On a Class of Equilibrium States under the Kubo-Martin-Schwinger Condition

II. Bosons

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Abstract. We study equilibrium states of a quantum Bose gas using Kubo-Martin-Schwinger boundary conditions, for a special class of time evolutions, namely the quasi-free evolutions. Under suitable restrictions, in particular positivity of the elementary excitation spectrum, we are able to describe the states fulfilling the Kubo-Martin-Schwinger conditions. In contrast to the Fermi case the solution is, in general, not unique; this is related to a possible Bose condensation.

§ 1. Introduction

In a previous paper [1], we looked for the solutions of the Kubo-Martin-Schwinger boundary condition for Fermi systems and for a special class of evolutions. It is our goal to do a similar study for the Bose systems.

We shall not discuss the importance of the K.M.S. boundary condition within the framework of the algebraic description of equilibrium states of statistical mechanics but only refer to the fundamental papers [2–4], where one can find the formulation of the K.M.S. boundary conditions we shall use.

Definition 1.1. *Let $\mathscr{A}$ be a C^*-algebra, $t \to \alpha_t$, a homomorphism of the additive group of reals into *-automorphisms of $\mathscr{A}$: a state ω of $\mathscr{A}$ is said to be a K.M.S. state with respect to $t \to \alpha_t$, if, $\forall A, B \in \mathscr{A}$*

$$t \in R \to \omega(A\alpha_t B)$$

can be extended to an analytic function in the strip $0 < \operatorname{Im} t < \beta$, continuous on the boundary and such that

$$\omega(A\alpha_z B)\big|_{z=i\beta} = \omega(BA).$$

We notice incidentally that a K.M.S. state is automatically an invariant state [3, 5].

Equilibrium states of Bose systems have been extensively studied, especially in the fundamental papers of Araki [6] and Robinson [7];

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their interest lies in the well-known fact that Bose systems can exhibit Bose-condensation. We shall come back to this point in the last section.

The second section is devoted to notation and definitions, in particular to those connected with the C^*-algebra we shall use and its quasi-free states.

In a third section we define the quasi-free evolutions and exhibit necessary and sufficient conditions for the existence of a state fulfilling K.M.S. boundary conditions (we shall call this the K.M.S. problem).

In the fourth section, we solve the K.M.S. problem completely for the previous class of evolutions.

In the next section, we look at the limiting cases, i.e. the limits, when the temperature goes to zero or to infinity, of the states that we have defined.

§ 2. The C^*-Algebra of a Bose System

For Fermi systems, one has a natural choice of C^*-algebra, namely the Clifford algebra built on the one particle states. The situation here is quite different and for convenience we shall choose as C^*-algebra the one which is described in [8]. For the sake of brevity, we shall only sketch the construction of this algebra.

⌗ i. The One Particle Space

The one particle space (H, σ) is a real symplectic space, i.e.
 i) H is a real linear space
 ii) σ is a real bilinear, antisymmetric, non-degenerated form, i.e.

$$\sigma : H \times H \to \mathbb{R},$$
$$\sigma(\varphi, \alpha\psi_1 + \beta\psi_2) = \alpha\sigma(\varphi, \psi_1) + \beta\sigma(\varphi, \psi_2),$$
$$\forall \varphi, \psi_1, \psi_2 \in H \qquad \forall \alpha, \beta \in \mathbb{R}, \tag{2.1}$$
$$\sigma(\varphi, \psi) = -\sigma(\psi, \varphi),$$
$$\sigma(\varphi, \psi) = 0, \quad \forall \varphi \in H \Leftrightarrow \psi = 0.$$

A real linear operator T from H onto H will be said to be symplectic iff

$$\sigma(T\varphi, T\psi) = \sigma(\varphi, \psi) \qquad \forall \varphi, \psi \in H. \tag{2.2}$$

As a special case, we can define a complex structure J of H as a symplectic operator such that

$$J^2 = -1, \tag{2.3}$$

$$\sigma(\varphi, J\varphi) \geqq 0 \qquad \forall \varphi \in H. \tag{2.4}$$

╫ ii. The Algebra $\overline{\Delta(H, \sigma)}$

One considers the set $\Delta(H, \sigma)$ of functions with complex values defined on H which are zero, except on a finite number of points. Equipped with its natural structure as a complex linear space, with the product

$$(a.b)(\psi) = \sum_{\varphi \in H} a(\varphi)\, b(\psi - \varphi)\, e^{-i\sigma(\varphi, \psi)} \tag{2.5}$$

$$\forall a, b \in \Delta(H, \sigma), \forall \psi \in H$$

and with the involution

$$a^*(\psi) = \overline{a(-\psi)} \qquad \forall a \in \Delta(H, \sigma) \qquad \forall \psi \in H, \tag{2.6}$$

$\Delta(H, \sigma)$ is a *-algebra.

$\Delta(H, \sigma)$ can be generated by the set of δ_ψ, $\psi \in H$ defined by:

$$\begin{aligned} \delta_\psi(\varphi) &= 0 \quad \text{if} \quad \varphi \neq \psi, \\ \delta_\psi(\psi) &= 1 \end{aligned} \tag{2.7}$$

and satisfying the following canonical commutation relations

$$\delta_\psi.\delta_\varphi = \exp(-i\sigma(\psi, \varphi))\, \delta_{\psi + \varphi} \qquad \forall \psi, \varphi \in H. \tag{2.8}$$

δ_0 is the identity of $\Delta(H, \sigma)$ and the δ_ψ's are unitary elements in $\Delta(H, \sigma)$, namely

$$\delta_\psi.(\delta_\psi)^* = \delta_0. \tag{2.9}$$

Let $\mathscr{R}_1(H, \sigma)$ the set of non-degenerated representations π of $\Delta(H, \sigma)$ such that the map

$$\lambda \in \mathbb{R} \to \pi(\delta_{\lambda\psi}) \tag{2.10}$$

is weakly (or strongly) continuous; all these representations are faithfull and consequently induce the same norm on $\Delta(H, \sigma)$. The closure $\overline{\Delta(H, \sigma)}$ of $\Delta(H, \sigma)$, with respect to this norm is a C^*-algebra, we choose it as the C^*-algebra of a Bose system.

Let us mention two types of *-automorphisms of $\overline{\Delta(H, \sigma)}$ which will be important in the sequel. Firstly the one particle *-automorphisms: for every symplectic operator T of H (see definition (2.2)) there exists a unique automorphism α_T of $\overline{\Delta(H, \sigma)}$ such that

$$\alpha_T \delta_\psi = \delta_{T\psi} \qquad \forall \psi \in H. \tag{2.11}$$

As a special choice, the group of *-automorphisms induced by the group of symplectic transformations T_α:

$$T_\alpha = \cos\alpha\, 1 + \sin\alpha\, J \qquad \alpha \in [0, 2\pi] \tag{2.12}$$

where J is a complex structure of H, corresponds, as we shall see later, to gauge transformations of the first kind, otherwise α_T corresponds to generalized homogeneous Bogoliubov transformations.

Finally, consider χ, a character of H considered as an additive group; then there exists a unique $*$-automorphism of $\Delta(H, \sigma)$, such that:

$$\alpha_\chi \delta_\psi = \chi(\psi) \, \delta_\psi \qquad \forall \, \psi \in H \, . \tag{2.13}$$

Such $*$-automorphisms occur in inhomogeneous Bogoliubov transformations; e.g. if ϱ is a linear form on H, one can define

$$\chi_\varrho(\psi) = \exp(i \varrho(\psi)) \qquad \forall \, \psi \in H \, . \tag{2.14}$$

In that case, the $*$-automorphism (2.13) can be extended to $\overline{\Delta(H, \sigma)}$.

