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Upper Bounds for Ising Model Correlation Functions

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Abstract. A Griffiths correlation inequality for Ising ferromagnets is refined and is used to obtain improved upper bounds for critical temperatures. It is shown that, for non-negative external fields, the mean field magnetization is an upper bound for the magnetization of Ising ferromagnets.

1. Introduction

For each nonempty subset R of an index set Λ define

$$\sigma_R = \prod_{i \in \Lambda} \sigma_i \tag{1.1}$$

where $\sigma_i = \pm 1, i \in \Lambda$, is a set of Ising spins. In a given configuration of spins $\{\sigma\} = \{\sigma_i : i \in \Lambda\}$, the interaction energy is defined by

$$E\{\sigma\} = -\sum_{R \subset \Lambda} J(R)\, \sigma_R . \tag{1.2}$$

Thermodynamic averages of functions $f = f\{\sigma\}$ are defined by

$$\langle f \rangle = \sum_{\{\sigma\}} f\{\sigma\} \exp(-\beta E\{\sigma\}) / \sum_{\{\sigma\}} \exp(-\beta E\{\sigma\}) \tag{1.3}$$

where sums extend over all configurations of spins. We denote

$$\sigma_R \sigma_S = \sigma_{RS} \tag{1.4}$$

where from the Definition (1.1) RS is the set-theoretic symmetric difference $R \cup S - R \cap S$.

For ferromagnetic pair interactions, i.e., $J(R)$ non-negative and zero unless R is a one or a two element subset of Λ (one element subsets corresponding to interactions with an external field), Griffiths [1, 2, 3] proved a number a correlation function inequalities which were subsequently generalized by Kelley and Sherman [4]. For the inter-

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action (1.2) with $J(R) \geqq 0$, these generalized Griffiths inequalities are

$$\text{I.} \qquad \langle \sigma_R \rangle \geqq 0\,, \tag{1.5}$$

$$\text{II.} \qquad \frac{\partial}{\partial \beta J(S)} \langle \sigma_R \rangle = \langle \sigma_{RS} \rangle - \langle \sigma_R \rangle \langle \sigma_S \rangle \geqq 0\,, \tag{1.6}$$

and for any $k \in R$

$$\text{III.} \qquad \langle \sigma_R \rangle \leqq \sum_{\substack{S \\ k \in S}} \tau(S) \langle \sigma_{RS} \rangle \tag{1.7}$$

where

$$\tau(S) = \tanh \beta J(S) \tag{1.8}$$

and the sum in (1.7) extends over sets $S \in \{A \subset \Lambda, J(A) > 0\}$.

It is to be noted that interactions with an external magnetic field $H > 0$ can be included in the above by taking $J(R) = H$ for all one element subsets R of Λ.

Ginibre [5] and Fortuin, Ginibre and Kasteleyn [6] have recently constructed a general framework in which inequalities of the type I and II are valid. Inequality II and its generalizations have been particularly useful in proving various existence theorems for phase transitions [1, 7] and for obtaining critical exponent inequalities [8]. The inequality III, which is the subject of this note, has been used primarily to obtain bounds for critical temperatures [3].

In the next section, we obtain a refinement of the inequality III and use it to obtain improved bounds for critical temperatures. In the final section, we show that, for non-negative external fields, the mean field magnetization is an upper bound for the magnetization of Ising ferromagnets with pair interactions.

2. Refinement of the Inequality (1.7)

In the following, we will make use of the identity

$$\exp[\beta J(R)\, \sigma_R] = \cosh \beta J(R)\, [1 + \tau(R)\, \sigma_R]\,, \tag{2.1}$$

where $\tau(R)$ is defined by (1.8). This result is easily proved by expanding the exponential and noting that $(\sigma_R)^2 = 1$. We will assume throughout that $J(R) \geqq 0$.

Writing

$$\exp(-\beta E\{\sigma\}) = \prod_{R \subset \Lambda} \exp[\beta J(R)\, \sigma_R] \tag{2.2}$$

and applying the identity (2.1) to the term in (2.2) corresponding to the subset $S \subset \Lambda$, we obtain immediately from the definition (1.3) that

$$\langle \sigma_R \rangle = [\langle \sigma_R \rangle_S + \tau(S) \langle \sigma_{RS} \rangle_S] [1 + \tau(S) \langle \sigma_S \rangle_S]^{-1} \tag{2.3}$$

where $\langle \ldots \rangle_S$ denotes an average for the system (1.2) with $J(S) = 0$. Interchanging R and RS in (2.3) gives

$$\langle \sigma_{RS} \rangle = [\langle \sigma_{RS} \rangle_S + \tau(S) \langle \sigma_R \rangle_S] [1 + \tau(S) \langle \sigma_S \rangle_S]^{-1} . \tag{2.4}$$

