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# Invariant Tensors in $SU(3)$

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**Abstract.** The construction of independent $SU(3)$ tensors out of octets of fields is considered by investigating numerically invariant $SU(3)$ tensors. A method of obtaining independent sets of these to any rank is discussed and also independent sets are explicitly displayed up to fifth rank. It is shown that this approach allows us to obtain relations among the invariant tensors, and useful new identities involving the $d_{ijk}$ and $f_{ijk}$ tensors are exhibited.

## I. Introduction

In $SU(3)$ or chiral $SU(3) \otimes SU(3)$ theories of elementary particles, the most common multiplets of particles which we have to handle are the octets or regular representations of $SU(3)$. When dealing with effective Lagrangians [1], especially with the terms in these Lagrangians which break the symmetry, and also when considering non-linear transformations under the chiral group, it is frequently necessary to build $SU(3)$ scalars and tensors out of these basic octets. In practice this means [2] that the quantities of importance are the tensors of $SU(3)/Z(3)$ and henceforth we shall restrict ourselves to this subset of the $SU(3)$ tensors. In practical calculations it is always of the greatest importance to be certain that all possible independent tensors have been written down. In a recent consideration [3] of a model of chiral symmetry breaking, involving three independent octets of fields, we were faced with just such a problem, and it was in the attempt to find a general effective way of dealing with this that the following work arose. This approach allowed us to handle the very complicated expressions which occur in the second and third orders in our calculation with relative ease, and this was not achieved satisfactorily with the usual techniques [2] which we used when we first attempted this problem. Our analysis consists of coupling the octets together with invariant $SU(3)$ tensors and for this purpose we have been able to develop a technique for deriving all these tensors to all orders. With the higher ranks the method is, as could be expected, unwieldy, although nonetheless possible in principle, and probably within the scope of computer techniques. We have obtained explicitly independent sets of invariant tensors up to the fifth rank.

In Section IV we have derived the necessary sets of independent invariant tensors which we required in the work mentioned above, and which will be extremely useful in similar situations. Section V deals with the question of choosing independent sets of higher rank tensors if these should be required, and Section VI contains results which apply to the construction of tensors from repeated octet vectors. Sections II and III describe respectively the basic ideas and methods for checking independence and obtaining identities on the invariant tensors. Some identities which have not previously been obtained are found by this approach and are displayed in Section IV.

Needless to say, this work can readily be generalised to the problem of forming tensors from the regular representations of $SU(n)$, but since this would in no way serve to clarify the arguments we confine ourselves to $SU(3)$ throughout the paper.

## II. Numerically Invariant $8\otimes 8\otimes \ldots \otimes 8$ Tensors for $SU(3)$

Consider a set of $r$ independent $SU(3)$ vectors $A_i, B_j \ldots G_k$. By vectors we mean that each transforms as the regular representation of $SU(3)$, i.e.

$$[Q_i, A_j] = i f_{ijk} A_k \tag{1}$$

where the $Q_i$ are the $SU(3)$ generators and $f_{ijk}$ are the structure constants.

Suppose that $H^{(\alpha)}_{ij\ldots k}: 1 \leqq \alpha \leqq \beta(r)$ are $\beta(r)$ numerically invariant $r^{\text{th}}$ rank tensors; that is, they are tensors which transform according to the $8_1 \otimes 8_2 \otimes \cdots \otimes 8_r$ representation of $SU(3)$, i.e.

$$[Q_p, H^{(\alpha)}_{ij\ldots k}] = i f_{pit} H^{(\alpha)}_{tj\ldots k} + i f_{pjt} H^{(\alpha)}_{it\ldots k} + \cdots + i f_{pkt} H^{(\alpha)}_{ij\ldots t} \tag{2}$$

and they are also just sets of numbers which are the same in all $SU(3)$ frames. This means, of course, that the left hand side of Eq. (2) is zero. Examples of such tensors are, say, $\delta_{ij}$ and $\varepsilon_{ijk}$ for $0(3)$ or $\delta_{ij}$, $f_{ijk}$ and $d_{ijk}$ of Gell-Mann [4] for $SU(3)$. The $H^{(\alpha)}_{ij\ldots k}$ have been chosen so that they are all independent and we assume that we have also found *all* the tensors of this particular rank; we emphasise that this implies all the independent tensors when we take order of indices into account. For example, the fourth rank tensors $\delta_{ij}\delta_{kl}$ and $\delta_{ik}\delta_{jl}$ are therefore counted as two different tensors; but the third rank tensors $f_{ijk}$ and $f_{jik}$ are not counted separately since they are linearly dependent: $f_{ijk} = -f_{jik}$.