⚌ iii. Some Special Representations of $\overline{\Delta(H, \sigma)}$

We had chosen $\overline{\Delta(H, \sigma)}$ as the C^*-algebra of Bose systems in order to take account of the canonical commutation relations (C.C.R.); let us sketch the link of this algebra to the (C.C.R.) (see [9]). Let us define $\mathscr{R}_\infty(H, \sigma)$ the set of states of $\overline{\Delta(H, \sigma)}$ such that

$$\lambda \in \mathbb{R} \to \omega(\delta_{\lambda \psi + \varphi}) \qquad \forall \, \psi, \varphi \in H \tag{2.15}$$

is infinitely differentiable.

Let π_ω the corresponding representation constructed à la G.N.S. in the Hilbert space H_ω; continuity of the map

$$\lambda \in \mathbb{R} \to \pi_\omega(\delta_{\lambda \psi}) \qquad \forall \, \psi \in H \tag{2.16}$$

ensures the existence of the unbounded operator $B_\omega(\psi)$, $\forall \, \psi \in H$, which is real linear with respect to $\psi \in H$, densely defined and such that,

$$\pi_\omega(\delta_\psi) = \exp(i B_\omega(\psi)) \qquad \forall \, \psi \in H \, . \tag{2.17}$$

Moreover one has

$$B_\omega(\psi) \, B_\omega(\varphi) - B_\omega(\varphi) \, B_\omega(\psi) \subseteqq 2 \, i \, \sigma(\psi, \varphi) \, I_{H_\omega} \tag{2.18}$$

and if ξ_ω is the cyclic vector corresponding to ω then all the correlation functions $\left(\xi_\omega \Big| \prod_{i=1}^{N} B_\omega(\varphi_i) \, \xi_\omega \right)$, $\varphi_i \in H$, exist.

(2.18) is not the usual form of C.C.R. Actually let J be a complex structure of H defined in (2.3) and (2.4), then H can be turned into a complex Hilbert space H^c if one defines

$$(\alpha + \beta i) \, \psi = (\alpha + \beta J) \, \psi \qquad \forall \, \psi \in H^c \qquad \forall \, \alpha + i \beta \in \mathbb{C} \tag{2.19}$$

for the multiplication by complex numbers and

$$h_J(\psi, \varphi) = \sigma(\psi, J \varphi) + i \sigma(\psi, \varphi) \qquad \forall \, \psi, \varphi \in H^c \tag{2.20}$$

for scalar (hermitian) product.

Moreover, one defines creation and annihilation operators

$$B_\omega^\pm(\varphi) = B_\omega(\varphi) \mp i B_\omega(J \varphi) \qquad \forall \, \varphi \in H^c \tag{2.21}$$

which are respectively linear and antilinear in $\varphi \in H^c$ and satisfy:

$$B_\omega^\pm(\varphi)\, B_\omega^\pm(\psi) - B_\omega^\pm(\psi)\, B_\omega^\pm(\varphi) \leqq 0\,, \tag{2.22}$$

$$B_\omega^-(\psi)\, B_\omega^+(\varphi) - B_\omega^+(\varphi)\, B_\omega^-(\psi) \leqq h_J(\psi, \varphi)\, I_{H_\omega}\,. \tag{2.23}$$

We shall not detail any longer the correspondance with the usual formalism, especially with regard to the action of the $*$-automorphisms we have defined, but we shall come back to the states.

Let ω be a state over $\Delta(H, \sigma)$, it necessarily satisfies the positivity condition

$$\omega\!\left(\!\left(\sum_{i=1}^N c_i \delta_{\psi_i}\right)^{\!*}\!\left(\sum_{j=1}^N c_j \delta_{\psi_j}\right)\!\right)$$
$$= \sum_{i,j=1}^N \bar{c}_i c_j e^{+i\sigma(\psi_i,\psi_j)} \omega(\delta_{\psi_j - \psi_i}) \geqq 0 \tag{2.24}$$

$\forall \psi_i \in H$ and $c_j \in C$.

Moreover, if the map

$$\mathbb{R} \ni \lambda \to \omega(\delta_{\lambda\psi + \varphi}) \qquad \forall \psi, \varphi \in H \tag{2.25}$$

is continuous, ω can be extended to a state of $\overline{\Delta(H, \sigma)}$.

We consider now the quasi-free states of $\overline{\Delta(H, \sigma)}$, let $\mathfrak{S}$ be the set of real symmetric positive scalar products S on H, such that

$$|\sigma(\psi, \varphi)|^2 \leqq S(\psi, \psi)\, S(\varphi, \varphi) \qquad \forall \psi, \varphi \in H\,. \tag{2.26}$$

The non-degeneracy of σ shows that $S \in \mathfrak{S}$ is strictly positive. Completion of H with respect to the norm deduced from S will be denoted by $\overline{H}^S$. Let σ' be the continuous extension of σ to $\overline{H}^S$, it still satisfies (2.26) for every pair of vectors of $\overline{H}^S$, and so, by Riesz's representation theorem, there exists in $\overline{H}^S$ an operator D_S such that

$$\sigma'(\psi, \varphi) = S(D_S \psi, \varphi) \qquad \forall \psi, \varphi \in \overline{H}^S, \tag{2.27}$$

and which satisfies, for the Hilbert structure of $\overline{H}^S$,

$$D_S^* = -D_S\,, \tag{2.28}$$

$$\|D_S\| \leqq 1\,. \tag{2.29}$$

The polar decomposition of D_S (which holds in the real case):

$$D_S = J|D_S| \tag{2.30}$$

furnishes a complex structure J of $(1 - E)\,\overline{H}^S$, where E is the projection over the kernel of D_S. Notice that

$$\sigma'(E\psi, \varphi) = S(D_S E\psi, \varphi) = 0, \forall \psi, \varphi \in \overline{H}^S \tag{2.31}$$

and the operator D_S restricted to $(1 - E)\,\overline{H}^S$ has an inverse $-A_S$ which is not necessarily bounded.

Quasi-free states are defined out of elements of $\mathfrak{S}$ by

$$\omega(\delta_\psi) = \exp\left(-\tfrac{1}{2} S(\psi, \psi)\right) \quad \psi \in H, S \in \mathfrak{S}. \tag{2.32}$$

As a special case Fock states are defined by those elements of $\mathfrak{S}$ such that D_S is a complex structure.

Later on, we shall be faced with the problem of construction of quasi-free states out of an operator D_S. An important case will be the one where:

$$D_S H \subset H. \tag{2.33}$$

Conversely, we shall need the following result.

Proposition (2.34). *Let D be a real linear operator in H such that:*

D 1 $\sigma(\psi, D\varphi) = -\sigma(D\psi, \varphi) \quad \forall \varphi, \psi \in H$.

D 2 $-\sigma(D\psi, \psi) \geqq 0 \quad \forall \psi \in H$.

D 3 *D is injective.*

D 4 *Let $-A$ be the mapping of $DH \to H$, the inverse of D, then*

$$\sigma(\psi, (A-D)\psi) \geqq 0 \quad \forall \psi \in DH.$$

D 5 *The bilinear form $S: DH \times H \to \mathbb{R}$ defined by*

$$S(D\psi, \varphi) = \sigma(\psi, \varphi)$$

can be extended to a symmetric positive bilinear non-degenerated form S' from $H \times H \to \mathbb{R}$.

Then S' is in $\mathfrak{S}$.

Proof. D 4 shows that the extension $\bar{D}$ to $\overline{DH}^{S'}$ of D exists, indeed if $\psi \in DH$

$$S'(\psi, \psi) = S(\psi, \psi) = \sigma(\psi, A\psi) \geqq \sigma(\psi, D\psi) = S'(D\psi, D\psi).$$

D 1 shows that $\bar{D}$ is antisymmetric.

