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Quantum Stochastic Processes II

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Abstract. We investigate properties of a class of quantum stochastic processes subject to a condition of irreducibility. These processes must be recurrent or transient and an equilibrium state can only exist in the former case. Every finite dimensional process is recurrent and it is possible to establish convergence in time to a unique equilibrium state. We study particularly the class of transition processes, which describe photon emissions of simple quantum mechanical systems in excited states.

§ 1. Introduction

In earlier papers [1–3] we have shown that in order to treat repeated measurements or measurements extended over a period of time, it is necessary to extend considerably the conventional description of measurement theory in quantum mechanics. In order to describe regorously the photon counting experiments being done in quantum optics, for example, it was necessary to develop a theory of quantum stochastic processes [3]. These are generalisations of classical Markov processes and can be analyzed in terms of, and constructed from, two infinitesmal generators. The first of these is the Hamiltonian of the quantum mechanical system, and the second is a stochastic kernel, describing how the measuring instrument interacts with that system. In the presence of the measuring instrument the system evolves according to a one-parameter strongly continuous semigroup of positive endomorphisms of a space of self-adjoint trace class operators.

In this paper we start the analysis of the properties of a class of (quantum stochastic) processes. A process is called irreducible if it cannot be restricted to any proper closed subspace of the underlying Hilbert space and we restrict attention throughout to the irreducible processes. As in classical probability theory the reducible processes are of a much more complex nature and cannot be "decomposed" as direct integrals of irreducible ones. A class of processes, called simple, has the property that certain order ideals associated with compact subsets of the value space of the process are finite-dimensional. We establish necessary and sufficient conditions on the infinitesmal generators of a process for it to be simple and irreducible. We also prove that the simple irreducible

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classical stochastic processes are precisely the irreducible continuous time Markov *chains* in the sense of [4].

Section 3 is concerned with proving that every simple irreducible process is either recurrent or transient. Here a process is called recurrent if for every initial state and every non-null open set in the value space the number of events in that set is infinite with probability one. A process is called transient if for every initial state and every open relatively compact set the number of occurences of events in that set has finite expectation. A process with an equilibrium state must be recurrent. Every finite-dimensional process is recurrent and it is possible to establish convergence in time from any initial state to a unique equilibrium state.

In Section 4 we consider the class of transition processes, behind which there is a rather simple physical model. This consists of an atom emitting photons in various directions at random times under external excitation. The Hilbert space used is that describing a finite number of electrons moving in a central potential. The irreducibility hypothesis means that an electron can jump from any energy level to any other energy level eventually, perhaps via intermediate levels. The hypothesis on the order ideals means that each of the energy levels of the atom must be finite-dimensional. For a spherically symmetrical atom the group $SU(2)$ has a unitary representation on the Hilbert space, and this can be used to restrict greatly the possible variety of structures.

As far as the interpreation of this model is concerned two attitudes are possible. One can argue that an exact treatment of the radiation of the atom requires a quantisation of the electromagnetic field to be included and that the one-parameter semigroup we obtain on the space of self-adjoint trace class operators on the small Hilbert space is just the "best approximation" to a one-parameter unitary group on the larger Hilbert space. While partially justified this overlooks the fact that any theory which describes the actual arrival of individual photons at counters at random times must incorporate something like a stochastic process. Indeed we have described in the earlier paper how to construct a quantum stochastic process on Fock space in order to describe the action of photon counters. Therefore the choice of Hilbert space depends on the level of accuracy desired, but in neither case can the evolution be properly represented by a one-parameter unitary group. If the photon density in the electromagnetic field is low and the counter is effectively decoupled from the radiating atom then either approach will give essentially the same probabilistic predictions.

In the final section we exhibit a class of processes which we call reversible, and hence find a simple and rather surprising condition on the infinitesimal generators of an infinite dimensional irreducible process for there to exist no equilibrium state. We illustrate this by an example of an elementary particle moving through a bubble chamber with periodic boundary conditions imposed.