If we contract the $r$ independent vectors $A_i, B_j, \ldots, G_k$ with the invariant tensors $H^{(\alpha)}_{ij\ldots k}$ we shall form scalars $\sigma^{(\alpha)}$, i.e.

$$\sigma^{(\alpha)} = H^{(\alpha)}_{ij\ldots k} A_i B_j \ldots G_k \qquad 1 \leqq \alpha \leqq \beta(r). \tag{3}$$

Furthermore, since the $A_i, \ldots, G_k$ are independent vectors and since the $H^{(\alpha)}_{ij\ldots k}$ are not only all independent, but also all the invariant tensors we can form of rank $r$, the $\sigma^{(\alpha)}$ will be all the scalars that we can build out of these vectors (each being used once and only once), and of course they will all be independent. We therefore deduce that out of $r$ independent $SU(3)$ vectors we can form $\beta(r)$ independent $SU(3)$ scalars, where $\beta(r)$ is the number of $r^{\text{th}}$ rank numerically invariant tensors.

Similarly all the independent vectors that it is possible to form out of the $(r-1)$ vectors $B_j \ldots G_k$ will be

$$V^{(\alpha)}_i = H^{(\alpha)}_{ij\ldots k} B_j \ldots G_k \qquad 1 \leqq \alpha \leqq \beta(r). \tag{4}$$

We can generalise this procedure to the problem of constructing all the independent $k^{\text{th}}$ rank $SU(3)$ tensors out of $(r-k)$ independent vectors, and in each case we shall find $\beta(r)$ independent tensors. Thus, for example, the number of scalars that we can form out of six independent vectors is the same as the number of vectors that we can form out of five independent vectors, which is in turn the number of second rank tensors that we can form out of four independent vectors, and so on.

Alternatively we can approach the problem in the following way. The scalars are the singlet representations of $SU(3)$. Therefore the number of scalars we can form out of $r$ independent vectors is the number of times that 1 appears in the decomposition of $8_1 \otimes 8_2 \otimes \cdots \otimes 8_r$ into irreducible representations. Suppose this number is $\beta(r)$. Now consider the $k^{\text{th}}$ rank tensors formed out of $(r-k)$ vectors; suppose we have $\gamma(r,k)$ of these. If we consider the irreducible parts of these $k^{\text{th}}$ rank tensors according to

$$8_1 \otimes 8_2 \otimes \cdots \otimes 8_k = \oplus \sum_i \varepsilon_i \sigma^i \tag{5}$$

where $\sigma^i$ are the irreducible representations of $SU(3)$ and $\varepsilon_i$ are their multiplicities in the decomposition then

$$\gamma(r,k) = \sum_i \varepsilon_i \lambda_i \tag{6}$$

where $\lambda_i$ is the number of times that the representation $\sigma^i$ appears in the decomposition of

$$8_{(k+1)} \otimes 8_{(k+2)} \otimes \cdots \otimes 8_r .$$

But the number of times that the irreducible representation $\mu$ appears in the decomposition of the representation $\nu$ is the same as the number of times that 1 appears in the decomposition of $\mu \otimes \bar{\nu}$, where $\bar{\nu}$ is the representation adjoint to $\nu$. (This can be deduced from the orthogonality theorems.) Hence $\lambda_i$ is the number of times that 1 appears in the decomposition of

$$\sigma^i \otimes 8_{(k+1)} \otimes 8_{(k+2)} \otimes \cdots \otimes 8_r$$

since the representation $8 \otimes 8 \otimes \cdots \otimes 8$ is self adjoint. Therefore Eq. (6) tells us that $\gamma(r, k)$ is the number of times that 1 appears in the decomposition of

$$\left(\oplus \sum_i \varepsilon_i \sigma^i\right) \otimes 8_{(k+1)} \otimes 8_{(k+2)} \otimes \cdots \otimes 8_r .$$

Using Eq. (5), $\gamma(r, k)$ is the number of times that 1 appears in the decomposition of

$$(8_1 \otimes 8_2 \otimes \cdots \otimes 8_k) \otimes 8_{(k+1)} \otimes 8_{(k+2)} \otimes \cdots \otimes 8_r$$

and this is $\beta(r)$. Hence,

$$\gamma(r, k) = \beta(r) \qquad \text{for all } k. \tag{7}$$

These two different ways of looking at the problem each have their respective merits. The first shows us that the construction of all independent vectors and tensors is trivial once we have found all the numerically invariant tensors, and the second gives us an explicit method of counting these tensors.

