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New "Coherent" States Associated with Non-Compact Groups*

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Abstract. Generalized "Coherent" States are the eigenstates of the lowering and raising operators of non-compact groups. In particular the discrete series of representations of $SO(2, 1)$ are studied in detail: the resolution of the identity and the connection with the Hilbert spaces of entire functions of growth (1, 1). Also discussed are the application to the evaluation of matrix elements of finite group elements and the contraction to the usual coherent states.

I. Introduction

The definition and use of coherent states associated with the Heisenberg algebra is well known (Section II). The purpose of this paper is to generalize this notion to the Lie algebra of non compact groups. In particular, we deal with the simplest semi-simple Lie algebra of $SO(2, 1)$ isomorphic to the algebra of $SU(1, 1)$ and $SL(2, R)$. We call generalized "coherent" states the eigenstates of the ladder operators in the discrete series of representations. Generalizations of these continuous bases will be indicated. The new "coherent" states are useful mathematically, aside from their intrinsic interest, in the evaluation of matrix elements of the finite transformations of the group and will have physical applications as the ordinary coherent states have.

II. Coherent States Associated with the Heisenberg Algebra

In this Section we review briefly, for reference purposes, some important properties of the usual coherent states[1] which are introduced

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[1] For more details see, for example, Klauder, J. R.: Ann. Phys. (N. Y.) **11**, 123 (1960); Glauber, J.: Phys. Rev. **131**, 2766 (1963), and Klauder, J., Sudarshan, E. C. G.: Quantum Optics. New York: Benjamin 1968.

as the eigenstates in an Hilbert space of the boson annihilation operator

$$a|z\rangle = z|z\rangle, \qquad [a, a^+] = \mathbb{1}, \tag{2.1}$$

where z is a complex eigenvalue. In terms of the eigenstates $|n\rangle$ of $a^+ a$ (or the Hamiltonian of the linear harmonic oscillator) one obtains

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle . \tag{2.2}$$

The factor in front of the sum is so chosen that the states are normalized

$$\langle z|z\rangle = 1 . \tag{2.3}$$

But they are not orthogonal to each other

$$\langle z'|z\rangle = \exp[-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|z'|^2 + z'^* z] \tag{2.4}$$

so that they form an over-complete linearly dependent set. The resolution of the identity holds in the form

$$\frac{1}{\pi} \int d^2 z\, |z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1\,;\; d^2 z = d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z) . \tag{2.5}$$

It has been recently shown [1] under which conditions countable subsets are complete. In particular one can take lattice points $z_{mn} = \gamma(m + in)$, $m, n = 0, \pm 1, \pm 2, \ldots$. For $0 < \gamma < \sqrt{\pi}$ the set of states $\{|z_{m,n}\rangle\}$ is still overcomplete; for $\gamma > \sqrt{\pi}$ it is not complete. For $\gamma = \sqrt{\pi}$ one obtains a complete set (von Neumann case).

An arbitrary vector f can be expanded

$$|f\rangle = \frac{1}{\pi} \int d^2 z\, |z\rangle \langle z|f\rangle . \tag{2.6}$$

The coefficients satisfy the equation

$$\langle z|f\rangle = \frac{1}{\pi} \int d^2 z' \langle z|z'\rangle \langle z'|f\rangle \tag{2.7}$$

so that $\langle z|z'\rangle$, Eq. (2.4), acts as a reproducing kernel.

The coherent states are special quantum states most closely approximating classical states in the sense that for them the uncertainty relation $\Delta p \Delta q \geqq \hbar/2$ has its minimum value. This can be seen by comparing the ground state wave function of the oscillator with the relation

$$\langle z|n\rangle = e^{-\frac{1}{2}|z|^2} \frac{z^{*n}}{\sqrt{n!}} .$$

III. Coherent States Associated with the Lie-Algebra of $SU(1, 1)$

$SU(1,1) \sim SO(2,1) \sim SL(2,R)$.

The Lie-algebra is defined by the commutation relations [2]

$$[L^+, L^-] = -L_{12}, [L_{12}, L^\pm] = \pm L^\pm . \tag{3.1}$$

The Casimir operator is given by

$$Q = -L_{12}(L_{12}-1) + 2L^+L^- = -L_{12}(L_{12}+1) + 2L^-L^+ . \tag{3.2}$$

Note that $L^\pm = \frac{1}{\sqrt{2}}(L_{13} \pm i L_{23})$. The fundamental spinor-representation of the algebra is

$$L_{12} = \tfrac{1}{2}\sigma_3, \quad L^\pm = \frac{i}{2\sqrt{2}}(\sigma_1 \pm i\sigma_2). \tag{3.3}$$

