tau^*(x^1)]\, F_2(x^1), \qquad Y = Y^*(x^1) + (\tau - \tau^*)\, F_3(x^1).$$

For $f(x^2) = \sin x^2$ and (3.21) the dependence of Y upon τ due to (3.22) has an oscillating character with a limited amplitude, while an amplitude of an oscillating function X of τ increases with increasing time. In the cases $f(x^2) = \sinh x^2$ and $f(x^2) = 1$ and also in the case $f(x^2) = \sin x^2$ with $\varphi \geqq 0$ values of X and Y approach infinity asymptotically as $|\tau| \to \infty$ in the following way

$$\varphi > 0: \; Y \approx [\varphi(x^1)]^{1/2} c\tau, \qquad X \approx Y\varphi'/2\varphi(\varphi \pm \alpha^2)^{1/2}, \quad (3.25)$$

$$\varphi = 0: \; Y \approx (9F_1/4)^{1/3}(c\tau)^{2/3}, \quad X \approx \pm YF_1'/3F_1. \quad (3.26)$$

The non-central extensions of algebras A_3 of the groups G_3 keep the condition $\lambda_2 \neq 0$. The noncentral extension of A_3 to A_4 for $f(x^2) = 1$ with the metric (2.9a) corresponds to a homogeneous axially symmetric model with a co-moving 3-space of constant negative curvature in accordance with (2.16). For such a metric an energy density of a magnetic field in virtue of (2.15b) would depend only upon τ in the contradiction with (3.12a) and (2.9a). Thus, such a non-central extension becomes impossible when a magnetic field is present [1]. If only matter is present ($B = 0$), such an extension is possible, and in this case in the system with (2.9a) one obtains that $R_1^0 \neq 0$ and $u^1(\tau) \neq 0$ in (3.4). The integration becomes possible after a transformation from (2.9a) into a synchronous and co-moving system with a metric (2.1), (2.1a) [21], and the result is given by

[1] The fact that homogeneous Bianchi type V space-times cannot admit a magnetic field is shown in [25].

(3.19), (3.19a), (3.15b), (3.20) with the values of functions

$\varphi(x^1) = \exp(-2x^1)$, $F = \text{const}$, $\tau_0(x^1) = K_0 x^1 + \text{const}$, $K_0 = \text{const}$; $B = 0$.

In the case when only matter is present ($B = 0$) the further extensions to G_6, which lead to Friedmann models, are possible with a certain choice of arbitrary functions in the solutions. For (2.2a) one obtains from (3.19), (3.19a) the open Friedmann model with (2.10) for $F = \text{const}$, $\tau_0 = \text{const}$ and

$$f(x^2) = 1: \varphi(x^1) = \exp(-2x^1). \tag{3.27}$$

For (2.2b) one obtains in (3.19) the open Friedmann model with (2.11) for $F = \text{const}$, $\tau_0 = \text{const}$ and

$$f(x^2) = \sinh x^2: \varphi(x^1) = \cosh^2 x^1. \tag{3.28}$$

For (2.2c) one comes in (3.19) to the open Friedmann model if $F = \text{const}$, $\tau_0 = \text{const}$ and

$$f(x^2) = \sin x^2: \varphi(x^1) = \sinh^2 x^1, \tag{3.29}$$

obtains the flat Friedmann model in (3.24) if $\tau_0 = \text{const}$ and

$$f(x^2) = \sin x^2: \varphi(x^1) = 0, \qquad F_1(x^1) = \text{const}(x^1)^3, \tag{3.30}$$

and has in (3.22) the closed Friedmann model if $\varphi(x^1) = -\sin^2 x^1$, $F = \text{const}$, $\tau_0 = \text{const}$.

In the case when matter and a magnetic field are present for the groups G_3 the choice of arbitrary functions in the solutions with $\varphi(x^1) \gtreqless 0$ is possible for which the solutions become asymptotically isotropic as $|\tau| \to \infty$.