Since the main techniques of the paper are adaptations of very standard methods in classical probability theory [4], three remarks about the ways in which quantum processes differ from classical processes are in order. Firstly quantum processes with *discrete* time turn out to be very difficult to analyse because very complicated multiple periodicities can occur. Secondly simple quantum stochastic processes can have non-trivial connected groups of symmetries, and hence these processes actually have a richer structure than the continuous time Markov chains. We include such a symmetry group systematically in our treatment, and without loss of generality since it can always be taken to be trivial. Thirdly irreducible quantum processes need not have equilibrium states even when they are very strongly recurrent, as we show by an example in Section 5.

§ 2. Irreducible and Simple Processes

We recall some of the notation of [3], which will be followed systematically here, and at the same time construct the family of processes with which we shall be concerned for the rest of the paper. Let $\mathscr{H}$ be a separable Hilbert space and $V = \mathscr{T}_s(\mathscr{H})$ the real ordered Banach space of self-adjoint trace class operators on $\mathscr{H}$ with the trace norm. The trace tr defines a strictly positive linear functional on V, which is called the state space. Let U be a unitary representation of a separable locally compact group G on $\mathscr{H}$ and define a representation α of G as a strongly continuous group of positive, trace-preserving automorphisms of V by

$$\alpha_g(\varrho) = U_g \varrho U_g^* \tag{2.1}$$

where $\varrho \in V$. Let X be a separable locally compact Hausdorff space, called the value space, and let there be a jointly continuous action $X \times G \to X$ of G on X. Suppose that a G-invariant measure dx on X with support equal to X is specified and that $A: X \to \mathscr{L}(\mathscr{H})$ is a strongly continuous map such that for all $x \in X$ and $g \in G$ there is a number $\lambda(x, g)$ of absolute value one such that

$$A_{xg} = \lambda(x, g)\, U_g^* A_x U_g\,. \tag{2.2}$$

Also suppose that there exists a constant K such that for all $\xi \in \mathscr{H}$

$$\int_X \|A_x \xi\|^2 dx \leqq K \|\xi\|^2\,. \tag{2.3}$$

Then it is easy to verify that the formula

$$\operatorname{tr}[\mathscr{J}(E, \varrho)\, B] = \int_E \operatorname{tr}[A_x \varrho A_x^* B]\, dx \tag{2.4}$$

where $E \subseteq X$ is a Borel set, $\varrho \in V$ and $B \in \mathscr{L}(\mathscr{H})$ defines a bounded stochastic kernel on X in the sense of [3]. Moreover if $g \in G$ then

$$\begin{aligned} \operatorname{tr}[\mathscr{J}(E_{g^{-1}}, \varrho)\, B] &= \int_{E_{g^{-1}}} \operatorname{tr}[A_x \varrho A_x^* B]\, dx \\ &= \int_E \operatorname{tr}[A_{xg^{-1}} \varrho A_{xg^{-1}}^* B]\, dx \\ &= \int_E \operatorname{tr}[|\lambda(x, g)|^2\, U_g A_x U_g^* \varrho\, U_g A_x^* U_g^* B]\, dx \\ &= \int_E \operatorname{tr}[A_x \alpha_{g^{-1}}(\varrho)\, A_x^* (U_g^* B U_g)]\, dx \\ &= \operatorname{tr}[\mathscr{J}(E, \alpha_{g^{-1}}(\varrho))\, U_g^* B U_g] \\ &= \operatorname{tr}[\alpha_g\{\mathscr{J}(E, \alpha_{g^{-1}}(\varrho))\}\, B] \end{aligned}$$

so

$$\mathscr{J}(E_{g^{-1}}, \varrho) = \alpha_g\{\mathscr{J}(E, \alpha_{g^{-1}}(\varrho))\} \tag{2.5}$$

and $\mathscr{J}$ is covariant with respect to G in the sense of [2].If R is the operator on $\mathscr{H}$ defined by

$$\operatorname{tr}[R\varrho] = \operatorname{tr}[\mathscr{J}(X, \varrho)] \tag{2.6}$$

then $0 \leqq R \leqq K1$ and $U_g R = R U_g$ for all $g \in G$.