The corresponding parametrization of the group is defined by the Euler angles such that the group element is

$$W = e^{i\mu L_{12}} e^{i\xi L_{23}} e^{i\nu L_{12}}$$

$$= \begin{pmatrix} e^{i\frac{\mu}{2}} & \\ & e^{-i\frac{\mu}{2}} \end{pmatrix} \begin{pmatrix} \cos\left(\varepsilon\frac{\xi}{2}\right) & \sin\left(\varepsilon\frac{\xi}{2}\right) \\ -\sin\left(\varepsilon\frac{\xi}{2}\right) & \cos\left(\varepsilon\frac{\xi}{2}\right) \end{pmatrix} \begin{pmatrix} e^{i\nu/2} & \\ & e^{-i\nu/2} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha = e^{i(\mu+\nu)/2}\cos\left(\varepsilon\frac{\xi}{2}\right), \quad \beta = e^{i(\mu-\nu)/2}\sin\left(\varepsilon\frac{\xi}{2}\right) \tag{3.4}$$

$\det W = 1$, $\varepsilon = i$ for $SU(1,1)$, $\varepsilon = 1$ for $SO(3)$.

1. Discrete Representations $D^+(\Phi)$

The discrete class of unitary representations in the Hilbert space with the basis vectors $|\Phi, m\rangle$ can be defined by the following relations [2]

$$\begin{aligned} L_{12}|\Phi, m\rangle &= (E_0 + m)|\Phi, m\rangle \\ L^+|\Phi, m\rangle &= \frac{1}{\sqrt{2}}[(\Phi + E_0 + m + 1)(E_0 - \Phi + m)]^{\frac{1}{2}}|\Phi, m+1\rangle \\ L^-|\Phi, m\rangle &= \frac{1}{\sqrt{2}}[(\Phi + E_0 + m)(E_0 - \Phi + m - 1)]^{\frac{1}{2}}|\Phi, m-1\rangle \\ Q|\Phi, m\rangle &= \Phi(\Phi+1)|\Phi, m\rangle . \end{aligned} \tag{3.5}$$

Here E_0 is an arbitrary invariant (coming from the universal covering group). For discrete representations E_0 and the Casimir invariant are not independent

$$\Phi + E_0 = 0 . \tag{3.6}$$

The spectrum of L_{12} is discrete and bounded below:

$$L_{12} - E_0 = L_{12} + \Phi = 0, 1, 2, 3 \ldots . \tag{3.7}$$

For unitary representations

$$\begin{gathered} \operatorname{Im} E_0 = 0, \ \Phi < 0, \ \langle \Phi, m | L^- L^+ | \Phi, m \rangle = \text{real} > 0 \\ -2\Phi = 1, 2, 3 \ldots . \end{gathered} \tag{3.8}$$

The representations with Φ and $-\Phi - 1$ are equivalent. The inner product in terms of the basis states is

$$\langle \Phi, m | \Phi, m' \rangle = \delta_{mm'} . \tag{3.9}$$

2. Diagonalization of L^- in $D^+(\varphi)$: "Coherent" States

We introduce the generalized coherent states as the eigenvectors of L^-. From (3.5)

$$L^- |\Phi, m\rangle = \frac{1}{\sqrt{2}} [(m(-2\Phi + m - 1)]^{\frac{1}{2}} |\Phi, m-1\rangle . \tag{3.10}$$

We define the eigenvectors $|z\rangle$ of L^- as linear combinations of the basis vectors $\{|\Phi, m\rangle\}$ which is a complete orthonormal set in the Hilbert space:

$$L^- |z\rangle = z |z\rangle, \qquad z = \text{any complex number}, \tag{3.11}$$

$$|z\rangle = \left(\sqrt{\Gamma(-2\Phi)}\right) \sum_{n=0}^{\infty} \frac{(\sqrt{2}\, z)^n}{[n!\, \Gamma(-2\Phi + n)]^{\frac{1}{2}}} |\Phi, n\rangle . \tag{3.12}$$

The adjoint states are given by

$$\langle z| = [\Gamma(-2\Phi)]^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\, z^*)^n}{[n!\, \Gamma(-2\Phi + n)]^{\frac{1}{2}}} \langle \Phi, n| . \tag{3.13}$$

Eq. (3.11) can directly be verified using (3.10).

The inner product of the new "coherent" states is

$$\langle z' | z \rangle = \Gamma(-2\Phi) \sum_{n=0}^{\infty} \frac{(2 z'^* z)^n}{n!\, \Gamma(-2\Phi + n)} = {}_0F_1(-2\Phi;\ 2 z'^* z) . \tag{3.14}$$

Hence the norm is given by

$$\| |z\rangle \|^2 = \langle z | z \rangle = \Gamma(-2\Phi) \sum_{n=0}^{\infty} \frac{|\sqrt{2}\, z|^{2n}}{n!\, \Gamma(-2\Phi + n)} = {}_0F_1(-2\Phi; 2|z|^2) \tag{3.15}$$

where

$$ {}_0F_1(c;z)=1+\frac{1}{c!}\,\frac{1}{1!}+\frac{1}{c(c+1)}\,\frac{z^2}{2!}+\frac{1}{c(c+1)(c+2)}\,\frac{z^3}{3!}+\cdots \quad (3.16) $$

is a confluent hypergeometric function (entire function in z).