For $\varphi > 0$ according to (3.25) such solutions, which approach the open Friedmann model as $|\tau| \to \infty$, are given by (3.19), (3.19a) with the functions $\varphi(x^1)$ which are chosen due to formulae (3.27), (3.28), (3.29) and with arbitrary functions $F_1(x^1)$ and $\tau_0(x^1)$.

For $\varphi = 0$ according to (3.26) the formulae (3.24), (3.24a) give solutions which approach with $|\tau| \to \infty$ the flat Friedmann model for $F_1(x^1)$ being chosen due to (3.30) and for an arbitrary $\tau_0(x^1)$.

In the case when the signature of V_2 is $(+\,-)$ and a metric has a form (2.17a), (2.17b), the construction of gravitational fields with matter becomes impossible since the invariance of the 4-velocity with respect to the transformations with (2.18)–(2.21) would lead to a physically senseless condition of vanishing of zero component of the 4-velocity. Nevertheless, in this case gravitational fields may be considered the sources of which are collinear magnetic and electric fields directed along x^1 with the energy-momentum tensor given by (3.12a). Calculations similar to the previous give for the metric (2.17a)

$$X = \pm[\mp\alpha^2 - \varphi(x^2)]^{-1/2}\partial Y/\partial x^2, \qquad \mp\alpha^2 = d^2 f(x^0)/f(dx^0)^2,$$

and for the metric (2.17b)

$$X = \pm[\mp\alpha^2 + \varphi(x^2)]^{-1/2}\partial Y/\partial x^2, \qquad \mp\alpha^2 = d^2 f(x^1)/f(dx^1)^2,$$

with an arbitrary function $\varphi(x^2)$, which obeys corresponding inequalities. In both cases (2.17a) (2.17b) the expressions for Y are given by ($A = \text{const}$)

$$\varphi > 0: \quad x^3 = \Phi(x^2) + [\varphi(x^2)]^{-3/2}\,[\pm(A^2 + B^2\varphi)^{1/2}\sinh\eta - A\eta],$$
$$Y = [\varphi(x^2)]^{-1}\,[\pm(A^2 + B^2\varphi)^{1/2}\cosh\eta - A];$$
$$\varphi < 0: \quad x^3 = \Phi(x^2) + |\varphi(x^2)|^{-3/2}\,[A\eta - (A^2 - B^2\varphi)^{1/2}\sin\eta],$$
$$Y = |\varphi(x^2)|^{-1}\,[A - (A^2 - B^2\varphi)^{1/2}\cos\eta];$$
$$\varphi = 0: \quad x^3 = \Phi(x^2) + \frac{\lambda(A^2\lambda^2 + 3B^2)}{6A}, \quad Y = \frac{A^2\lambda^2 + B^2}{2A}, \quad X = \mp\frac{\lambda A\Phi'(x^2)}{Y},$$

where $\Phi(x^2)$ is an arbitrary function.

In the case of the group G_4 with (2.17a), (2.17b), (2.21) the solution for $f = 1$ is given by [7]

$$Y = (A^2\eta^2 + B^2)/2A, \qquad x^3 = \eta(A^2\eta^2 + 3B^2)/6A, \qquad X = \text{const}\,\eta/Y. \quad (3.31)$$

For $f(x^0) = \sinh x^0$ in (2.17a), (2.21) and for $f(x^1) = \sin x^1$ in (2.17b), (2.21) the solution is

$$Y = \pm(A^2 + B^2)^{1/2}\cosh\eta - A, \qquad x^3 = \pm(A^2 + B^2)^{1/2}\sinh\eta - A\eta,$$
$$X = \text{const}\sinh\eta/Y.$$

For $f(x^0) = \sin x^0$ in (2.17a), (2.21) and for $f(x^1) = \sinh x^1$ in (2.17b), (2.21) the solution is given by

$$Y = A - (A^2 - B^2)^{1/2}\cos\eta, \qquad x^3 = A\eta - (A^2 - B^2)^{1/2}\sin\eta,$$
$$X = \text{const}\sin\eta/Y. \quad (3.32)$$

For the metric (2.17b) one thus obtains the static solutions for magneto-gravitational configurations. In this case formulae (3.31) and (3.32) with $A = B$ are reduced to previously studied solutions [26, 27].

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