Now let H_0 be a self-adjoint operator on $\mathscr{H}$ such that $H_0 U_g = U_g H_0$ for all $g \in G$. As in [3] we define $Z = iH_0 - \frac{1}{2} R$ and let B_t be the semigroup on $\mathscr{H}$ with infinitesmal generator Z. S_t is the semigroup on V given by $S_t(\varrho) = B_t \varrho B_t^*$. By Theorem 4.7 of [3] $\mathscr{J}$ and Z are the infinitesmal generators of a unique quantum stochastic process $\mathscr{E}$ on X, V. Moreover $\mathscr{E}$ is covariant with respect to G in the following sense. G has an action on each of the sample spaces X_t induced from its action on X and for all $t \geqq 0$, $\varrho \in V$ and $E \subseteq X_t$, $\mathscr{E}$ satisfies

$$\mathscr{E}_t(E_{g^{-1}}, \varrho) = \alpha_g\{\mathscr{E}_t(E, \alpha_{g^{-1}}(\varrho))\}\,. \tag{2.7}$$

This equation implies that the semigroups S and T on V defined in [3] satisfy

$$T_t \alpha_g = \alpha_g T_t\,; \qquad S_t \alpha_g = \alpha_g S_t \tag{2.8}$$

for all $t \geqq 0$ and $g \in G$. For the rest of the paper the term *process* will mean a quantum stochastic process constructed from given $H_0, A: X \to \mathscr{L}(\mathscr{H})$, $U: G \to \mathscr{L}(\mathscr{H})$ as described above.

In order to formulate further conditions on processes we define an *order ideal* on a state space V as a subset $I \subseteq V^+$ such that

(i) if $x, y \in I$ and $\alpha, \beta \geqq 0$ then $\alpha x + \beta y \in I$;

(ii) if $0 \leqq x \leqq y \in I$ then $x \in I$.

In an abstract state space order ideals do not have very good properties, but for the case $V = \mathscr{T}_s(\mathscr{H})$ the following results are well known and easy to prove. The closure of an order ideal is an order ideal and every finite-dimensional order ideal I is closed, the set $\{\varrho \in I : \mathrm{tr}[\varrho] = 1\}$ forming a compact base for I in the sense of [5]. There is a one-one correspondence between closed order ideals $I \subseteqq V^+$ and closed subspaces $\mathscr{K} \subseteqq \mathscr{H}$ given by

$$I = \{\varrho \in \mathscr{T}_s(\mathscr{H})^+ : \mathrm{supp}(\varrho) \subseteqq \mathscr{K}\} . \tag{2.9}$$

If I is a closed order ideal then $W = I - I$ is a closed subspace of V and even a state space. If $S \subseteqq V^+$ is any set then

$$I = \left\{\varrho \in V^+ : \varrho \leqq \sum_{r=1}^{n} \alpha_r s_r \quad \text{for some} \quad \alpha_r \geqq 0, s_r \in S, n\right\}$$

is the smallest order ideal containing S and is called the order ideal *generated by S.*

Now suppose $\mathscr{E}$ is a process on X, V and $T_t(\varrho) \in I$ for all $\varrho \in I$ and $r \geqq 0$, where I is a certain closed order ideal. Then for all $t > 0$ and all Borel sets $E \subseteqq X_t$ and all $\varrho \in I$

$$0 \leqq \mathscr{E}_t(E, \varrho) \leqq \mathscr{E}_t(X_t, \varrho) = T_t(\varrho) \in I$$

so $\mathscr{E}_t(E, \varrho) \in I$. Therefore it is possible to define a *restriction* of the process to the state space $W = I - I$. The process is called *irreducible* if $\dim V > 1$ and $\mathscr{E}$ has no proper restriction. We comment that our notion of restriction does not agree with that introduced in [6]. Indeed in the sense of [6] the state space V has no proper restrictions because the von Neumann algebra $\mathscr{L}(\mathscr{H})$ has no proper central projections.

Proposition 1. *If* $\dim \mathscr{H} > 1$ *the process* $\mathscr{E}$ *is irreducible if and only if there is no proper closed subspace* $\mathscr{K}$ *of* $\mathscr{H}$ *such that* $A_x \mathscr{K} \subseteqq \mathscr{K}$ *for all* $x \in X$ *and*

$$(1 - Z)\{\mathscr{K} \cap \mathscr{D}(Z)\} = \mathscr{K} \tag{2.10}$$

where $\mathscr{D}(Z)$ *is the domain of the infinitesmal generator* Z.