## III. Relations between Invariant Tensors

Suppose that $K_{ij\ldots k}$ is a numerically invariant tensor of the $r^{\text{th}}$ rank. We know that the $\beta(r)$ tensors $H^{(\alpha)}_{ij\ldots k}$ can be taken as a basis of such tensors. Hence

$$K_{ij\ldots k} = x_\beta H^{(\beta)}_{ij\ldots k} \qquad \text{(summation on } \beta) \tag{8}$$

define

$$Q^{\alpha\beta} = Q^{\beta\alpha} = H^{(\alpha)}_{ij\ldots k} H^{(\beta)}_{ij\ldots k} \qquad \text{(summation on } i, j \ldots k) \tag{9}$$

and let

$$y_\alpha = K_{ij\ldots k} H^{(\alpha)}_{ij\ldots k} . \tag{10}$$

Then contracting both sides of Eq. (8) with $H^{(\alpha)}_{ij\ldots k}$ gives us

$$Q^{\alpha\beta} x_\beta = y_\alpha . \tag{11}$$

These are equations which we can solve for the $x_\alpha$, because $Q^{\alpha\beta}$ and $y_\alpha$ are known quantities which we determine from Eqs. (9) and (10). The Eqs. (11) have a unique solution only when $Q^{\alpha\beta}$ is non-singular and this is obviously the condition that our $H^{(\alpha)}_{ij\ldots k}$ are independent, for a singular $Q^{\alpha\beta}$ indicates that at least one of the $H^{(\alpha)}_{ij\ldots k}$ is linearly dependent on the others.

## IV. Invariant Tensors of the Second, Third, Fourth and Fifth Ranks

We shall now choose specific sets of invariant tensors for the $H^{(\alpha)}$ for the lowest few ranks. The last part of section II tells us that the number of $r^{\text{th}}$ rank tensors is $\beta(r)$ where this is the number of 1's in the decomposition of

$$(\otimes 8)^r \equiv 8_1 \otimes 8_2 \otimes \cdots \otimes 8_r .$$

First we notice that if we take the convention

$$(\otimes 8)^\circ = 1 \tag{12}$$

(which we need for compatibility in $(\otimes 8)^r \otimes (\otimes 8)^s = (\otimes 8)^{r+s}$), then $\beta(0)=1$. This, of course, is the completely trivial case. The zeroth rank tensors are just elements of the complex field and these are a one dimensional space. The other $\beta(r)$ can be evaluated by actually performing the reduction of $(\otimes 8)^r$ by the method of Young Tableaux for instance. We list the results below for ranks up to six.

$$\begin{aligned}
\beta(0) &= \ \ 1\\
\beta(1) &= \ \ 0\\
\beta(2) &= \ \ 1\\
\beta(3) &= \ \ 2\\
\beta(4) &= \ \ 8\\
\beta(5) &= \ 32\\
\beta(6) &= 145 .
\end{aligned} \tag{13}$$

For the lowest ranks the results are already well known. That $\beta(1)$ equals zero is a statement that there is no numerically invariant vector; that $\beta(2)$ equals one means that there is one second rank tensor which we know to be $\delta_{ij}$, and $\beta(3)$ equals two tells us that there are two third rank tensors. We know that $d_{ijk}$ and $f_{ijk}$ are numerically invariant tensors and we know that they are independent (from their symmetry properties). We may therefore take

$$\begin{aligned}
H^{(1)}_{ijk} &= f_{ijk}\\
H^{(2)}_{ijk} &= d_{ijk} .
\end{aligned} \tag{14}$$

The statement (2) that $H^{(1)}$ and $H^{(2)}$ are invariant tensors gives the well known relations

$$\begin{aligned} f_{ijp}f_{klp}+f_{ilp}f_{jkp}+f_{ikp}f_{ljp}=0 \\ f_{ijp}d_{klp}+f_{ilp}d_{jkp}+f_{ikp}d_{ljp}=0. \end{aligned} \tag{15}$$

From the usual normalisation of $f_{ijk}$ and $d_{ijk}$, namely,

$$\begin{aligned} f_{ipq}f_{jpq}&=3\,\delta_{ij} \\ d_{ipq}d_{jpq}&=\tfrac{5}{3}\,\delta_{ij} \end{aligned}$$

we can calculate

$$Q^{\alpha\beta}(3)=\begin{bmatrix} 24 & 0 \\ 0 & \frac{40}{3} \end{bmatrix}$$

and proceed as in Eqs. (8)...(11) to calculate identities. For instance, consider

$$K_{ijk}=f_{piq}f_{qjr}f_{rkp}.$$

then

$$\begin{aligned} y_1 &= f_{piq}f_{qjr}f_{rkp}f_{ijk} \\ &= f_{piq}f_{qjr}(f_{pkj}f_{rik}+f_{pki}f_{jrk}) \\ &= -y_1+(-3\,\delta_{kq})(3\,\delta_{kq}) \\ \therefore y_1 &= -36. \\ y_2 &= f_{piq}f_{qjr}f_{rkp}d_{ijk}=f_{piq}f_{qjr}(f_{pkj}d_{rik}+f_{pki}d_{jrk})=-y_2 \\ \therefore y_2 &= 0. \end{aligned}$$