The norm can also be written in terms of the Bessel functions of integer order (for $2\Phi=$ integer)

$$ \langle z|z\rangle=\Gamma(-2\Phi)\,(i\sqrt{2}|z|)^{2\Phi+1}J_{-2\Phi-1}(2\sqrt{2}\,i|z|) \quad (3.15') $$

because of the relation

$$ J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\;{}_0F_1(\nu+1;\;-z^2/4). \quad (3.17) $$

Again we see that the coherent states are overcomplete, and do not form an orthonormal set. Two vectors $|z'\rangle$ and $|z\rangle$ are orthogonal if the entire function ${}_1F_0(-2\Phi, 2z'^*z)$ has a zero at the point $2z'^*z$. There is only qualitative information about the location of such zeros. Clearly, if z'^*z is a positive number the entire function has no zeros. Also it is not very useful to normalize the vectors $|z\rangle$ by dividing it with the square root of the norm (3.15).

The connection to the Hilbert spaces of entire functions of the exponential type of growth (1, 1) will be treated in Section VI.

3. *The Adjoint Operator* L^+

In the *unitary* discrete representation $D^+(\Phi)$, we have immediately from $L^+=(L^-)^\dagger$

$$ \langle z|\,L^+=z^*\langle z|\,. \quad (3.18) $$

Hence

$$ \langle z'|\,L^+L^-\,|z\rangle=z'^*z\langle z'|z\rangle\,. \quad (3.19) $$

Consequently, we can directly verify that the Casimir operator acts as

$$ Q|z\rangle=(-L_{12}(L_{12}-1)+2L^+L^-)\,|z\rangle=\Phi(\Phi+1)\,|z\rangle\,, \quad (3.20) $$

as it should.

We also easily obtain the coefficients $\langle\Phi,m|z\rangle$

$$ \langle\Phi,m|z\rangle=[\Gamma(-2\Phi)]^{\frac{1}{2}}\frac{\sqrt{2}\,z^m}{[m!\,\Gamma(-2\Phi+m)]^{\frac{1}{2}}}\,. \quad (3.21) $$

4. Resolution of the Identity

The problem here consists in finding a weight function $\sigma(z)$ such that

$$\int d\sigma(z)\,|z\rangle\,\langle z| = \sum_{m=0}^{\infty} |\Phi, m\rangle\,\langle m, \Phi| = \mathbb{1}. \tag{3.22}$$

Let $|f\rangle$ and $|g\rangle$ be two arbitrary vectors in $\mathscr{H}$; then Eq. (3.22) means that

$$\langle f|g\rangle = \int d\sigma(z)\,\langle f|z\rangle\,\langle z|g\rangle\,. \tag{3.23}$$

We shall now determine $\sigma(z)$. Let

$$d\sigma(z) = \sigma(r)\, r\, d\, r\, d\theta,\ r = |z|\,.$$

Then

$$\begin{aligned}
&\langle f|g\rangle \\
&= \int_0^\infty r\sigma(r)dr \int_0^{2\pi} d\theta \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\sqrt{2}z^*)^m(\sqrt{2}z)^n\,\Gamma(-2\Phi)}{(m!\,n!\,\Gamma(-2\Phi+n)\,\Gamma(-2\Phi+m))^{\frac{1}{2}}}\,\langle f|n\rangle\,\langle m|g\rangle \\
&= \frac{\Gamma(-2\Phi)}{\sqrt{2}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\langle f|n\rangle\,\langle m|g\rangle}{(m!\,n!\,\Gamma(-2\Phi+n)\,\Gamma(-2\Phi+m))^{\frac{1}{2}}} \\
&\qquad \times \int_0^\infty \int_0^{2\pi} (\sqrt{2}r)^{m+n+1}\, e^{i\theta(n-m)}\,\sigma(r)\,dr\,d\theta \\
&= \frac{2\pi}{\sqrt{2}}\,\Gamma(-2\Phi) \sum_{n=0}^{\infty} \frac{\langle f|n\rangle\,\langle n|g\rangle}{n!\,\Gamma(-2\Phi+n)} \int_0^\infty (\sqrt{2}r)^{2n+1}\,\sigma(r)\,dr\,.
\end{aligned} \tag{3.24}$$

Hence we must have

$$\frac{2\pi\Gamma(-2\Phi)}{\sqrt{2}} \int_0^\infty (\sqrt{2}r)^{2n+1}\,\sigma(r)\,dr = \Gamma(n+1)\,\Gamma(-2\Phi+n)\,. \tag{3.25}$$