Let D' the operator defined on $\bar{H}^{S'}$ by

$$D' = \bar{D} P \tag{2.35}$$

where P is the projection on $\overline{DH}^{S'}$. One has

$$S'(D'\psi, D\varphi) = S'(\bar{D}P\psi, D\varphi)$$
$$\hspace{5cm} \forall \varphi, \psi \in H$$
$$= -S'(P\psi, D^2\varphi)$$

due to the antisymmetry of $\bar{D}$ and the fact that

$$\bar{D}\big|_{DH} = D\big|_{DH}$$

which implies that

$$S'(D'\psi, D\varphi) = - S'(\psi, D^2\varphi) = - S'(D^2\varphi, \psi)$$

since $D^2\varphi \in DH$.

Moreover, by D 5,

$$S'(D'\psi, D\varphi) = - \sigma(D\varphi, \psi) = \sigma(\psi, D\varphi)$$
$$= S(D\psi, D\varphi)$$

so that $D = D'$ on H, since $(\bar{D} - D)\psi \in \overline{DH}^{S'}$ is orthogonal to DH for ψ in H. Therefore the bilinear form σ' on $\bar{H}^{S'}$,

$$\sigma'(\psi, \varphi) = S'(D'\psi, \varphi) \tag{2.36}$$

coincides with σ on H.

Moreover $\|D'\|_{S'} \leq 1$ since $\|\bar{D}\|_{S'} \leq 1$ on $\overline{DH}^{S'}$, and S' fulfills the relation (2.26), hence the result follows.

§ 3. Quasi-free Evolutions and Special Solution of the K.M.S. Problem

We define the class of evolutions that we shall deal with.

Definition (3.1). *A quasi-free evolution of $\overline{\Delta(H, \sigma)}$ is an homomorphic mapping of the additive group of reals into the group of one-particle *-automorphisms of $\overline{\Delta(H, \sigma)}$ defined in* (2.11).

We denote by

$$t \in \mathbb{R} \to T_t \tag{3.2}$$

the corresponding homomorphism into the symplectic group of operators of H.

We shall restrict ourselves however to evolutions which satisfy the following condition;

E 1 *the mapping:*

$$t \in \mathbb{R} \to \sigma(\varphi, T_t\psi) \qquad \forall \psi, \varphi \in H$$

is continuous and bounded.

The necessity of these requirements rests on the following remarks.

Remark (3.3). *If* $t \to \sigma(\varphi, T_t\psi)$ *is not continuous, there exists no invariant state ω of $R_\infty(H, \sigma)$ such that*

$$t \in \mathbb{R} \to \omega(\delta_\varphi \alpha_t \delta_\psi)$$

is continuous.

The proof is obvious and quite similar to that of the:

Remark (3.4). *If the mapping*

$$t \in \mathbb{R} \to \sigma(\varphi, T_t\psi)$$

is not bounded, then there exists no $R_\infty(H, \sigma)$ *invariant state and, consequently, no K.M.S. state.*

Proof. Suppose that $\omega \in R_\infty(H, \sigma)$ and ω is α_t-invariant.

Let π_ω, H_ω, Ω_ω be the representation in the Hilbert space H_ω with cyclic vector Ω_ω constructed à la G.N.S. Let B_ω be the associated field operator, then

$$S(\psi, \varphi) = \tfrac{1}{2}\left(\Omega_\omega | B_\omega(\psi) B_\omega(\varphi) + B_\omega(\varphi) B_\omega(\psi) | \Omega_\omega\right) \tag{3.5}$$

is a symmetric bilinear form. T_t is orthogonal with respect to this form. Moreover, due to the positivity condition, S satisfies

$$|\sigma(\psi, T_t\varphi)|^2 \leqq S(\psi, \psi)\, S(\varphi, \varphi) \tag{3.6}$$

hence we get a contradiction.

Another condition will be necessary:

E 2   *the set h in H*

$$h = \{\psi \in H;\ \forall\, t \in R,\ T_t\psi = \psi\} \tag{3.7}$$

is reduced to $\{0\}$.

Indeed direct application of the K.M.S. boundary condition implies that for $\psi \in h$ and $\varphi \in H$, $\varphi \neq 0$

$$\omega(\delta_\psi \delta_\varphi \delta_{-\psi}) = \omega(\delta_\varphi) \tag{3.8}$$

so using (2.8)

$$\omega(\delta_\varphi)\,(e^{2i\sigma(\psi,\varphi)} - 1) = 0 \qquad \forall\, \varphi \in H\,. \tag{3.9}$$

If $\psi \neq 0$, choose a $\varphi \in H$, $\varepsilon > 0$ such that $0 < |\lambda| < \varepsilon$ implies

$$0 < \lambda\,\sigma(\varphi, \psi) < \pi$$

and so

$$\omega(\delta_{\lambda\varphi}) = 0\,,$$

that is, ω is not a Weyl state.

The next assumption is a little more difficult to justify within our general framework; nevertheless, its physical meaning is clear in the important case of a translation invariant evolution (see Section 6).

E 3   *For every ψ and φ in H and every real function $\hat{f}$ whose Fourier transform is in $\mathscr{D}$, there exists an element $T(\hat{f})\,\psi$ in H such that*

$$\sigma(\varphi, T(\hat{f})\,\psi) = \int\limits_{-\infty}^{+\infty} \hat{f}(t)\, \sigma(\varphi, T_t\psi)\, dt\,.$$

We assume, moreover, that H is the linear closure of the vectors $T(\hat{f})\,\psi$.

Notice that E 3 is fulfilled if H is $\sigma(H)$ – quasi-complete (cf. [9], p. 295 and [10] Prop. 21, Corr. 1).

With the help of the previous hypothesis, we have the following well-known lemma (compare e.g. to [4]):

Lemma 3.10 *For every φ and ψ in H the function*

$$t \in R \to \sigma(\varphi, T_t \psi)$$

can be extended to an analytic function. Furthermore, if ψ is of the form $T(\hat{f}) \xi, \xi \in H, \hat{f}$ a real function whose Fourier transform is in $\mathscr{D}$, then the function

$$z \to \int\limits_{-\infty}^{+\infty} dt \, \sigma(\varphi, T_t \xi) \, \widehat{f e_z}(t) \,,$$

with $e_z(\omega) = \exp(-2\pi i z \omega)$, realizes the extension.

The next lemma is close to some results obtained in [1].

Lemma 3.11. *There exist operators, $U_\beta, V_\beta, D_\beta$ and Z in H, such that for $\beta > 0$*

$$U_\beta \sum_{i=1}^{N} T(\hat{f}_i) \, \psi_i = \sum_{i=1}^{N} T(\widehat{f_i^{U_\beta}}) \, \psi_i \,,$$

$$V_\beta \sum_{i=1}^{N} T(\hat{f}_i) \, \psi_i = \sum_{i=1}^{N} T(\widehat{f^{V_\beta}}) \, \psi_i \,,$$

$$D_\beta \sum_{i=1}^{N} T(\hat{f}_i) \, \psi_i = \sum_{i=1}^{N} T(\widehat{f_i^{D_\beta}}) \, \psi_i \,,$$

$$Z \sum_{i=1}^{N} T(\hat{f}_i) \, \psi_i = \sum_{i=1}^{N} T(\widehat{f^Z}) \, \psi_i \,;$$

$\forall \psi_i \in H$ and $\hat{f}_i$ real functions whose Fourier transforms f_i are in $\mathscr{D}$. Where

$$f^{U_\beta}(\omega) = \operatorname{ch}(2\pi\beta\omega) \, f(\omega) \,,$$

$$f^{V_\beta}(\omega) = -i \operatorname{sh}(2\pi\beta\omega) \, f(\omega) \,,$$

$$f^{D_\beta}(\omega) = -i \operatorname{th}(\pi\beta\omega) \, f(\omega) \,,$$

$$f^Z(\omega) = -2i\pi\omega f(\omega) \,.$$

Notice that $\widehat{f^{U_\beta}}, \widehat{f^{V_\beta}}, \widehat{f^{D_\beta}}$ and $\widehat{f^Z}$ are still real functions with Fourier transform in $\mathscr{D}$.