Proof. Suppose $I \subseteqq V^+$ is a closed invariant ideal corresponding to the subspace $\mathscr{K} \subseteqq \mathscr{H}$. Suppose $\xi \in \mathscr{H}$ and f_n is a sequence of positive continuous functions on X with $\int_X f_n(x)\, dx = 1$ and such that the support of f_n decrease to $\{x\}$. For every $t > 0$ we define $\mathscr{J}_t$ on X, V as in [3] and obtain

$$\begin{aligned} 0 \leqq \mathscr{J}_t(f_n, \xi \otimes \bar{\xi}) &\leqq t^{-1} \mathscr{E}_t(X, \xi \otimes \bar{\xi}) \\ &= t^{-1} T_t(\xi \otimes \bar{\xi}) \end{aligned}$$

so

$$\begin{aligned} (A_x \xi) \otimes (A_x \xi)^- &= \lim_{n \to \infty} \mathscr{J}(f_n, \xi \otimes \bar{\xi}) \\ &= \lim_{n \to \infty} \lim_{m \to \infty} \mathscr{J}_{m^{-1}}(f_n, \xi \otimes \bar{\xi}) \end{aligned}$$

lies in I. Therefore $A_x\xi \in \mathscr{K}$. Also for all $t \geqq 0$

$$0 \leqq (B_t\xi) \otimes (B_t\xi)^- = S_t(\xi \otimes \bar{\xi}) \leqq T_t(\xi \otimes \bar{\xi})$$

so the one-parameter semigroup of contractions B leaves $\mathscr{K}$ invariant. The equivalence of the last statement with Eq. (2.10) is a straightforward consequence of the general theory of one-parameter semigroups [7].

The converse proposition is obtained in a similar manner by examining the construction of $\mathscr{E}$ from $\mathscr{J}, Z$ in Theorem 4.7 of [3].

If $\mathscr{E}$ is any process on X, V and $\varrho \in V^+$ we define a positive measure P_ϱ with total mass $\mathrm{tr}[\varrho]$ on X_t for $t < \infty$ by

$$P_\varrho(E) = \mathrm{tr}[\mathscr{E}_t(E, \varrho)] \tag{2.11}$$

for all Borel sets $E \subseteqq X_t$. There measures are compatible under the natural maps $\pi: X_t \to X_s$ defined for $\infty \geqq t \geqq s$, and therefore there is a measure, which we also denote by P_ϱ, on X_∞ compatible with all of them. For each set $E \subseteqq X_\infty$ the map $\varrho \to P_\varrho(E)$ is positive, linear and bounded with norm not greater than one. Instead of regarding $\mathscr{E}_t$ as defined on X_t we regard it as defined on the σ-field $\mathscr{F}_t$ of Borel sets in X_∞ which are inverse images under $\pi: X_\infty \to X_t$ of Borel sets in X_t. The $\mathscr{F}_t$ form an increasing family of σ-fields in X_∞ and the union generates the σ-field $\mathscr{F}$ of all Borel sets in X_∞. If $\omega \in X_\infty$ is a sample point containing at least n events we denote by $x_n(\omega)$ and $t_n(\omega)$ the place and time of the nth event. If a sample point ω does not contain n events we adopt the convention $t_n(\omega) = +\infty$. It will be convenient to introduce the following notation for certain Borel subsets of X_∞, where U is any Borel subset of X.

$$A_t^n = \{t_n(\omega) \leqq t, t_{n+1}(\omega) > t\}\,. \tag{2.12}$$

$$B_t^{n,U} = \{x_i(\omega) \in U \text{ on at least } n \text{ occasions with } t_i(\omega) \leqq t\} \tag{2.13}$$

except for $t = \infty$ we demand all $t_i(\omega) < \infty$.

$$L_t^U = \bigcup_{i=1}^{\infty} \{t_i(\omega) \leqq t, x_i(\omega) \in U, t_{i+1}(\omega) > t\}\,. \tag{2.14}$$

If t is an integral multiple of $1/n$ we define

$$C_t^n = \left\{\text{if } \frac{r-1}{n} < t_i(\omega) \leqq \frac{r}{n} \text{ for any } r \leqq nt \text{ and } i \text{ then } t_{i+1}(\omega) > \frac{r}{n}\right\} \tag{2.15}$$

From [3] we have the estimate

$$\mathrm{tr}[\mathscr{E}_t(B_t^{n,X}, \varrho)] \leqq \frac{K^n t^n}{n!}\,\mathrm{tr}[\varrho] \tag{2.16}$$

for all $\varrho \in V^+$, where K is a bound on the total interaction rate of the process. The following lemma allows us to use the sets C_t^n to construct approximate discrete skeletons to the continuous time process $\mathscr{E}$.