Eq. (3.25) is a Mellin transform. We start from the formula [3]:

$$\int_0^\infty 2x^{\alpha+\beta} K_{\alpha-\beta}(2x^{\frac{1}{2}})\, x^{s-1} dx = \Gamma(2\alpha+s)\,\Gamma(2\beta+s) \tag{3.26}$$

where

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\pi\nu} \tag{3.27}$$

is the modified Bessel function of the third kind and

$$I_\nu(z) = \sum_0^\infty n\,\frac{(\frac{1}{2}z)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \tag{3.28}$$

is the modified Bessel function of the first kind. Substituting $x^{\frac{1}{2}} = \sqrt{2}r$, $\alpha = \frac{1}{2}$, $2\beta = -2\Phi$ in (3.26) and rearranging terms we obtain

$$4\sqrt{2} \int_0^\infty (\sqrt{2}r)^{-2\Phi-1} K_{\frac{1}{2}+\Phi}(2\sqrt{2}r)(\sqrt{2}r)^{2n+1}\,dr = \Gamma(n+1)\,\Gamma(-2\Phi+n).$$

Thus, comparing with (3.25) we obtain finally the desired weight function

$$\sigma(r) = \frac{4}{\pi\Gamma(-2\Phi)} (\sqrt{2}r)^{-2\Phi-1} K_{\frac{1}{2}+\Phi}(2\sqrt{2}r),$$
$$\sigma(r) > 0, \quad r > 0.$$

There is no problem of convergence of the integrals for $r \to 0$. The change of summation and integration in (3.24) may be rigorously justified by taking finite limit on the integration and going to the limit.

5. Diagonalization of L^+ in $D^-(\Phi)$

The unitary discrete series of representations $D^-(\Phi)$ are bounded above. Instead of (3.6), (3.7), and (3.10) we have

$$\begin{aligned} D^-(\Phi): \Phi - E_0 &= 0 \\ \operatorname{Im} E_0 = 0, \Phi < 0 \;\; 2\Phi &= -1, -2, \ldots \\ L_{12} - E_0 &= 0, -1, -2, \ldots, \end{aligned} \tag{3.29}$$

$$L^+|\Phi, -m\rangle = \frac{1}{\sqrt{2}} [m(-2\Phi+m-1)]^{\frac{1}{2}} |\Phi, -m+1\rangle, \; m > 0.$$

Consequently we can find eigenstates of L^+ analogous to Section II. It is easy to verify that

$$L^+|z\rangle = z|z\rangle$$

$$|z\rangle = [\Gamma(-2\Phi)]^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\sqrt{2}z)^m}{(m!\,\Gamma(-2\Phi+m))^{\frac{1}{2}}} |\Phi, -m\rangle \tag{3.30}$$

and that these states have all the properties of the eigenstates of L^- in $D^+(\Phi)$.

6. Remark

Continuous basis for $SO(2,1) \simeq SU(1,1) \simeq SL(2,R)$ have been studied before by a number of authors [4]. With the exception of Vilenkin, who considered this problem in a different context, in all these studies a non-compact, self-adjoint generator has been diagonalized. What we did here amounts to a diagonalization of a non-compact, non self-adjoint generator (lowering or raising operators). This was easy in the case of the discrete series D^+ or D^-. In the case of the principal or the supplementary series, however, this cannot be done in the Hilbert space: these representations are unbounded both below and above, hence the corresponding eigenvectors of L^- and L^+ would have infinite norm. In these cases appeal must be made to more general spaces, such as the rigged Hilbert spaces. (This problem will be studied elsewhere.)

IV. Matrix Elements of Finite Group Transformations

1. First we calculate the matrix elements of L_{12} in the coherent state basis [L^- is diagonal in $D^+(\Phi)$]. We have immediately, using (3.21),

$$\begin{aligned}\langle z'|L_{12}|z\rangle &= \sum_{n=0}^{\infty} \langle z'|L_{12}|\Phi, n\rangle \langle \Phi, n|z\rangle \\ &= \sum_{n=0}^{\infty} (-\Phi+n) \langle z'|\Phi, n\rangle \langle \Phi, n|z\rangle \\ &= -\Phi\langle z'|z\rangle + \Gamma(-2\Phi) \sum_{n=0}^{\infty} \frac{n(2z'^*z)^n}{n!\,\Gamma(-2\Phi+n)} \\ &= -\Phi\, {}_0F_1(-2\Phi, 2z'^*z) - \frac{z'^*z}{\Phi}\, {}_0F_1(-2\Phi+1; 2z'^*z)\end{aligned} \tag{4.1}$$

2. In this Section we evaluate the matrix elements of a general group element (3.4) in the continuous "coherent" basis $|z\rangle$. We can change the parameters such that

$$\begin{aligned}W &= \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \begin{pmatrix}e^{\frac{a}{2}} & 0\\ 0 & e^{-\frac{a}{2}}\end{pmatrix} \begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}, \quad a, b, c \text{ real. } (SL(2R)) \\ &= e^{-ibL^+} e^{aL_{12}} e^{-icL^-}\end{aligned} \tag{4.2}$$

det $W=1$ is satisfied.