$$\therefore \begin{bmatrix} 24 & 0 \\ 0 & \frac{40}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -36 \\ 0 \end{bmatrix}$$

i.e. $x_1=-\frac{3}{2}$ and $x_2=0$. Hence

$$f_{piq}f_{qjr}f_{rkp}=-\tfrac{3}{2}f_{ijk}. \tag{16}$$

In a similar manner we obtain

$$d_{piq}f_{qjr}f_{rkp}=-\tfrac{3}{2}\,d_{ijk}, \tag{17}$$

$$d_{piq}d_{qjr}f_{rkp}=\tfrac{5}{6}\,f_{ijk}, \tag{18}$$

$$d_{piq}d_{qjr}d_{rkp}=\lambda\,d_{ijk}, \tag{19}$$

where in Eq. (19) $\lambda$ is a number which will be fixed when we deduce an identity similar to Eqs. (15) for the $d_{ijk}$ tensors, by consideration of the independent fourth rank tensors.

Since $\beta(4)$ equals eight, there are eight independent fourth rank tensors. We shall attempt to form these out of outer products of $\delta_{ij}$, $f_{ijk}$ and $d_{ijk}$, and products with contractions. Consider the set

$$\begin{aligned}
H^{(1)}_{ijkl} &= \delta_{ij}\delta_{kl} \\
H^{(2)}_{ijkl} &= \delta_{ik}\delta_{jl} \\
H^{(3)}_{ijkl} &= \delta_{il}\delta_{jk} \\
H^{(4)}_{ijkl} &= d_{ijm}d_{klm} \\
H^{(5)}_{ijkl} &= d_{ikm}d_{jlm} \\
H^{(6)}_{ijkl} &= d_{ijm}f_{klm} \\
H^{(7)}_{ijkl} &= d_{ikm}f_{jlm} \\
H^{(8)}_{ijkl} &= d_{ilm}f_{jkm}
\end{aligned} \tag{20}$$

which consists of some of the simplest tensors we can build. Again we construct the matrix $Q^{\alpha\beta}$ defined in Eq. (9) and to do this we use the identities in Eqs. (16)...(19).

We find

$$Q^{\alpha\beta}(4) = \begin{bmatrix}
64 & 8 & 8 & 0 & \frac{40}{3} & 0 & 0 & 0 \\
8 & 64 & 8 & \frac{40}{3} & 0 & 0 & 0 & 0 \\
8 & 8 & 64 & \frac{40}{3} & \frac{40}{3} & 0 & 0 & 0 \\
0 & \frac{40}{3} & \frac{40}{3} & \frac{200}{9} & \frac{40\lambda}{3} & 0 & 0 & 0 \\
\frac{40}{3} & 0 & \frac{40}{3} & \frac{40\lambda}{3} & \frac{200}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 40 & -20 & 20 \\
0 & 0 & 0 & 0 & 0 & -20 & 40 & -20 \\
0 & 0 & 0 & 0 & 0 & 20 & -20 & 40
\end{bmatrix} \tag{21}$$

which explicitly contains the number $\lambda$ from Eq. (19). For general values of $\lambda$, $Q^{\alpha\beta}(4)$ is non-singular. The $H^{(\alpha)}_{ijkl}$ are therefore eight independent fourth rank tensors, thus by means of our earlier technique, we may write down any fourth rank numerically invariant tensor in terms of the eight we have chosen above. First we consider $d_{ilm}d_{jkm}$ and obtain the identity

$$\begin{aligned}
&(1-7\lambda)\, d_{ijm}d_{klm} + (1-7\lambda)\, d_{ikm}d_{jlm} + (8+7\lambda)\, d_{ilm}d_{jkm} \\
&\qquad = \left(\frac{\lambda+5}{3}\right)\delta_{ij}\delta_{kl} + \left(\frac{\lambda+5}{3}\right)\delta_{ik}\delta_{jl} - 3\lambda\,\delta_{il}\delta_{jk}\,.
\end{aligned} \tag{22}$$

We then contract both sides of this equation with $f_{ijp}$ and use Eq. (18) to arrive at

$$\lambda = -\tfrac{1}{2}\,.$$

Putting this value in Eq. (22) gives

$$d_{ijm} d_{klm} + d_{ikm} d_{jlm} + d_{ilm} d_{jkm} = \tfrac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \tag{23}$$

Also if we take for $K_{ijkl}$ in Eq. (8) the tensor $f_{ilm} f_{jkm}$ we obtain

$$f_{ilm} f_{jkm} = \tfrac{2}{3} (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) + d_{ijm} d_{klm} - d_{ikm} d_{jlm}. \tag{24}$$

Eqs. (23) and (24) are of course well established identities [2] on $f_{ijk}$ and $d_{ijk}$ tensors. It is interesting to note that we have deduced these relations merely from the symmetry properties of $f$ and $d$, our count of invariant octet coupling tensors, and the relations (15) which are the equations which state that $f_{ijk}$ and $d_{ijk}$ are invariant tensors.