Lemma 2. *If $t>0$ is an integer, $\varrho \in V$, $E \in \mathscr{F}_t$ and $n \geqq K$ then*

$$\|\mathscr{E}_t(E \cap C_t^n, \varrho) - \mathscr{E}_t(E, \varrho)\| \leqq \frac{K^2 t}{2n} \|\varrho\| . \tag{2.17}$$

Proof. Define $W(\varrho) = \mathscr{E}_{1/n}(X_{1/n} - B_{1/n}^{2,X}, \varrho)$ so that $\mathscr{E}_t(C_t^n, \varrho) = W^{tn}(\varrho)$. Suppose first that $\varrho \in V^+$. If $n \geqq K$ then Eq. (2.16) yields

$$\operatorname{tr}[\mathscr{E}_t(C_t^n, \varrho)] \geqq \left(1 - \frac{K^2}{2n^2}\right)^{nt} \operatorname{tr}[\varrho] \geqq \left(1 - \frac{K^2 t}{2n}\right) \operatorname{tr}[\varrho] .$$

Therefore

$$\begin{aligned} \|\mathscr{E}_t(E \cap C_t^n, \varrho) - \mathscr{E}_t(E, \varrho)\| &= \operatorname{tr}[\mathscr{E}_t(E - E \cap C_t^n, \varrho)] \\ &\leqq \operatorname{tr}[\mathscr{E}_t(X - C_t^n, \varrho)] \\ &\leqq \frac{K^2 t}{2n} \|\varrho\| . \end{aligned}$$

For a general element $\varrho \in V$ we can solve $\varrho = \varrho_1 - \varrho_2$ where $\varrho_1, \varrho_2 \in V^+$ and $\|\varrho\| = \|\varrho_1\| + \|\varrho_2\|$. Therefore

$$\begin{aligned} \|\mathscr{E}_t(E \cap C_t^n, \varrho) - \mathscr{E}_t(E, \varrho)\| &\leqq \frac{K^2 t}{2n} \{\|\varrho_1\| + \|\varrho_2\|\} \\ &= \frac{K^2 t}{2n} \|\varrho\| . \end{aligned}$$

We call a Borel set $E \subseteqq X$ *null* if $P_\varrho\{B_\infty^{1,E}\} = 0$ for all $\varrho \in V^+$.

Lemma 3. *An open set $U \subseteqq X$ is null if and only if $A_x = 0$ for all $x \in U$. If $\mathscr{E}$ is irreducible then either U is null or $P_\varrho\{B_\infty^{1,U}\} > 0$ for all non-zero $\varrho \in V^+$; moreover the set X cannot be null.*

Proof. Suppose $A_x \xi \neq 0$ for some $x \in U$ and hence for x in a subset of positive measure in U. Then for $\varrho = \xi \otimes \bar{\xi}$

$$\mathscr{J}(U, \varrho) = \int_{x \in U} (A_x \xi) \otimes (A_x \xi)^- \, dx \neq 0 .$$

In the notation of [3]

$$\begin{aligned} \mathscr{J}(U, \varrho) &= \lim_{t \to 0} \mathscr{J}_t(U, \varrho) \\ &= \lim_{t \to 0} t^{-1} \mathscr{E}_t(\{0 < t_1(\omega) \leqq t, x_1(\omega) \in U, t_2(\omega) > t\}, \varrho) \end{aligned}$$

and so by choosing t suitably small

$$P_\varrho\{B_\infty^{1,U}\} \geqq P_\varrho\{0 < t_1(\omega) \leqq t, x_1(\omega) \in U, t_2(\omega) > t\} > 0 .$$