Then

$$\begin{aligned}&\langle z'|e^{-ibL^+} e^{aL_{12}} e^{-icL^-}|z\rangle \\ &\quad = e^{i(bz'^*-cz)} \sum_{n=0}^{\infty} \langle z'|e^{aL_{12}}|\Phi, m\rangle \langle \Phi, m|z\rangle \\ &\quad = e^{i(bz'^*-cz)} e^{-a\Phi} \Gamma(-2\Phi) \sum_{m=0}^{\infty} e^{am} \frac{(2z'^*z)^m}{m!\,\Gamma(-2\Phi+m)}.\end{aligned}$$

Or, with $e^a=\delta^{-2}$,

$$= (\delta)^{2\Phi} e^{i(bz'^*-cz)}\, {}_0F_1\left(-2\Phi; \frac{2z'^*z}{\delta^2}\right). \tag{4.3}$$

V. Contraction to the Usual Coherent States

Referring to Eq. (3.1) we define ($L_3 \equiv L_{12}$)

$$L^{+\prime} = \sqrt{\varepsilon}\, L^+ , \; L^{-\prime} = \sqrt{\varepsilon}\, L^- , \; L_3' = \varepsilon L_3 , \; \varepsilon > 0 . \tag{5.1}$$

Then

$$[L^{+\prime}, L^{-\prime}] = -L_3', \; [L_3', L^{\pm\prime}] = \pm \varepsilon L^{\pm\prime} . \tag{5.2}$$

Hence in the limit $\varepsilon \to 0$,

$$[L^{+\prime}, L^{-\prime}] = -L_3', \; [L_3', L^{\pm\prime}] = 0 . \tag{5.3}$$

This contracted algebra is isomorphic to the Heisenberg algebra: We have the correspondence with the boson creation and annihilation operators:

$$\begin{gathered} L^{-\prime} \to a, \; L^{+\prime} \to a^+, \; L_3' \to I \\ [a, a^+] = 1, \; [I, a^+] = [I, a] = 0 . \end{gathered} \tag{5.4}$$

Next we consider the matrix elements. From (3.5) for $D^+(\Phi)$:

$$\begin{aligned} \langle \Phi, m' | \, L_3 \, | \Phi, m \rangle &= (-\Phi + m)\, \delta_{m'm} \\ \langle \Phi, m' | \, L^+ \, | \Phi, m \rangle &= \frac{1}{\sqrt{2}} \, [(m+1)(-2\Phi + m)]^{\frac{1}{2}} \, \delta_{m'+1,m} \\ \langle \Phi, m' | \, L^- \, | \Phi, m \rangle &= \frac{1}{\sqrt{2}} \, [m(-2\Phi + m - 1)]^{\frac{1}{2}} \, \delta_{m'-1,m} \, . \end{aligned} \tag{5.5}$$

We evaluate the limit of these matrix elements as [5]

$$\varepsilon \to 0, \Phi \to -\infty, \quad \text{but} \quad \varepsilon \Phi \to -1 , \tag{5.6}$$

i.e. through a sequence of representations, and obtain

$$\begin{aligned} \langle \Phi, m' | \, L_3' | \Phi, m \rangle &= \varepsilon(-\Phi + m)\, \delta_{m'm} \to \delta_{m'm} \\ \langle \Phi, m' | \, L^{+\prime} | \Phi, m \rangle &= \frac{1}{\sqrt{2}} \, (\varepsilon (m+1)(-2\Phi + m))^{\frac{1}{2}} \to (m+1)^{\frac{1}{2}} \, \delta_{m'-1,m} \\ \langle \Phi, m' | \, L^{-\prime} | \Phi, m \rangle &= \frac{1}{\sqrt{2}} \, [\varepsilon m(-2\Phi + m - 1)]^{\frac{1}{2}} \to m^{\frac{1}{2}} \, \delta_{m'+1,m} \, . \end{aligned} \tag{5.7}$$

These limits are precisely the matrix elements of the boson operators

$$\begin{gathered} \langle m' | I | m \rangle = \delta_{m',m} ; \; \langle m' | a^+ | m \rangle = (m+1)^{\frac{1}{2}} \, \delta_{m'-1,m'} \\ \langle m' | a | m \rangle = m^{\frac{1}{2}} \, \delta_{m'+1,m} \, . \end{gathered} \tag{5.8}$$

Now we can evaluate the limit of our coherent states. Because the limit of an infinite linear combinations of states $|\Phi, m\rangle$ is not defined, we evaluate the limit of the norm. The eigenstates of $L^{-\prime}=\sqrt{\varepsilon}\,L^{-}$ are

$$|z'\rangle=\sum_{m=0}^{\infty}\frac{(\sqrt{2}\,z')^m(\Gamma(-2\Phi))^{\frac{1}{2}}}{(\varepsilon^m m!\,\Gamma(-2\Phi+m))^{\frac{1}{2}}}\,|\phi,m\rangle \tag{5.9}$$

with the norm

$$\langle z'|z'\rangle_\Phi=\Gamma(-2\Phi)\sum_{n=0}^{\infty}\frac{(2|z|^2)^n}{n!\,\varepsilon^n\,\Gamma(-2\Phi+n)}={}_0F_1\left(-2\Phi,\frac{2|z|^2}{\varepsilon}\right).$$