At this level, the simplicity of this approach has been obscured by our desire to obtain well known identities from a minimum amount of initial information. Now that the groundwork has been completed, however, the more complicated expressions can be deduced with relative ease. For instance, two identities which are not well known and which are not easy to deduce from the well known relations are:

$$36 d_{piq} d_{qjm} d_{mkt} d_{tlp} = 13 \delta_{ij} \delta_{kl} - 7 \delta_{ik} \delta_{jl} + 13 \delta_{il} \delta_{jk} - 6 d_{ikm} d_{jlm}, \tag{25}$$

$$12 d_{piq} d_{qjm} d_{mkt} f_{tlp} = -7 d_{ijm} f_{klm} + d_{ikm} f_{jlm} + 9 d_{ilm} f_{jkm}. \tag{26}$$

If we were to assume knowledge of Eqs. (23) and (24) we could summarise our procedure for picking the independent fourth rank tensors as follows. After writing down all the straightforward outer products of lower rank tensors (which were $H^{(1)} \ldots H^{(3)}$) we wrote down all the products of $f_{ijk}$ and $d_{ijk}$ with the minimum number of contractions (in this case one contraction). Using the Eqs. (15), (23) and (24) we were able to find among these just the required number of independent ones to give us eight altogether.

For the fifth rank tensors the treatment is exactly the same. The simple outer products are of the form

$$\begin{aligned} H^{(1)}_{ijklm} &= \delta_{ij} d_{klm} \\ H^{(11)}_{ijklm} &= \delta_{ij} f_{klm}. \end{aligned} \tag{27}$$

There are ten of each of these (this being the number of ways of choosing two from five) so we have found $H^{(\alpha)}_{ijklm}$ for $1 \leqq \alpha \leqq 20$. We look for the remaining twelve which we require among the set of contracted $f$ and $d$ tensors with the minimum number of contractions. These are obviously the tensors of the form

$$g_{ijp} g_{pkt} g_{tlm}$$

or as we shall write henceforth

$$g_{\cdot\cdot\smile\cdot\smile\cdot\cdot}\; g \; g \tag{28}$$

where $g_{ijk}$ stands for either $d_{ijk}$ or $f_{ijk}$. There are fifteen combinations of indices which we number in the following way.

$$\begin{array}{rlll}
(1) & ijp & pkt & tlm \\
(2) & ikp & pjt & tlm \\
(3) & kjp & pit & tlm \\
(4) & ijp & plt & tkm \\
(5) & ijp & pmt & tkl \\
(6) & ikp & plt & tjm \\
(7) & ikp & pmt & tjl \\
(8) & kjp & plt & tim \\
(9) & kjp & pmt & til \\
(10) & ilp & pjt & tkm \\
(11) & jlp & pit & tkm \\
(12) & imp & pjt & tkl \\
(13) & jmp & pit & tkl \\
(14) & ilp & pkt & tjm \\
(15) & mip & pkt & tjl\,.
\end{array} \tag{29}$$

First consider the products $f_{\cdot\cdot\smile\cdot\smile\cdot\cdot}\, f \, f$ which we can write as $F(1)\,\dots\,F(15)$, according to the numbering convention in Eqs. (29). Applying Eqs. (15) gives us ten relations among the $F(i)$. These are

$$\begin{aligned}
&F(1)-F\ \ (4)+F\ \ (5)=0\\
&F(2)-F\ \ (6)+F\ \ (7)=0\\
&F(9)+F(10)-F(14)=0\\
&F(8)+F(12)+F(15)=0\\
&F(3)-F\ \ (8)+F\ \ (9)=0\\
&F(5)-F(12)+F(13)=0\\
&F(4)-F(10)+F(11)=0\\
&F(7)-F(11)+F(15)=0\\
&F(6)-F(13)-F(14)=0\\
&F(1)-F\ \ (2)-F\ \ (3)=0.
\end{aligned} \tag{30}$$

We deduce that all the expressions to the right in Eqs. (30) namely $F(5)$, $F(7), F(14), F(15), F(9), F(13), F(11)$ and $F(3)$ depend on their predecessors and can be discarded. In fact $F(14)$ and $F(15)$ appear twice in this position and eliminating them from the equations gives

$$F(12) = -F(2) + F(4) + F(6) - F(8) - F(10) \tag{31}$$

in both cases. We may therefore discard $F(12)$ as well; the only independent tensors of this form are $F(1)$, $F(2)$, $F(4)$, $F(6)$, $F(8)$ and $F(10)$.