Conversely suppose $\varrho \in V^+$ and

$$\begin{aligned} 0 &< P_\varrho\{B_\infty^{1,U}\} \\ &= \sum_{n=1}^{\infty} P_\varrho\{x_i(\omega) \notin U \quad \text{for} \quad i<n \quad \text{and} \quad x_n(\omega) \in U\} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P_\varrho\{x_i(\omega) \notin U \quad \text{for} \quad i<n, x_n(\omega) \in U, m-1 < t_n(\omega) \leqq m\} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lim_{r\to\infty} P_\varrho\{\omega \in C_m^r, x_i(\omega) \notin U \quad \text{for} \quad i<n, \\ &\quad x_n(\omega) \in U, m-1 < t_n(\omega) \leqq m\} \end{aligned}$$

by Lemma 2. Therefore for suitable m, n, r and $\varrho' \in V^+$

$$\begin{aligned} 0 &< \operatorname{tr}\left[\mathscr{E}_{1/r}\left(\left\{x_1(\omega) \in U, t_1(\omega) \leqq \frac{1}{r}, t_2(\omega) > \frac{1}{r}\right\}, \varrho'\right)\right] \\ &= \operatorname{tr}[\mathscr{J}_{1/r}(U, \varrho')] \\ &= \int_0^{1/r} \operatorname{tr}[S_{1/r-s}\mathscr{J}(U, S_s\varrho')]\, ds \end{aligned}$$

by Eq. (4.5) of [3]. Putting $\varrho'' = S_s\varrho'$ for some $0 < s < \dfrac{1}{r}$ we obtain

$$0 < \mathscr{J}(U, \varrho'') = \int_U A_x \varrho'' A_x^* dx .$$

It follows that $A_x \neq 0$ for some $x \in U$.

Now suppose that $\mathscr{E}$ is irreducible and U is an open set which is not null. The set J of $\varrho \in V^+$ such that $P_\varrho\{B_\infty^{1,U}\} = 0$ is a norm closed order ideal which is invariant because if $\varrho \in J$

$$\begin{aligned} P_{T_t\varrho}\{B_\infty^{1,U}\} &= P_\varrho\{x_i(\omega) \in U \quad \text{some } i \text{ with} \quad t < t_i(\omega) < \infty\} \\ &\leqq P_\varrho\{B_\infty^{1,U}\} = 0 . \end{aligned}$$

Since $J \neq V^+$ we must have $J = 0$.

Finally suppose X is a null set. Then $A_x = 0$ for all $x \in X$ so $\mathscr{J} = 0$ and $R = 0$. Therefore $Z = iH_0$ and $T_t(\varrho) = e^{itH_0}\varrho e^{-itH_0}$. Since we have $\dim \mathscr{H} > 1$ there exists a proper projection P commuting with H_0 and then $J = \{\varrho \in V^+ : P\varrho = \varrho P = \varrho\}$ is a proper closed invariant order ideal, which contradicts irreducibility.

Returning to the general situation we say that a process $\mathscr{E}$ on X, V is *simple* if it satisfies the condition

(S) *for every open $U \subseteqq X$ with compact closure the order ideal V_U^+ generated by the set of $\mathscr{E}_t(L_t^U, \varrho)$ where $\varrho \in V^+$ and $t > 0$ is finite-dimensional.*

Proposition 4. *A process $\mathscr{E}$ is simple if and only if for every open relatively compact set $U \subseteqq X$ there exists a finite dimensional subspace $\mathscr{K} \subseteqq \mathscr{H}$ such that $A_x \mathscr{H} \subseteqq \mathscr{K}$ for all $x \in U$, $\mathscr{K} \subseteqq \mathscr{D}(Z)$ and $Z\mathscr{K} \subseteqq \mathscr{K}$.*

Proof. This is just a slight modification of Proposition 1 and Lemma 3.

Before develooing the implications of our hypotheses we interpret them for the case of classical Markov processes, slightly generalising the situation of [3]. The notation used for the remainder of this section will not be referred to elsewhere. We start with the space X as before but now define the state space V as the space of all bounded signed measures on X. If $\mathscr{B}$ is the σ-field of Borel sets in X a Markov process can be constructed from a bounded stochastic kernel $K: X \times \mathscr{B} \to \mathbb{R}$. The total interaction rate is the bounded Borel function $R(x) = K(x, X)$ and the semigroup S is given by

$$(S_t \mu)(E) = \int_E e^{-R(x)t} \mu(dx) \geqq e^{-\alpha t} \mu(E)$$

for all $\mu \in V^+$, where α is any bound on R. As we have formulated them the definitions of simplicity and irreducibility apply to processes defined on completely general state spaces, and therefore to this situation. We observe that the order ideal generated by a set $S \subseteqq V^+$ can be finite dimensional only when there exists a finite set $Y \subseteqq X$ such that $\mu(X - Y) = 0$ for all $\mu \in S$.