Combining (2.3) and (2.4) then gives

$$\begin{aligned} \langle \sigma_R \rangle &= \tau(S) \langle \sigma_{RS} \rangle + [(1 - \tau(S)^2) \langle \sigma_R \rangle_S] [1 + \tau(S) \langle \sigma_S \rangle_S]^{-1} \\ &\leqq \tau(S) \langle \sigma_{RS} \rangle + (1 - \tau(S)^2) \langle \sigma_R \rangle_S \end{aligned} \tag{2.5}$$

where use has been made of (1.5), i.e., $\langle \sigma_S \rangle_S \geqq 0$.

Let now $S_1, S_2, \ldots, S_n$ be any family of subsets of Λ. By repeated iteration of (2.5), we obtain

$$\begin{aligned} \langle \sigma_R \rangle \leqq \tau(S_1) \langle \sigma_{RS_1} \rangle &+ (1 - \tau(S_1)^2) \tau(S_2) \langle \sigma_{RS_2} \rangle_{S_2} \\ &+ \cdots + (1 - \tau(S_1)^2) \ldots (1 - \tau(S_n)^2) \langle \sigma_R \rangle_{S_1, S_2, \ldots S_n} , \end{aligned}$$

i.e.

$$\begin{aligned} \langle \sigma_R \rangle \leqq & \sum_{j=1}^{n} \tau(S_j) \left\{ \prod_{i=0}^{j-1} (1 - \tau(S_i)^2) \right\} \langle \sigma_{RS_j} \rangle_{S_1, \ldots, S_{j-1}} \\ & + \prod_{i=1}^{n} (1 - \tau(S_i)^2) \langle \sigma_R \rangle_{S_1, \ldots, S_n} , \end{aligned} \tag{2.6}$$

where $\tau(S_0) \equiv 0$.

It is to be noted that if the family $\{S_i\} = \mathscr{A}$, the set of subsets of Λ excluding R such that $S_i \cap R \neq \phi$ and $J(S_i) > 0$, $i = 1, 2, \ldots n$,

$$\langle \sigma_R \rangle_{S_1, \ldots, S_n} = \tau(R) . \tag{2.7}$$

Also, because of the monotonicity property (1.6)

$$\langle \sigma_{RS_j} \rangle_{S_1, \ldots, S_{j-1}} \leqq \langle \sigma_{RS_j} \rangle . \tag{2.8}$$

It follows that if $\{S_i, i = 1, 2, \ldots, n\} = \mathscr{A}$ and $S_{n+1} = R$, (2.6), (2.7), and (2.8) give

$$\langle \sigma_R \rangle \leqq \sum_{j=1}^{n+1} \tau(S_j) \left\{ \prod_{i=0}^{j-1} (1 - \tau(S_i)^2) \right\} \langle \sigma_{RS_j} \rangle \tag{2.9}$$

where use has been made of $\langle \sigma_{RR} \rangle = \langle \sigma_R^2 \rangle = 1$. Obviously, the best inequality from (2.9) is obtained by choosing an ordering for $S_1, S_2, \ldots, S_n$ which minimizes the right hand side.

For a set of N pair-wise interacting spins in the presence of an external magnetic field H, we choose $R = \{r\}$ $(J(R) = H)$, $S_j = \{r, s_j\}$

$j=1, 2, \ldots, N-1, S_N = R$, such that $s_i \neq s_j, i \neq j$, and $s_j \neq r$. From (2.9) (with $n = N-1$), we then obtain

$$\begin{aligned}\langle\sigma_r\rangle \leqq & \prod_{j=1}^{N-1} (1-\tau(r,s_j)^2) \tanh \beta H \\ & + \sum_{j=1}^{N-1} \tau(r,s_j) \left\{ \prod_{i=0}^{j-1} (1-\tau(r,s_i)^2) \right\} \langle\sigma_{s_j}\rangle ,\end{aligned} \tag{2.10}$$

where $\tau(r,s) = \tanh \beta J_{rs}$, J_{rs} is the coupling constant between spins r and s, and $\tau(r, s_0) \equiv 0$.