Now because

$$\begin{gathered}\lim_{\substack{\varepsilon\to 0\\ \Phi\to-\infty\\ \varepsilon\Phi\to -1}}\frac{\Gamma(-2\Phi)}{\varepsilon^n\,\Gamma(-2\Phi+n)}=\frac{1}{2^n},\\ \lim_{\substack{\varepsilon\to 0\\ \Phi\to-\infty}}\langle z'|z'\rangle_\Phi=\sum_n\frac{|z|^{2n}}{n!}=e^{|z|^2}\end{gathered} \tag{5.10}$$

which is exactly the square of the norm of the usual coherent states (see Eqs. (2.2) and (2.4)). Or in terms of the Bessel functions we have the new relation

$$\begin{aligned}\lim_{\substack{\varepsilon\to 0\\ \Phi\to-\infty}}{}_0F_1\left(-2\Phi,\frac{2}{\varepsilon}|z|^2\right)&=\lim_{\substack{\varepsilon\to 0\\ \Phi\to-\infty}}\Gamma(-2\Phi)\left[\sqrt{\frac{2}{\varepsilon}}\,i|z|\right]^{2\Phi+1}J_{-2\Phi-1}\left(\frac{2\sqrt{2}}{\sqrt{\varepsilon}}\,i|z|\right)\\ &=e^{|z|^2}.\end{aligned} \tag{5.11}$$

VI. The Connection with the Hilbert Spaces of Entire Functions of the Exponential Type

1.

Let $|f\rangle$ denote an arbitrary vector of the Hilbert space and let us consider the function[2]

$$f_\Phi(z)\equiv\langle f|z\rangle_\Phi=\sqrt{\Gamma(-2\Phi)}\sum_{n=0}^{\infty}\frac{(z)^n}{\sqrt{n!\,\Gamma(-2\Phi+n)}}\langle f|\Phi,n\rangle \tag{6.1}$$

with

$$\sum_{n=0}^{\infty}|\langle f|\Phi,n\rangle_\Phi|^2<\infty.$$

[2] In this section we prefer to write z instead of $\sqrt{2}z$ for $|z\rangle$. Consequently the measure changes slightly as indicated in (6.3). Furthermore we shall not write the label ϕ when no confusion arises.

As it can be easily seen, $f_\Phi(z)$ is an entire analytic function of order 1 and type 1 (exponential type), i.e., growth (1, 1). It is clear that $f_\Phi(z)$ uniquely determines $|f\rangle$ and *vice versa.* It is obvious at this point that we can state a connection between the eigenstates of L^- in $D^+(\Phi)$ (or of L^+ in $D^-(\Phi)$) and the Hilbert spaces of entire analytic functions or growth (1, 1), in the same way as the usual coherent states are connected to the Segal-Bargmann [6, 7] space of entire functions of growth ($\frac{1}{2}$, 2).

We introduce the countable set of Hilbert spaces $\mathscr{F}_\Phi$, whose elements are entire analytic functions. For each Φ, the inner product is defined by[2]

$$(f,g)_\Phi=\int \bar{f}(z)\,g(z)\,d\sigma_\Phi(z)\,, \tag{6.2}$$

$$d\sigma_\Phi(z)=\frac{4}{2\pi\Gamma(-2\Phi)}\,r^{-2\Phi-1}K_{\frac{1}{2}+\Phi}(2r)\,r\,d\theta\,dr\,. \tag{6.3}$$

f belongs to $\mathscr{F}_\Phi$ if and only if $(f,f)_\Phi<\infty$; its norm is $\|f\|_\Phi=\sqrt{(f,f)_\Phi}$. Let $f(z)$ be an entire function with the power series $\sum_n c_n z^n$. The norm in terms of the expansion coefficients is given by

$$(f,f)_\Phi^{\frac{1}{2}}=\left[[\Gamma(-2\Phi)]^{-1}\sum_0^\infty |c_n|^2\,n!\,\Gamma(-2\Phi+n)\right]^{\frac{1}{2}}. \tag{6.4}$$

Every set of coefficients c_n for which the sum in (6.4) converges defines an entire function $f\in\mathscr{F}_\Phi$. From the linearity we get the inner product of two functions f, g:

$$\begin{aligned}(f,g)_\Phi&=[\Gamma(-2\Phi)]^{-1}\sum_{n=0}^\infty \bar{c}_n b_n n!\,\Gamma(-2\Phi+n)\\ g(z)&=\sum_{n=0}^\infty b_n z^n\,.\end{aligned} \tag{6.5}$$