Proof. Let us consider $\sum_{i=1}^{N} T(\hat{f}_i) \, \psi_i = 0$ then

$$u \in \mathbb{R} \to \sigma\left(\varphi, T_u \sum_{i=1}^{N} T(\hat{f}_i) \, \psi_i\right) = 0 \quad \forall \varphi \in H \qquad (3.12)$$

is the restriction, to $u \in \mathbb{R}$, of an entire function, which is consequently zero everywhere in the complex plane. If for $u = i\beta$, one looks at the real

and imaginary parts of this function:

$$\sigma\left(\varphi, \sum_{i=1}^{N} T(\widehat{f^{U_\beta}})\,\psi_i\right) = 0 \quad \forall \varphi \in H,$$

$$\sigma\left(\varphi, \sum_{i=1}^{N} T(\widehat{f^{V_\beta}})\,\psi_i\right) = 0 \quad \forall \varphi \in H$$

then the result follows from the non-degeneracy of σ.

For Z we consider the derivative at $u = 0$ of (3.12). This is zero. The result then follows from the fact that the $\widehat{f_k^Z}$'s are in $\mathscr{S}$.

In order to prove the existence of D_β we shall need the following lemma:

Lemma 3.13. *Let U_β and V_β be defined as previously;*

i) $U_\beta\psi = \psi \Leftrightarrow V_\beta\psi = 0 \Leftrightarrow \psi = 0 \quad \psi \in H,$

ii) $\sigma(U_\beta\varphi, \psi) = \sigma(\varphi, U_\beta\psi) \quad \psi, \varphi \in H,$

iii) $\sigma(V_\beta\varphi, \psi) = -\sigma(\varphi, V_\beta\psi) \quad \psi, \varphi \in H.$

Proof. ii and iii are obvious from Lemma 3.11.

Assume now that $V_\beta\psi = 0$, then for any real function $\hat{f}$ with Fourier transform in $\mathscr{D}$

$$\sigma\big(T(\hat{f})\,\varphi, V_\beta\psi\big) = 0 \quad \forall \varphi \in H$$

from iii):

$$\int_{-\infty}^{+\infty} dt\, \widehat{f^{V_\beta}}(t)\, \sigma(T_t\varphi, \psi) = 0.$$

The previous equality extends by linearity to every f in $\mathscr{D}$.

Using the fact that $\sigma(T_t\varphi, \psi)$ is continuous, it is the Fourier transform of a distribution whose support is reduced to zero. Its Fourier transform is at most a polynomial in t; the boundedness of $\sigma(T_t\varphi, \psi)$ shows that it is actually a constant

$$\sigma(T_t\psi, \varphi) = \sigma(\psi, \varphi) \quad \forall t \in R.$$

Finally, from the regularity of σ, $T_t\psi = \psi$ and $\psi = 0$.

Now, conversely if $U_\beta\psi = \psi$ one has from $U_\beta^2 + V_\beta^2 = I$ that

$$V_\beta^2\psi = 0$$

and so since V_β is injective, $\psi = 0$.

In order to complete the proof of the Lemma 3.11, it suffices to remark that:

$$(U_\beta - 1)\,H \subset V_\beta H$$

and so one can see that:

$$D_\beta = V_\beta^{-1}(U_\beta - I). \tag{3.14}$$

The next lemma reveals some interesting properties of D_β:

Lemma 3.15. *The operator D_β satisfies:*

i) $\sigma(D_\beta \psi, \varphi) = -\sigma(\psi, D_\beta \varphi),$

ii) D_β *is injective,*

iii) $D_\beta T_t - T_t D_\beta = 0 \quad \forall t \in \mathbb{R}.$

The proof of these properties is rather obvious.

We shall need the operator D_β in order to define an invariant quasi-free state, but the conditions of Proposition 2.34 are not necessarily satisfied and we shall make the following restriction on the evolution,

E 4 D_β *satisfies* D_2 *for at least one value* $\beta_0 > 0$ *of* β.

We shall come back in Section 6 to the physical meaning of this assumption, but we now give an equivalent condition whose physical meaning is evident.

E 4′   *The evolution* T_t *satisfies*

$$\frac{d}{dt}\, \sigma(\psi, T_t \psi)\big|_{t=0} \geqq 0\,.$$

The Condition E 4 can be extended immediately to the:

Lemma 3.16. *Under the hypothesis* E 4

i) D_β *satisfies* D_2 *for every* $\beta > 0,$

ii) D_β *satisfies* D_4 *for every* $\beta > 0.$

Indeed E 4 can be written for every finite family $\varphi_k, k = 1, 2, \ldots, N$ in H and every finite family $\hat{f}_k$ of real functions whose Fourier transforms are in $\mathscr{D}$.

$$i \sum_{i,j=1}^{N} \int_{-\infty}^{+\infty} dt\, \sigma(\varphi_i, T_t \varphi_j)\, \widehat{\overline{f}_i f_j \mathrm{th}\, \pi \beta_0}(t) \geqq 0\,. \tag{3.17}$$

One obtains 3.16 i) if one chooses:

$$f_i(\omega) = g_i(\omega) \sqrt{\frac{\mathrm{th}(\beta \pi \omega)}{\mathrm{th}(\beta_0 \pi \omega)}}$$

with $\hat{g}_i$ real and g_i still in $\mathscr{D}$.

In order to prove 3.16 ii) one takes

$$f_i(\omega) = g_i(\omega)\, (\mathrm{ch}(\pi \beta \omega))^{-1}\,.$$

Furthermore, it follows that

$$D_\beta H = D_{\beta_0} H$$

and the injectivity of D_β implies that $D_\beta^{-1} D_{\beta_0}$ is everywhere defined in H.

We shall need another restriction on the evolution, namely:

E 5   *There exists a $\beta_0 > 0$ such that D_{β_0} satisfies* D 5.

We shall come back in Section 6 to the physical meaning of this property. As previously, we can extend to every β Condition E 5.

Lemma 3.18. *If D_β satisfies* E 5 *for a β_0 then it satisfies* E 5 *for every $\beta > 0$.*

We shall denote by $\mathscr{S}'_\beta$ the extension to $H \times H$ of the bilinear form $\mathscr{S}_\beta$ defined on $D_\beta H \times H$ by:

$$\mathscr{S}_\beta(D_\beta \psi, \varphi) = \sigma(\psi, \varphi) \qquad \forall \varphi, \psi \in H .$$

The next proposition is quite important:

Proposition 3.19. *If $t \to T_t$ satisfies* E 1, E 2, E 3, E 4, *and* E 5, *D_β and $\mathscr{S}'_\beta$ satisfy:*

i)   *$\mathscr{S}'_\beta$ is invariant, i.e.*

$$\mathscr{S}'_\beta(T_t \psi, T_t \varphi) = \mathscr{S}'_\beta(\psi, \varphi) \qquad \forall \varphi, \psi \in H .$$

ii)   *The mapping*

$$t \in \mathbb{R} \to \mathscr{S}'_\beta(T_t \psi, \varphi) \qquad \forall \varphi \in \bar{H}^{\mathscr{S}_\beta}, \ \forall \psi \in H$$

can be extended to an entire function such that

$$\gamma > 0 \qquad \mathscr{S}'_\beta(T_t \psi, \varphi)\big|_{t = i\gamma} = \mathscr{S}'_\beta(U_\gamma \psi, \varphi) + i\mathscr{S}'_\beta(V_\gamma \psi, \varphi) .$$

iii)   *The mean (in the sense of Godement [11]) of the continuous and bounded function*

$$t \in \mathbb{R} \to \mathscr{S}'_\beta(T_t \psi, \varphi) \qquad \forall \varphi, \psi \in \bar{H}^{\mathscr{S}_\beta}$$

is $\mathscr{S}'_\beta((I - P_\beta) \psi, \varphi)$ where P_β is the projection on $\overline{D_\beta H}^{\mathscr{S}_\beta} \subset \bar{H}^{\mathscr{S}_\beta}$.

iv)   *The bilinear form on $H \times H$*

$$\bar{\mathscr{T}}_\beta(\varphi, \psi) = \mathscr{S}'_\beta(P_\beta \varphi, P_\beta \psi)$$

is in $\mathfrak{S}$; and it is the only extension of $\mathscr{S}_\beta$ such that the continuous extension $\bar\sigma$ of σ to $\bar{H}^{\mathscr{T}_\beta}$ is regular.