For a translationally invariant system $\langle\sigma_k\rangle = m_N(H, \beta)$ is the magnetization per spin for all k. It follows from (2.10) that if

$$G(\beta) = \sum_{j=1}^{N-1} \tau(r,s_j) \left\{ \prod_{i=0}^{j-1} (1-\tau(r,s_i)^2) \right\} < 1 \tag{2.11}$$

the spontaneous magnetization $m_0(\beta) = \lim_{H\to 0+} \lim_{N\to\infty} m_N(H, \beta)$ vanishes, and hence that a solution of

$$G(\beta_0) = 1 , \quad \beta_0 = (kT_0)^{-1} \tag{2.12}$$

gives an upper bound T_0 for the critical temperature T_c.

For example, if there are nearest neighbor interactions only on a lattice with coordination number q,

$$G(\beta) = \tanh(\beta J) \sum_{j=0}^{q-1} (1 - \tanh^2 \beta J)^j \tag{2.13}$$

where J is the coupling constant between nearest neighbor spins. For the square lattice ($q=4$), (2.12) and (2.13) give $\tanh(\beta_0 J) = 0.29\ldots$, which is to be compared with the mean field value 0.25 [3], Fisher's [9] self-avoiding walk bound $0.37\ldots$, and the exact value $\sqrt{2}-1 = 0.414\ldots$. The bounds obtained from (2.12) and (2.13) of course improve with increasing coordination number.

3. Mean Field Bound for the Magnetization

For a set of N Ising spins with ferromagnetic pair interactions only in the presence of an external magnetic field $H \geqq 0$, the choice $R = \{r\}$, $S = \{r, s\}$ in (2.3) gives

$$\langle\sigma_r\rangle = [\langle\sigma_r\rangle_S + \tanh(\beta J_{rs}) \langle\sigma_s\rangle_S] [1 + \tanh(\beta J_{rs}) \langle\sigma_r \sigma_s\rangle_S]^{-1} \tag{3.1}$$

where $J_{rs} \geqq 0$ is the coupling constant between spins r and s. From the monotonicity property (1.6), $\langle\sigma_r\sigma_s\rangle_S \geqq \langle\sigma_r\rangle_S \langle\sigma_s\rangle_S$. Also, since

$0 \leqq \langle\sigma_s\rangle_S \leqq 1$, $\langle\sigma_s\rangle_S \tanh\beta J_{rs} \leqq \tanh(\beta J_{rs}\langle\sigma_s\rangle_S)$. Using these results in (3.1), we obtain

$$\langle\sigma_r\rangle \leqq \tanh(\beta J_{rs}\langle\sigma_s\rangle_S + g(\langle\sigma_r\rangle_S)) \tag{3.2}$$

where $g(z) = \tanh^{-1} z$. Hence, since $\langle\sigma_s\rangle_S \leqq \langle\sigma_s\rangle$,

$$g(\langle\sigma_r\rangle) \leqq \beta J_{rs}\langle\sigma_s\rangle + g(\langle\sigma_r\rangle_S)\,. \tag{3.3}$$

Iterating (3.3) until all bonds $J_{r,s} > 0$ have been eliminated, we then obtain, using (2.7)

$$g(\langle\sigma_r\rangle) \leqq \sum_{s \neq r} \beta J_{rs}\langle\sigma_s\rangle + \beta H\,,$$

i.e.

$$\langle\sigma_r\rangle \leqq \tanh\left(\sum_{s \neq r} \beta J_{rs}\langle\sigma_s\rangle + \beta H\right) \tag{3.4}$$

For a translationally invariant system $\langle\sigma_r\rangle = m_N(H, \beta)$ is the magnetization per spin for all r. Taking the limit $N \to \infty$ in (3.4), we then obtain

$$0 \leqq m \leqq \tanh(\beta\alpha m + \beta H), \qquad \text{for} \qquad H \geqq 0\,, \tag{3.5}$$

where $m = \lim_{N \to \infty} m_N(H, \beta)$, and from translational invariance,

$$\alpha = \sum_{s \neq r} J_{rs} \tag{3.6}$$

is independent of r.

The positive solution of

$$m^* = \tanh(\beta\alpha m^* + \beta H)\,, \qquad H \geqq 0 \tag{3.7}$$

is the mean field magnetization. From (3.5) we then obtain

$$0 \leqq m \leqq m^*\,, \qquad \text{for} \qquad H \geqq 0\,. \tag{3.8}$$

Notice also, from (3.8), that the mean field critical temperature T^* given from (3.7) by $\beta^* = (kT^*)^{-1} = \alpha^{-1}$ is necessarily an upper bound for the true critical temperature.

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