An orthonormal set of vectors in $\mathscr{F}_\Phi$ is given by

$$u_{n,\Phi}(z)=[\Gamma(-2\Phi)]^{\frac{1}{2}}\frac{z^n}{\sqrt{n!\,\Gamma(-2\Phi+n)}}\,. \tag{6.6}$$

For any function $f\in\mathscr{F}_\Phi$,

$$(u_n,f)_\Phi=c_n\frac{\sqrt{n!\,\Gamma(-2\Phi+n)}}{\sqrt{\Gamma(-2\Phi)}}\,.$$

Eq. (6.4) expresses the completeness of the system $u_m(\Phi, z)$. The Schwarz inequality gives

$$|f(z)|^2 \leqq \left(\sum_{n=0}^{\infty} |c_n z^n|\right)^2 \leqq \left(\sum_{n=0}^{\infty} |c_n|^2 n!\,\Gamma(-2\Phi+n)\,[\Gamma(-2\Phi)]^{-1}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{|z|^2}{n!\,\Gamma(-2\Phi+n)}\,\Gamma(-2\Phi)\right).$$

from which

$$|f(z)|^2 \leqq \|f\|_\Phi^2\, {}_0F_1(-2\Phi; |z|^2)$$

or

$$|f(z)| \leqq \|f\|_\Phi ({}_0F_1(-2\Phi; |z|^2))^{\frac{1}{2}}. \tag{6.7}$$

As a consequence, strong convergence in $\mathscr{F}_\Phi$ implies pointwise convergence, because

$$|f(z) - g(z)| \leqq [{}_0F_1(-2\Phi; |z|^2)]^{\frac{1}{2}} \|f - g\|_\Phi \tag{6.8}$$

for any $f, g \in \mathscr{F}_\Phi$ and from (6.7) we see that the convergence is uniform on any compact set.

2. *The Principal Vector and the Reproducing Kernel*

Following Bargmann [6] we can introduce the *principal vectors* and the reproducing kernel for each $\mathscr{F}_\Phi$. For a fixed complex number a, the mapping $f \to f(a)$ defines a bounded linear functional. Because f is an element of an Hilbert space $\mathscr{F}_\Phi$, the functional is of the form

$$f(a) = (e_a, f)_\Phi, \tag{6.9}$$

where e_a is uniquely defined in $\mathscr{F}_\Phi$. The vectors e_a are called the principal vectors of $\mathscr{F}_\Phi$ and they behave like a continuous set of orthonormal vectors. In particular

$$(f, g)_\Phi = \int (f, e_a)_\Phi (e_a, g)_\Phi\, d\sigma_\Phi(a). \tag{6.10}$$

This expression corresponds to the expression for the resolution of the identity in terms of the eigenstates $|a\rangle_\Phi$ of L^- in $D^+(\Phi)$ (or L^+ in $D^-(\Phi)$) (see (3.23)). Thus $(f, e_a)_\Phi$ corresponds to $\langle f|a\rangle_\Phi$ which is the functional representative of the abstract vector $|f\rangle$. The vectors e_a are complete: i.e. their finite linear combinations are dense in $\mathscr{F}_\Phi$, because the only vector orthogonal to all of them is $f \equiv 0$. This is due to the fact that $(e_a, f)_\Phi$ is an entire function of a. In integral form, (6.10) reads

$$f(a) = \int W_\Phi(a, z)\, f(z)\, d\sigma_\Phi(z) \tag{6.11}$$

and $W_\Phi(a, z)$ is the "reproducing kernel". Because $e_a(z) = (e_z, e_a)_{\Phi}$, we have

$$W(a, z)_\Phi = \overline{W(z, a)_\Phi} = (e_a, e_z)_\Phi \tag{6.12}$$

and $W(a, z)$ is analytic in a and $\bar{z}$. It is the analog of the delta function in the usual Hilbert space of quantum mechanics. In terms of any complete orthonormal discrete set $v_1, v_2 \ldots$, we have, by (6.9):

$$e_a = \sum_{n=0}^{\infty} (v_n, e_a)\, v_n = \sum_{n=0}^{\infty} \overline{v_n(a)}\, v_n \tag{6.13}$$

and since strong convergence implies pointwise convergence, we have

$$e_a(z) = \sum_{n=0}^{\infty} \overline{v_n(a)}\, v_n(z)\,, \tag{6.14}$$

irrespective of the choice of the system $\{v_n\}$. Using the set $u_{n,\Phi}(z)$ (6), we find

$$\begin{aligned} e_a(z) &= \sum_{n=0}^{\infty} u_n(a)\, u_n(z) \\ &= \Gamma(-2\Phi) \sum_{o}^{\infty} \frac{(\bar{a}z)^n}{n!\,\Gamma(-2\Phi+n)} = {}_0F_1(-2\Phi; \bar{a}z) \end{aligned} \tag{6.15}$$

or

$$W(a, z)_\Phi = {}_0F_1(-2\Phi; a\bar{z})\,.$$

Likewise from (9) and (7) we find again

$$\|e_a\|_\Phi = ({}_0F_1(-2\Phi; a\bar{a}))^{\frac{1}{2}}\,. \tag{6.16}$$

We have thus the reproducing formula

$$\begin{aligned} f(a) &= \int W_\Phi(a, z)\, f(z)\, d\sigma_\Phi(z) \\ &= \int {}_0F_1(-2\Phi; a\bar{z})\, f(z)\, d\sigma_\Phi(z)\,, \qquad \forall f \in \mathscr{F}_\Phi. \end{aligned} \tag{6.17}$$