Proof. Let us define T'_t through:

$$T'_t = \bar{T}_t P_\beta + (1 - P_\beta) \qquad \forall t \in \mathbb{R} \tag{3.20}$$

where $\bar{T}_t$ is the continuous extension to $\overline{D_\beta H}^{\mathscr{S}_\beta}$ of T_t restricted to $D_\beta H$. It is an orthogonal operator and

$$t \in \mathbb{R} \to T'_t$$

is a homomorphism of the additive group of reals into the orthogonal operators of $\bar{H}^{\mathscr{S}_\beta}$, so in order to prove i) it suffices to prove that

$$T_t'|_H = T_t \qquad t \in \mathbb{R}.$$

First notice that since $t \in \mathbb{R} \to (\exp(2i\pi\omega t) - 1)\,\mathrm{cth}(\pi\beta\omega)$ is infinitely differentiable, one has: $(T_t - 1)\,\psi \in D_\beta H$ if $\psi \in H$ and

$$
\begin{aligned}
\mathscr{S}_\beta'(\overline{T_t}P_\beta\psi - P_\beta\psi,\, D_\beta\varphi) \\
&= \mathscr{S}_\beta'(\overline{T_t}P_\beta\psi,\, D_\beta\varphi) - \mathscr{S}_\beta'(P_\beta\psi,\, D_\beta\varphi) \\
&= \mathscr{S}_\beta'(P_\beta\psi,\, T_{-t}D_\beta\varphi) - \mathscr{S}_\beta'(P_\beta\psi,\, D_\beta\varphi) \\
&= \mathscr{S}_\beta'(\psi,\, T_{-t}D_\beta\varphi) - \mathscr{S}_\beta(\psi,\, D_\beta\varphi) \\
&= \mathscr{S}_\beta'(T_t\psi - \psi,\, D_\beta\varphi) = \mathscr{S}_\beta((T_t - 1)\,\psi,\, D_\beta\varphi).
\end{aligned}
$$

Hence the result follows.

In order to prove ii) one has for $\varphi = D_\beta \xi \in D_\beta H$

$$\mathscr{S}_\beta'(T_t\psi,\, \varphi) = -\sigma(T_t\psi,\, \xi)$$

thus the analyticity is a consequence of 3.10. Similarly if $\varphi \in (1 - P_\beta)\,\bar{H}^{\mathscr{S}_\beta}$:

$$
\begin{aligned}
\mathscr{S}_\beta'(T_t\psi,\, (1 - P_\beta)\,\xi) &= \mathscr{S}_\beta'((T_t - I)\,\psi,\, (1 - P_\beta)\,\xi) \\
&\quad + \mathscr{S}_\beta'(\psi,\, (1 - P_\beta)\,\xi) \\
&= \mathscr{S}_\beta((T_t - 1)\,\psi,\, (1 - P_\beta)\,\xi) + \mathscr{S}_\beta'(\psi,\, (1 - P_\beta)\,\xi) \\
&= \mathscr{S}_\beta'(\psi,\, (1 - P_\beta)\,\xi)
\end{aligned}
$$

since $(T_t - 1)\,\psi \in D_\beta H$. We get the analyticity and the desired expression if we notice that

$$\mathscr{S}_\beta'((1 - P_\beta)\,\xi,\, (U_\gamma - I)\,T_t\psi) = 0,$$

$$\mathscr{S}_\beta'((1 - P_\beta)\,\xi,\, V_\gamma T_t\psi) = 0 \qquad \forall\,\psi,\,\xi \in H \qquad \gamma > 0$$

for $\varphi \in D_\beta H \oplus (1 - P_\beta)\,\bar{H}^{\mathscr{S}_\beta}$.

In the general case, if $\varphi \in \bar{H}^{\mathscr{S}_\beta}$ and if φ_n is a sequence in $D_\beta H \oplus (1 - P_\beta)\,\bar{H}^{\mathscr{S}_\beta}$ converging to φ, the functions

$$(t, \gamma) \in \mathbb{R} \times \mathbb{R} \to \mathscr{S}_\beta'(\varphi_n,\, U_\gamma T_t\psi) + i\mathscr{S}_\beta'(\varphi_n,\, V_\gamma T_t\psi)$$

converge uniformly on every compact to the same expression with φ_n replaced by φ.

Indeed using i) and the fact that

$$\gamma > 0 \to \|U_\gamma\psi\|^2 = \|\psi\|^2 + \|V_\gamma\psi\|^2$$

is increasing, its derivative

$$\gamma \in \mathbb{R} \to 2\mathscr{S}_\beta'(\psi,\, U_\gamma V_\gamma\psi) = 2\mathscr{S}_\beta(\psi,\, U_\gamma V_\gamma\psi)$$

is positive, hence ii) follows ([12], 9.12.1).

In order to prove iii), let us show that $1 - P_\beta$ is the projection onto the subspace of $\bar{H}^{\mathscr{S}_\beta}$ which is pointwise invariant with respect to T'_t. If $P_\beta \psi = 0$, $\psi \in \bar{H}^{\mathscr{S}_\beta}$, then

$$T'_t \psi = \bar{T}_t P_\beta \psi + \psi - P_\beta \psi = \psi \qquad \forall t \in \mathbb{R}.$$

Conversely, if $T'_t \psi = \psi$, $\psi \in \bar{H}^{\mathscr{S}_\beta}$; the analytic function defined in ii) vanishes for all real t and therefore everywhere. In particular

$$\mathscr{S}'_\beta(V_\beta \varphi, \psi) = 0 \qquad \forall \varphi \in H$$

therefore ψ is orthogonal to $V_\beta H = D_\beta H$ and $P_\beta \psi = 0$.

Moreover let M be the mean in the sense of Godement, then the bilinear form:

$$(\psi, \varphi) \in \bar{H}^{\mathscr{S}_\beta} \times \bar{H}^{\mathscr{S}_\beta} \to M(\mathscr{S}'_\beta(\psi, T'_t \varphi))$$

is continuous, thence there exists an operator C in $\bar{H}^{\mathscr{S}_\beta}$ such that

$$\mathscr{S}'_\beta(\psi, C\varphi) = M(\mathscr{S}'_\beta(\psi, T'_t \varphi))$$

with the properties

$$C^* = C = C^2 = T'_t C = C T'_t.$$

Consequently for $\forall \psi \in \bar{H}^{\mathscr{S}_\beta}$,

$$T'_t C \psi = C \psi$$

and

$$C \leqq 1 - P_\beta.$$

On the other hand, if $P_\beta \psi = 0$, one has $C \psi = \psi$ so

$$P_\beta \geqq 1 - C$$

and iii) follows.

In order to complete the proof, notice that

$$\sigma'(P_\beta \psi, P_\beta \varphi) = \sigma(\psi, \varphi) \qquad \forall \psi, \varphi \in H$$

so that $\mathscr{S}_\beta \in \mathfrak{S}$; moreover $\bar{\sigma}$ is non-degenerate since $P_\beta|H$ is injective (cf. E 2), $P_\beta H$ is dense in $\overline{D_\beta H}^{\mathscr{S}_\beta}$ and $\sigma'|_{\overline{D_\beta H}}{}^{\mathscr{S}_\beta}$ is non-degenerate. The uniqueness comes from the density of $D_\beta H$ in H for the norm induced by $\mathscr{S}_\beta$.