Now consider the expressions $D(i): i = 1, \ldots, 15$ which are products of the form $d_{\cdot\cdot\cdot}\,d_{\cdot\cdot\cdot}\,d_{\cdot\cdot\cdot}$. We shall use the equivalence relation

$$A \sim B$$

to mean that $A = B$ to within expressions we have already chosen as independent 5$^{\text{th}}$ rank invariant tensors. An application of Eq. (23) leads to

$$\begin{aligned}
&D(1) + D(4) + D(5) \sim 0\\
&D(2) + D(6) + D(7) \sim 0\\
&D(9) + D(10) + D(14) \sim 0\\
&D(8) + D(12) + D(15) \sim 0\\
&D(3) + D(8) + D(9) \sim 0\\
&D(5) + D(12) + D(13) \sim 0\\
&D(4) + D(10) + D(11) \sim 0\\
&D(7) + D(11) + D(15) \sim 0\\
&D(6) + D(13) + D(14) \sim 0\\
&D(1) + D(2) + D(3) \sim 0.
\end{aligned} \tag{32}$$

These equations for the $D(i)$ are the same as Eqs. (30) for the $F(i)$ apart from the positive sign appearing everywhere and the equivalence sign replacing the equality sign. Consequently we discard the expressions $D(5)$, $D(7)$, $D(14)$, $D(15)$, $D(9)$, $D(13)$, $D(11)$ and $D(3)$ because these are linearly dependent on their predecessors (and perhaps on some of the $H^{(\alpha)}_{ijklm}$ for $1 \leqq \alpha \leqq 20$). Again, since $D(14)$ and $D(15)$ appear twice in this position, we may eliminate them to find other relations. This time we obtain two independent equations:

$$\begin{aligned}
&D(4) + D(6) + D(8) \sim 0\\
&D(2) + D(10) + D(12) \sim 0.
\end{aligned} \tag{33}$$

Hence we may discard $D(8)$ and $D(12)$ as well, and deduce that the only independent tensors of this form are

$$D(1),\ D(2),\ D(4),\ D(6) \quad \text{and} \quad D(10).$$

All we have yet to consider are the mixed products of $f_{ijk}$ and $d_{ijk}$ which must be $(ddf)$ or $(ffd)$ in some orders. If the odd one out is not in the middle of the product $g_{\cdot\cdot\smile}\,g_{\cdot\smile}\,g_{\cdot\cdot}$, then the other two $g_{ijk}$ tensors, which must be either both $f$ tensors or both $d$ tensors, can be written as a product of two tensors which are the same as the odd one out by Eqs. (15), (23) and (24). These expressions are therefore equivalent to the $F(i)$ or the $D(i)$ and can be discarded. If the odd one out is in the middle, then the product is either

$$f_{..p}d_{p.t}f_{t..} \quad \text{or} \quad d_{..p}f_{p.t}d_{t..}\,.$$

The latter can be written using Eqs. (15) as

$$d_{..p}f_{..t}d_{t.p}+d_{..p}f_{..t}d_{t.p}$$

which may be discarded by the arguments above ($f_{..t}$, the odd one out, is now on the end of the product in each term). We are just left with the former expression which we call $E(i)$ according to the earlier notation. If we apply Eqs. (15) to the $E(i)$ we find that they are all equivalent. Hence there is just one more independent tensor which is a mixed product of $f$ and $d$ tensors and this we take to be $E(1)$.