This could have been established also by looking at the corresponding formula in terms of the "coherent" states:

$$f(a) = \langle f | a \rangle = \int d\sigma_\Phi(z) \langle f | z \rangle \langle z | a \rangle\,, \tag{6.18}$$

from which we have the formal equality between the reproducing kernel and the inner product of two "coherent" states $|a\rangle_\Phi$ and $|z\rangle_\Phi$. Any bounded linear operator may be represented by means of the principal vectors as an integral transform. This may be adopted directly from Bargmann's case [6] and we shall not discuss here. Furthermore one can also carry an analysis similar to the one used in Ref. [1] concerning the problem of the „characteristic" sets or, in other words, the problem of the completeness of a countable subset of „coherent" states.

3. Realization of the Algebra $SO(2,1) \simeq SU(1,1) \simeq SL(2,R)$ in the Hilbert Spaces $\mathscr{F}_\Phi$ of Entire Functions

In the last section we have equipped the class of entire functions of growth (1,1) with a countable set of inner products $(f,f)_\Phi$ making them into a countable set of Hilbert spaces. We want to study in this section a realization of the generators of our algebra in these spaces. We consider here the case $D^+(\Phi)$. A similar analysis applies to the case $D^-(\Phi)$.

For each ϕ, we introduce the linear operators, acting on $\mathscr{F}_\Phi$:

$$\begin{aligned} \mathscr{L}_{12}^{(\Phi)} &\equiv z\frac{d}{dz} - \Phi, \\ \mathscr{L}_{+}^{(\Phi)} &\equiv \frac{1}{\sqrt{2}} z, \\ \mathscr{L}_{-}^{(\Phi)} &\equiv \frac{1}{\sqrt{2}}\left(-2\Phi\frac{d}{dz} + z\frac{d^2}{dz^2}\right). \end{aligned} \tag{6.19}$$

They satisfy the required commutation rules. In each $\mathscr{F}_\Phi$, $\mathscr{L}_{12}^{(\Phi)}$ is automatically diagonal in the orthonormal basis (6.6)

$$u_n^\Phi(z) = \sqrt{\Gamma(-2\Phi)}\ \frac{z^n}{\sqrt{n!\,\Gamma(-2\Phi+n)}}.$$

Furthermore, we have

$$\begin{aligned} \mathscr{L}_{+}^{\Phi} u_n^\Phi(z) &= \frac{1}{\sqrt{2}}\sqrt{(n+1)(-2\Phi+n)}\, u_{n+1}^\Phi(z) \\ \mathscr{L}_{-}^{\Phi} u_n^\Phi(z) &= \frac{1}{\sqrt{2}}\sqrt{n(-2\Phi+n-1)}\, u_{n-1}(z). \end{aligned} \tag{6.20}$$

It is easy to verify the unitarity of the realization:

$$(f, \mathscr{L}_{-}^{\Phi} g)_\Phi = (\mathscr{L}_{+}^{\Phi} f, g) \qquad \forall f, g \in \mathscr{F}_\Phi, \tag{6.21}$$

i.e. $(\mathscr{L}_{+}^{\Phi})^\dagger = \mathscr{L}_{-}^{\Phi}$.

By inspection of the confluent hypergeometric differential equation

$$z\frac{d^2}{dz^2}\varphi(z) - 2\Phi\frac{d}{dz}\varphi(z) - \sqrt{2}\lambda\,\varphi(z) = 0 \tag{6.22}$$

we find the eigenvectors of $\mathscr{L}_{-}^{\Phi}$, in $\mathscr{F}_\Phi$, to be

$$\varphi_\lambda(z) = {}_0F_1(-2\Phi, \sqrt{2}\lambda z) \tag{6.23}$$

with eigenvalues $\sqrt{2}\lambda$ where λ is any complex number.

We can understand now the meaning of $e_a^\Phi(z)$ in $\mathscr{F}_\lambda$: as it was to be expected $e_{\sqrt{2}\lambda}(z)$ with eigenvalue $\sqrt{2}\lambda$ are just the eigenvectors of $\mathscr{L}_{-}^{\Phi}$ or, in other words, they are the "coherent" states in $\mathscr{F}_\Phi$.

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