Let us give now the central result of this section:

Theorem 3.21. *If $t \to \alpha_t$ is a quasi-free evolution of $\overline{A(H, \sigma)}$ which satisfies E 1, E 2, E 3, E 4, and E 5, then there exists at least one K.M.S. state ω_β for every $\beta > 0$. It is the quasi-free state defined by*

$$\omega_\beta(\delta_\psi) = \exp(-\tfrac{1}{2} \mathscr{S}'_\beta(\psi, \psi)) \qquad \forall \psi \in H.$$

Proof. The existence of ω_β follows from E 5, the analyticity from 3.19 ii) and 3.10. (Actually the result is stronger: it is an entire function.)

Moreover, we have the identities:

$$\mathcal{S}'_\beta(\varphi, U_\beta\psi) - \sigma(\varphi, V_\beta\psi) = \mathcal{S}'_\beta(\varphi, \psi) \tag{3.22}$$

$$\mathcal{S}'_\beta(\varphi, V_\beta\psi) + \sigma(\varphi, U_\beta\psi) = -\sigma(\varphi, \psi) \qquad \forall \varphi, \psi \in H \tag{3.23}$$

hence the result follows.

This theorem means that under suitable conditions on the evolution (physically on the energy) we have at least one solution of the K.M.S. problem.

Notice that the uniqueness of $\mathcal{S}'_\beta$ is by no means ensured by the previous assumptions (cf. D 5 and E 5); $\mathcal{F}_\beta$ may furnish another possibility. We shall take advantage of this fact in the next section; in the last section, we give its physical meaning.

§ 4. The General Solution

In order to get the general solution of the K.M.S. problem in our case, let us remark that the relations (3.22) and (3.23) only define $\mathcal{S}'_\beta$ on $V_\beta H \times H$, and so if there exist many extensions, there will be no uniquess for the K.M.S. state. More precisely, we have the theorem:

Theorem 4.1. Let $t \in \mathbb{R} \to \alpha_t$, a quasi-free evolution of $\overline{\Delta(H, \sigma)}$ which satisfies E 1, E 2, E 3, E 4, and E 5; the general solution of the K.M.S. boundary condition satisfying (2.15) is of the form

$$\omega(\delta_\psi) = \int_{\mathbb{C}} \exp(-\tfrac{1}{2}\mathcal{F}_\beta(\psi, \psi) + i\varrho(\psi)) \, dm(\varrho) \qquad \forall \psi \in H$$

where ϱ is a real linear form on H, invariant with respect to T_t, and $dm(\varrho)$ is a positive measure of total mass equal to one on a space $\mathbb{C}$ of real invariant linear forms on H.

The proof requires some lemmas:

Lemma 4.2. Let ω be a K.M.S. state with respect to $t \to \alpha_t$; then the function $\Phi : H \to \mathbb{C}$ defined through

$$\omega(\delta_\varphi) = \exp(-\tfrac{1}{2}\mathcal{S}'_\beta(\varphi, \varphi)) \, \Phi(\varphi) \qquad \forall \varphi \in H \tag{4.3}$$

satisfies,

$$\Phi(T_t\psi + \varphi) = \Phi(\psi + \varphi) \qquad \forall \varphi, \psi \in H, \, t \in \mathbb{R} \tag{4.4}$$

conversely, let $\Phi : H \to \mathbb{C}$ satisfying (4.4), if ω defined through (4.3) is a state, then ω is a K.M.S. state.

Indeed, if one uses Theorem 3.21, the K.M.S. boundary condition implies that $t \in \mathbb{R} \to \Phi(T_t\psi + \varphi) \, \forall \psi, \varphi \in H$, is the continuous boundary value of an analytic function in the strip $0 < \operatorname{Im} t < \beta$ which satisfies

$$\Phi(T_t, \psi + \varphi)\big|_{t' = t + i\beta} = \Phi(T_t\psi + \varphi).$$

Moreover

$$|\Phi(T_t\psi + \varphi)| \leqq e^{\frac{1}{2}(\|\psi\| + \|\varphi\|)^2}.$$

Therefore we are in a position to complete the proof by an argument, identical to that used in (3.13) and in [5] to prove the invariance of the K.M.S. state.

On the other hand, the converse result is obvious.

The next lemma is very important and, in a sense, is connected to the usual replacement of the field operators at zero momentum by their mean value.

Lemma 4.5. *The function Φ_0 defined by*

$$\Phi_0(\psi) = \exp\left(-\tfrac{1}{2}\mathscr{S}'_\beta((1 - P_\beta)\,\psi,\,\psi)\right)\Phi(\psi) \qquad \forall\,\psi \in H$$

is of positive type, i.e.

$$\sum_{i,j=1}^{N} \bar{\lambda}_i\lambda_j\Phi_0(\psi_i - \psi_j) \geqq 0$$

$\forall\,\psi_i \in H$ *and* $\lambda_i \in \mathbb{C}$.

Proof. The positivity of ω implies that

$$\sum_{i,j}^{N} \bar{\lambda}_i\lambda_j\exp(h_\beta(\psi_i,\,\psi_j))\,\Phi(\psi_i - \psi_j) \geqq 0$$

where

$$h_\beta(\varphi,\,\psi) = \mathscr{S}'_\beta(\varphi,\,\psi) + i\,\sigma(\psi,\,\varphi) \qquad (4.6)$$

Thus, from Lemma 4.2:

$$\sum_{i,j=1}^{N} \bar{\lambda}_i\lambda_j\exp(h_\beta([\psi_i],\,[\psi_j]))\,\Phi(\psi_i - \psi_j) \geqq 0$$

where $[\psi_i]$ denotes any convex combination of translated ψ_i, i.e.

$$[\psi_i] = \sum_{k=1}^{K} \alpha_k T_{t_k}\psi_i \qquad 0 \leqq \alpha_k \leqq 1 \qquad \psi_i \in H \qquad (4.7)$$

$$\sum_{k=1}^{K} \alpha_k = 1 \qquad t_k \in \mathbb{R}. \qquad (4.8)$$

Finally, we use proposition 3.19 iii) to get the result. By standard arguments, one shows the existence of a compact space $\mathfrak{C}$ of characters of H (considered as an additive group) and of a positive measure on $\mathfrak{C}$ of total mass one, such that

$$\Phi_0(\psi) = \int_{\mathfrak{C}} dm(\varrho)\,\exp(i\,\varrho(\psi)).$$

(In our special case, one can replace the characters by exponentials of linear forms (cf. 2.14)). Thus

$$\omega(\delta_\psi) = \exp\left(-\tfrac{1}{2}\mathscr{S}'_\beta(\psi,\psi)\right)\Phi(\psi)$$
$$= \exp\left(-\tfrac{1}{2}\mathscr{S}'_\beta(\psi,\psi) + \tfrac{1}{2}\mathscr{S}'_\beta((1-P_\beta)\,\psi,\psi)\right)\Phi_0(\psi)$$
$$= \exp\left(-\tfrac{1}{2}\bar{\mathscr{S}}_\beta(\psi,\psi)\right)\int_{\mathfrak{C}} dm(\varrho)\exp(i\varrho(\psi))$$

and Theorem 4.1 follows.

The previous theorem gives some information on the structure of the state. We now discuss the structure in more detail.