Hence an independent set of fifth rank invariant tensors is

$$\begin{array}{ll}
H^{(1)}_{ijklm}=\delta_{ij}d_{klm} & H^{(17)}_{ijklm}=\delta_{jm}f_{ikl}\\
H^{(2)}_{ijklm}=\delta_{ik}d_{jlm} & H^{(18)}_{ijklm}=\delta_{kl}f_{ijm}\\
H^{(3)}_{ijklm}=\delta_{jk}d_{ilm} & H^{(19)}_{ijklm}=\delta_{km}f_{ijl}\\
H^{(4)}_{ijklm}=\delta_{il}d_{jkm} & H^{(20)}_{ijklm}=\delta_{lm}f_{ijk}\\
H^{(5)}_{ijklm}=\delta_{jl}d_{ikm} & H^{(21)}_{ijklm}=f_{ijp}f_{pkt}f_{tlm}\\
H^{(6)}_{ijklm}=\delta_{im}d_{jkl} & H^{(22)}_{ijklm}=f_{ikp}f_{pjt}f_{tlm}\\
H^{(7)}_{ijklm}=\delta_{jm}d_{ikl} & H^{(23)}_{ijklm}=f_{ijp}f_{plt}f_{tkm}\\
H^{(8)}_{ijklm}=\delta_{kl}d_{ijm} & H^{(24)}_{ijklm}=f_{ikp}f_{plt}f_{tjm}\\
H^{(9)}_{ijklm}=\delta_{km}d_{ijl} & H^{(25)}_{ijklm}=f_{kjp}f_{plt}f_{tim}\\
H^{(10)}_{ijklm}=\delta_{lm}d_{ijk} & H^{(26)}_{ijklm}=f_{ilp}f_{pjt}f_{tkm}\\
H^{(11)}_{ijklm}=\delta_{ij}f_{klm} & H^{(27)}_{ijklm}=f_{ijp}d_{pkt}f_{tlm}\\
H^{(12)}_{ijklm}=\delta_{ik}f_{jlm} & H^{(28)}_{ijklm}=d_{ijp}d_{pkt}d_{tlm}\\
H^{(13)}_{ijklm}=\delta_{jk}f_{ilm} & H^{(29)}_{ijklm}=d_{ikp}d_{pjt}d_{tlm}\\
H^{(14)}_{ijklm}=\delta_{il}f_{jkm} & H^{(30)}_{ijklm}=d_{ijp}d_{plt}d_{tkm}\\
H^{(15)}_{ijklm}=\delta_{jl}f_{ikm} & H^{(31)}_{ijklm}=d_{ikp}d_{plt}d_{tjm}\\
H^{(16)}_{ijklm}=\delta_{im}f_{jkl} & H^{(32)}_{ijklm}=d_{ilp}d_{pjt}d_{tkm}\,.
\end{array} \qquad (34)$$

As before we may write any other fifth rank invariant tensors in terms of the above set. To do this in general, however, we shall need to solve thirty-two simultaneous equations. This would give at the same time, of course, a further check on the independence of the tensors (34), because by solving these thirty-two equations we would also be revealing whether $Q^{\alpha\beta}$ was singular or not.

## V. Invariant Tensors of Arbitrary Rank

The above methods of dealing with the lower rank tensors are evidently possible only because of the simplicity of the problem at these levels. With the higher ranks, it is a hopeless task to try to pick an independent set just by manipulating Jacobi identity relations. We shall therefore indicate a general method by which one can obtain independent sets of tensors to all orders; this method is of course extremely tedious, and we suggest it more for the sake of completeness rather than for its practical value – when attempting to form independent scalars and tensors out of $SU(3)$ vectors, one will be unlikely to require more than the contents of the previous section.

We write down a spanning set for the $r^{\text{th}}$ rank tensors and choose from these an independent set in the usual way. Suppose we have already chosen $m$ independent tensors from the spanning set. We shall call these $H^{(1)}_{ij\ldots k} \ldots H^{(m)}_{ij\ldots k}$. We pick the next simplest tensor, $J_{ij\ldots k}$ say, from the spanning set, and form an $(m+1)$ dimensional matrix from $H^{(1)}, \ldots, H^{(m)}$ and $J$ as in Eq. (9). If this matrix is non-singular, $J$ is not linearly dependent on $H^{(1)}, \ldots, H^{(m)}$ and we can write $H^{(m+1)} = J$; if the matrix is singular, $J$ is dependent on $H^{(1)}, \ldots, H^{(m)}$ and can be discarded. We continue in this way until we have obtained $\beta(r)$ independent tensors; then these tensors must be a basis of numerically invariant $r^{\text{th}}$ rank tensors.

## VI. Symmetry in Pairs of Indices

Once all the independent invariant tensors have been found it is a straightforward job to write down all the scalars, vectors or tensors which can be formed out of a set of $SU(3)$ vectors. In practice, however, this set of vectors from which we wish to form our tensors may have vectors repeated and this will lead to less independent tensors than would have arisen if all the initial vectors had been different. On the other hand, instead of forming general $8 \otimes 8$ tensors, we may only require certain irreducible parts of these tensors. In each of these special cases it is useful to know how many independent tensors we expect to find, and this can be done using our earlier methods. We shall merely give brief descriptions of the ideas in these approaches; the proofs run on the same lines as those in section II.

It is well known that, if we decompose the outer product of two $SU(3)$ octets according to

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27,$$

then those representations that are symmetric under interchange of the octets in the outer product are the singlet, the symmetric octet and the 27, i.e.

$$(8 \otimes 8)_S = 1 \oplus 8_S \oplus 27. \tag{35}$$

In a similar manner we can show that the totally symmetric parts in a product of three octets are given by

$$(8 \otimes 8 \otimes 8)_S = 1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 64. \tag{36}$$

and that the totally symmetric parts in a product of four octets are given by

$$(8 \otimes 8 \otimes 8 \otimes 8)_S = 1 \oplus 8 \oplus 8 \oplus 27 \oplus 27 \oplus 35 \oplus \overline{35} \oplus 64 \oplus 125. \tag{37}$$

If we are forming $k^{\text{th}}$ rank tensors out of $(r-k)$ independent vectors, we know that this is the number of times that 1 appears in the decomposition of

$$8_1 \otimes 8_2 \otimes \cdots \otimes 8_r.$$

But, if two of the vectors we start with are the same, the number of independent vectors will be the number of times that 1 appears in

$$\begin{aligned} &8_1 \otimes 8_2 \otimes \cdots \otimes 8_{(r-2)} \otimes (8_{(r-1)} \otimes 8_r)_S \\ &\qquad = (\otimes 8)^{(r-2)} \otimes (1 \oplus 8 \oplus 27) \qquad \text{(using Eq. (35))}. \end{aligned}$$

If three are the same, the required number of tensors will be the number of 1's in

$$(\otimes 8)^{r-3} \otimes (8 \otimes 8 \otimes 8)_S = (\otimes 8)^{r-3} \otimes (1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 64)$$
$$\text{(using Eq. (36))}$$

and similarly if more than one of the vectors is repeated.

We shall illustrate this with the fifth rank invariant tensors, the number of which is 32. Hence there are 32 independent scalars formed out of $A_i$, $B_j$, $C_k$, $D_l$ and $E_m$. The number of scalars built out of $A_i$, $B_j$, $C_k$, $D_l$ and $D_m$ is the number of times that 1 appears in the decomposition of

$$8 \otimes 8 \otimes 8 \otimes (1 \oplus 8 \oplus 27) \tag{38}$$

and this is easily found to be sixteen. It is an amusing exercise to verify that, if the invariant tensors given in Eqs. (34) are in fact contracted against

$A_i, B_j, C_k, D_l$ and $D_m$, then only sixteen independent scalars do emerge. (The elimination of the dependent quantities is easily done using Eqs. (30) and (32).) Also the number of scalars built out of $A_i, A_j, A_k, B_l$ and $B_m$ is the number of 1's in

$$(8 \otimes 8 \otimes 8)_S \otimes (8 \otimes 8)_S \tag{39}$$

which is the number in

$$(1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 64) \otimes (1 \oplus 8 \oplus 27) \tag{40}$$

which is three. These are, of course,

$$d_{ijk} A_i A_j A_k B_l B_l,\ d_{ijk} A_i A_j B_k A_l B_l,\ d_{ijk} A_i B_j B_k A_l A_l. \tag{41}$$

Finally suppose we are forming $(8 \otimes 8)_{ij}$ tensors out of $A_k, B_l$ and $C_m$; there are 32 of these, this being the number of 1's in

$$(8_i \otimes 8_j) \otimes 8_k \otimes 8_l \otimes 8_m. \tag{42}$$

If we want to know how many of these 32 tensors are in each of the irreducible representations in the decomposition of $(8 \otimes 8)_{ij}$ we merely write the expression (42) as

$$(1_{ij} \oplus 8^S_{ij} \oplus 8^A_{ij} \oplus 10_{ij} \oplus \overline{10}_{ij} \oplus 27_{ij}) \otimes 8_k \otimes 8_l \otimes 8_m \tag{43}$$

and see that the $32 (8 \otimes 8)$ tensors can be rearranged into two singlets, eight symmetric octets, eight anti-symmetric octets, four 10's, four $\overline{10}$'s and six 27's. The singlets and octets can easily be written explicitly – the two scalars are $\delta_{ij}$ multiplied by each of the two scalars formed from $A_i, B_j$ and $C_k$, namely

$$f_{ijk} A_i B_j C_k,\ d_{ijk} A_i B_j C_k \tag{44}$$

and the two sets of octets are $d_{ijk}$ and $f_{ijk}$ contracted into the set of eight vectors formed from $A_l, B_m$ and $C_n$. Using the set (20) we may write

$$8^A_{ij} = \begin{matrix} f_{ijk}(\delta_{kl}\delta_{mn})\, A_l B_m C_n, \\ f_{ijk}(\delta_{km}\delta_{ln})\, A_l B_m C_n, \\ f_{ijk}(\delta_{kn}\delta_{lm})\, A_l B_m C_n, \\ f_{ijk}(d_{klp} d_{mnp})\, A_l B_m C_n, \\ f_{ijk}(d_{kmp} d_{lnp})\, A_l B_m C_n, \\ f_{ijk}(d_{klp} f_{mnp})\, A_l B_m C_n, \\ f_{ijk}(d_{kmp} f_{lnp})\, A_l B_m C_n, \\ f_{ijk}(d_{knp} f_{lmp})\, A_l B_m C_n. \end{matrix} \tag{45}$$

and similarly for $8^S_{ij}$.

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