Let us first state the lemma which is a slight generalization of a result in [6]:

Lemma 4.9. *Let* $\pi_\beta H_\beta \Omega_\beta$ *(resp.* $\pi'_\beta H'_\beta \Omega'_\beta$*) be the representation, the Hilbert space and the cyclic vector constructed out of the state defined in* (4.1) *(resp. by* $\omega_\beta(\delta_\psi) = \exp\left(-\tfrac{1}{2}\bar{\mathscr{S}}_\beta(\psi,\psi)\right)$*), then* π_β *is unitarily equivalent to the representation*

$$\varpi_\beta(\delta_\psi) = \pi'_\beta(\delta_\psi) \otimes \hat{\psi}$$

in $H' \otimes L_2(\mathfrak{C}, m)$, *where,*

$$(\hat{\psi}f)(\varrho) = \exp(i\varrho(\psi))f(\varrho) \qquad \forall f \in L_2(\mathfrak{C}, m)$$

and $\forall \varrho \in \mathfrak{C}$. *This last representation has for cyclic vector*

$$\Omega'_\beta \otimes 1 \, .$$

Finally, using the previous lemma, we have

Theorem 4.10. *The decomposition appearing in Theorem 4.1 is the central decomposition (see* [13, 14]*) of the state. As a special case, the solutions of the K.M.S. problem are primary if and only if m is concentrated on a point, and every two primary solutions differ only by a gauge transformation of the second kind.*

The proof follows from the explicit construction of the representation (cf. (4.9)) and from the fact that π'_β is factorial.

For the sake of completeness, we state a further theorem:

Theorem 4.11. *If the evolution* $t \to T_t$ *does not satisfy the conditions* E 1, E 2, E 3, E 4, *and* E 5, *then there exists no solution of the K.M.S. problem, such that*

$$\lambda \in \mathbb{R} \to \omega(\delta_{\lambda\varphi + \psi})$$

is twice differentiable.

Indeed let us assume such a state ω exists; let π, Ω and H be the cyclic vector, the Hilbert space and the representation constructed in the usual

way. Let $B_\omega(\psi)$ be the field operator. One has the relation

$$\int \left(\Omega | (B_\omega(T_t\varphi)\, B_\omega(\psi) - B_\omega(\psi)\, B_\omega(T_t\varphi)) | \Omega \right) \hat{f}(t)\, dt$$
$$= i \int \widehat{f^{D_\beta}}(t) \left(\Omega | (B_\omega(T_t\varphi)\, B_\omega(\psi) + B_\omega(\psi)\, B_\omega(T_t\varphi)) | \Omega \right) dt$$

(cf. [16]) so that $\mathscr{S}'_\beta$ defined by:

$$2\mathscr{S}'_\beta(\varphi, \psi) = (\Omega | B_\omega(\varphi)\, B_\omega(\psi) + B_\omega(\psi)\, B_\omega(\varphi) | \Omega)$$

is in $\mathfrak{S}$ and, moreover, the corresponding operator D satisfies the conditions D 1, D 2, D 3, D 4, and D 5.

§ 5. Limiting Cases

In this section we shall study the behaviour (as $\beta \to \infty$ or $\beta \to 0$). of the states that we obtained in the previous section. These cases cannot be studied directly by the K.M.S. boundary condition, but nevertheless, one can expect to get an integral of Fock states for $\beta \to \infty$ and the central state for $\beta \to 0$ (cf. [15]).

More precisely, we first prove

Lemma 5.1. *If* $t \in \mathbb{R} \to T_t$ *satisfies* E 1, E 2, E 3, E 4, *and* E 5, *then the mapping*

$$t \in \mathbb{R} \to \sigma(\varphi, T_t\psi) \qquad \forall\, \varphi, \psi \in H$$

is the Fourier transform of a bounded measure $\mu_{\varphi,\psi}$ *such that*
— $\mu_{\varphi,\psi}$ *has no mass on* $\{0\}$.
— *The bilinear form on* $H \times H$

$$\mathscr{S}_\infty(\varphi, \psi) = \mu_{\varphi,\psi}(-i\varepsilon) \tag{5.2}$$

where ε *is the sign-function, is symmetric, real and invariant.*

Proof. Existence of $\mu_{\varphi,\psi}$ is obvious; the existence of $\mathscr{S}_\infty$ follows from the fact that $\mu_{\varphi,\psi}$ has no mass on $\{0\}$ (cf. E 2); the symmetry of $\mathscr{S}_\infty$ follows from:

$$\mu_{\psi,\varphi} = -\mu^v_{\varphi,\psi} = -\overline{\mu_{\varphi,\psi}}. \tag{5.3}$$

The next lemma is rather obvious.

Lemma 5.4. $\mu_{D_\beta\psi,\varphi} = \mu_{\psi,\varphi} \cdot i\, th(\pi\beta.)$,

$$\mu_{D_{\bar{\beta}}^{-1} D_{\beta_0}\psi,\varphi} = \mu_{\psi,\varphi} \cdot \frac{th(\pi\beta_0.)}{th(\pi\beta.)} \qquad \forall\, \varphi, \psi \in H.$$

Moreover, we have

Lemma 5.5. i) $\beta \geqq \beta_0 \Rightarrow \mathscr{S}_{\beta_0}(\psi, \psi) \geqq \mathscr{S}_\beta(\psi, \psi)$,
ii) $\mathscr{S}_{\beta_0}(\psi, \psi) \geqq \mathscr{S}_\infty(\psi, \psi)$.

The first inequality follows from

$$\mathscr{F}_{\beta_0}((1 - D_\beta^{-1} D_{\beta_0}) \psi, \psi) \geqq 0 ,$$

indeed the function

$$\omega \in \mathbb{R} \to 1 - \frac{th \, \pi \beta_0 \omega}{th \, \pi \beta \omega}$$

is positive as long as $\beta > \beta_0$.

On the other hand, since the extension is unique (cf. Proposition 3.19 iv)

$$\mathscr{F}_{\beta_0}(D_\beta^{-1} D_{\beta_0} \psi, \psi) = \mathscr{F}_\beta(\psi, \psi)$$

In order to prove the second point, let us remark that

$$\lim_{\beta \to \infty} \sigma(\psi, D_\beta \psi) = \lim_{\beta \to \infty} - \mu_{D_\beta \psi, \psi}(1)$$

$$= \lim_{\beta \to \infty} \mu_{\psi, \psi}(- i \, th(\pi \beta.))$$

$$= \mu_{\psi, \psi}(- i \varepsilon) = \mathscr{F}_\infty(\psi, \psi)$$

using Lebesgue's Theorem. So we only need to show that:

$$\sigma(\psi, D_\beta \psi) \leqq \mathscr{F}_{\beta_0}(\psi, \psi) .$$

But

$$\sigma(\psi, D_\beta \psi) \leqq \mathscr{F}_\beta(D_\beta \psi, D_\beta \psi) \leqq \mathscr{F}_\beta(\psi, \psi) \leqq \mathscr{F}_{\beta_0}(\psi, \psi)$$

if $\beta \geqq \beta_0$.

Lemma 5.6. $\lim_{\beta \to \infty} \mathscr{F}_\beta(\psi, \psi) = \mathscr{F}_\infty(\psi, \psi) \qquad \forall \psi \in H$

for $\psi = D_{\beta_0} \varphi$ the result follows from a calculation very similar to the one performed in Lemma 5.5. For the general case, let $\{\beta_i\}_{i=1\ldots}$ be an increasing sequence going to infinity. One then makes use of the equi-continuity of the corresponding sequence in 5.6 with respect to ψ, of the continuity of $\mathscr{F}_\infty$ (with respect to $\overline{\mathscr{F}_{\beta_1}}$), and of the density of $D_{\beta_1} H$ in H (see e.g. [12] 7.5.5 for a similar argument).

It is clear that $\mathscr{F}_\infty$ is in $\mathfrak{S}$ ([12] 3.15.4); $\mathscr{F}_\infty$ is moreover strictly positive (σ is regular).

Lemma 5.7. *The linear operator J from $\bar{H}^{\mathscr{F}_\infty}$ to $\bar{H}^{\mathscr{F}_\infty}$ defined by*

$$\mathscr{F}_\infty(J \psi, \varphi) = \sigma(\psi, \varphi) \qquad \forall \psi, \varphi \in H$$

is a complex structure.

The proof is clear from (5.2) and Lemma 5.5 ii).

Theorem 5.8. *Let ω_β the K.M.S. state defined by (4.1); then*

$$\lim_{\beta \to \infty} \omega_\beta(\delta_\psi) = \int_{\mathfrak{C}} dm(\varrho) \exp\left(- \tfrac{1}{2} \mathscr{F}_\infty(\psi, \psi) + i \varrho(\psi)\right)$$

The proof follows without any difficulty from the previous lemma. We are still left with the limit $\beta \to 0$ of the previously defined states. We shall need a state which is rather unfamiliar since it does not define any field operator, namely the central state of $\overline{\Delta(H, \sigma)}$.[1]

Theorem 5.9. *The central state* ω_0 *of* $\Delta(H, \sigma)$ *defined by*

$$\omega_0(\delta_\varphi) = 0 \quad if \quad \varphi \neq 0$$

extends by continuity to $\overline{\Delta(H, \sigma)}$.

Proof. What we have to prove is that for a $a \in \Delta(H, \sigma)$

$$|\omega_0(a)| \leqq \|a\| \; .$$

Actually the result rests on the following inequality:

if $\quad a = a_0 \delta_0 + \sum_{i=1}^{N} a_i \delta_{\psi_i}, \quad \psi_i \neq 0$

then:

$$|a_0| < \|a\| + \varepsilon \quad \varepsilon > 0 \; .$$

Indeed let J a complex structure of H, and $\omega_\lambda (\lambda \geqq 1)$ the state of $\overline{\Delta(H, \sigma)}$ such that

$$\omega_\lambda(\delta_\psi) = \exp\left(-\frac{\lambda^2}{2} S_J(\psi, \psi)\right) \quad \forall \psi \in H$$

then given $\varepsilon > 0$, one can choose $\lambda > 1$ such that

$$\left| \sum_{i=1}^{N} a_i \exp\left(-\frac{\lambda^2}{2} S_J(\psi_i, \psi_i)\right) \right| < \varepsilon$$

and

$$|\omega_\lambda(a) - a_0| < \varepsilon$$

We can now state the theorem (cf. [15]):

Theorem 5.10. *Let* ω_β *defined in* (4.1); *then*

$$\lim_{\beta \to \infty} \omega_\beta = \omega_0$$

the limit being understood in the weak sense.

The proof is quite standard.

§ 6. Translation Invariant Quasi-free Evolutions

The aim of this section is two fold: firstly, we shall come back to the physical meaning of the various assumptions we have made about the evolution; secondly, we give an application of the previous results to

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[1] A. Verbeure told us that he got a similar result (private communication).

the physically important case of translation invariant quasi-free evolutions precisely defined by

Definition 6.1. *Let $L_2(\mathbb{R}_n, dx)$ be considered as a symplectic space; the symplectic form is, as usual,*

$$\sigma(f, g) = \text{Im} \int dx \, \bar{f}(x) \, g(x) \tag{6.2}$$

We shall denote by J_0 the complex structure of $L_2(\mathbb{R}_n)$ corresponding to the multiplication by i. Translation τ_x of $\mathbb{R}_n$ by $x \in \mathbb{R}_n$ acts as a symplectic operator of $L_2(\mathbb{R}_n, dx)$.

Definition 6.3. *Let H be a symplectic subspace of $L_2(\mathbb{R}_n, dx)$ invariant under both translations and J_0. A quasi-free evolution, corresponding to the group of symplectic operators $t \in \mathbb{R} \to T_t$, is diagonalized iff*

$$T_t J_0 = J_0 T_t \qquad \forall t \in \mathbb{R}. \tag{6.4}$$

It is translation invariant iff:

$$\tau_x T_t = T_t \tau_x \qquad \forall t \in \mathbb{R} \qquad \forall x \in \mathbb{R}_n. \tag{6.5}$$

In what follows, we choose for H the space $\hat{\mathscr{D}}$ of Fourier transforms of infinitely differentiable functions with compact support. A quasi-free evolution, diagonalized and translation invariant is induced by the following group of symplectic operators

$$(\widehat{T_t f})(p) = \exp(i \omega(p) \, t) \, \hat{f}(p) \qquad \forall f \in \mathscr{D} \tag{6.6}$$

if we assume E 1 (cf. [17]).

Definition 6.7. *$p \to \omega(p)$ will be called the spectrum of evolution.*

Notice that our choice of $H = \hat{\mathscr{D}}$ implies that $p \to \omega(p)$ is infinitely differentiable. Another choice of H would give different restrictions (e.g. $H = \mathscr{S}$ implies that $p \to \omega(p) \in \mathcal{O}_M$ cf. [18]).

One might think that the infinitely differentiability of $\omega(p)$ is a condition too drastic for some physical situations (e.g. for phonons). In such a case, one may choose for H the subset of elements of $\hat{\mathscr{D}}$ whose Fourier transforms have their supports in the complement of the origin and so $\omega(p)$ is infinitely differentiable only in the complement of the origin.

Assumption E 1 is then automatically satisfied. E 2 corresponds to the fact that the set

$$Q = \{p \in \mathbb{R}_n; \, \omega(p) = 0\} \tag{6.8}$$

is of zero measure.

Assumption E 3 is realized by our choice of H; indeed, one can see that:

$$\widehat{(T(\hat{f})\,\psi)}(p) = f\left(\frac{\omega(p)}{2\pi}\right)\hat{\psi}(p) \qquad \psi \in H \,. \tag{6.9}$$

Hence

$$
\begin{cases}
\widehat{U_\beta\psi}(p) = ch(\beta\omega(p))\,\hat{\psi}(p) \qquad \psi \in H \,, \\[2mm]
\widehat{V_\beta\psi}(p) = -i\,sh(\beta\omega(p))\,\hat{\psi}(p) \qquad \psi \in H \,, \\[2mm]
\widehat{D_\beta\psi}(p) = -i\,th\left(\frac{\beta\omega(p)}{2}\right)\hat{\psi}(p) \qquad \psi \in H \,, \\[2mm]
\widehat{Z\psi}(p) = -i\,\omega(p)\,\hat{\psi}(p) \qquad \psi \in H \,.
\end{cases}
\tag{6.10}
$$

Condition E 4 expresses the "positivity of the energy", namely

$$\omega(p) \geqq 0 \,. \tag{6.11}$$

Notice that $\omega(p)$ actually contains the chemical potential μ

$$\omega(p) = \varepsilon(p) - \mu \tag{6.12}$$

so that (6.11) implies:

  i) there exists a number such that $\varepsilon(p) \geqq a$,
  ii) $\mu \leqq a$.

It is worthwhile to stress that $\omega(p) > 0$ implies that E 5 is automatically satisfied, indeed $D_\beta H = H$. Otherwise, the problem corresponding to this requirement turns out to be the problem of the extension of the bilinear form

$$\psi, \varphi \to \int cth(\beta\omega(p))\,\overline{\hat{\psi}(p)}\,\hat{\varphi}(p)\,dp \tag{6.13}$$

defined on pairs of functions with support contained in the complement of Q; this classical problem is not yet completely solved. One can only give sufficient conditions; e.g. if $P\dfrac{1}{\omega(p)}$ can be defined as a principal value, the extension in (6.13) is given in an obvious way.

On the contrary in those situations where $\omega(p)$ goes too rapidly to zero near a point of Q, one could expect that the corresponding occupation number would go too rapidly to infinity and the corresponding density of the non-condensed phase would be infinite.

The condensation phenomena as expressed by Theorem 4.1 can occur only if Q is not empty and consequently the chemical potential is equal to the greatest lower bound of $\varepsilon(p)$ since then any distribution concentrated on Q is a linear invariant form.

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