Proposition 5. *If $\mathscr{E}$ is an irreducible simple Markov process on X then X is discrete and $\mathscr{E}$ is an irreducible Markov chain in the classical sense. $\mathscr{E}$ cannot have any connected group of symmetries.*

Proof. Let U be any open set in X with compact closure, let $x \in X$ and let μ be the measure $E \to K(x, E \cap U)$. Then

$$\begin{aligned} 0 \leqq e^{-\alpha t} \mu &\leqq \mathscr{E}_t(\{t_1(\omega) \leqq t, x_1(\omega) \in U, t_2(\omega) > t\}, \varepsilon_x) \\ &\leqq \mathscr{E}_t(L_t^U, \varepsilon_x) \in V_U^+ . \end{aligned}$$

Therefore there exists a finite set $X_U \subseteqq U$ such that for all $x \in X$ the measure $E \to K(x, E \cap U)$ has support in X_U. We take X_U to be the smallest such set and observe that the union X' of the X_U as U runs through all open sets with compact closure, is discrete and therefore countable. Also for all $x \in X$, $K(x, X - X') = 0$. If $I = \{\mu \in V^+ : \mu(X - X') = 0\}$ then I is a closed invariant order ideal in V^+, so $I = V^+$ by irreducibility, and $X = X'$. The closed ideals of V^+ correspond one-one with the subsets of X and the process is irreducible if and only if there is no subspace of X invariant under K. The last statement of the proposition is obvious.

§ 3. Recurrence and Transience

Throughout this section we suppose $\mathscr{E}$ is a simple irreducible process on X, V and that U, W are two given open subsets of X with compact closures. If $E \subseteq X$ is a Borel set we say the process is *m times recurrent* at E for the state $\varrho \in V^+$ if

$$P_\varrho\{B_\infty^{m,E}\} = \mathrm{tr}[\varrho]\,. \tag{3.1}$$

Lemma 6. *If $\mathscr{E}$ is singly recurrent at U for all $\varrho \in V_U^+$ then it is infinitely recurrent at U for all $\varrho \in V$, unless U is null.*

Proof. Suppose that for some integer m and all $\varrho \in V_U^+$ we have established that $P_\varrho\{B_\infty^{m,U}\} = \mathrm{tr}[\varrho]$. If $\varepsilon > 0$ then by the compactness of $\{\varrho \in V_U^+ : \mathrm{tr}[\varrho] = 1\}$ and Lemma 2 there exist integers t and n such that for all $\varrho \in V_U^+$

$$P_\varrho\{B_t^{m,U} \cap C_{2t}^n\} \leqq \left(1 - \frac{\varepsilon}{2}\right) \mathrm{tr}[\varrho]\,.$$

Therefore

$$\begin{aligned}
P_\varrho\{B_\infty^{2m,U}\} &\geqq P_\varrho\{B_{2t}^{2m,U} \cap C_{2t}^n\} \\
&\geqq \sum_{p,q} P_\varrho\Big\{ x_i(\omega) \in U \quad \text{for the } m\text{th time where } \frac{q-1}{n} < t_i(\omega) \leqq \frac{q}{n} \leqq t\,, \\
&\qquad x_j(\omega) \in U \quad \text{for the } 2m\text{th time where } \frac{p+q+1}{n} < t_j(\omega) \\
&\qquad \leqq \frac{p+q}{n} \leqq 2t,\, \omega \in C_{2t}^n \Big\}\,.
\end{aligned} \tag{3.2}$$

If we define

$$\varrho_q = \mathscr{E}_{q/n}\left(\left\{\omega \in C_{q/n}^n,\, x_i(\omega) \in U \quad \text{for the } n\text{th time where } \frac{q-1}{n} < t_i(\omega) \leqq \frac{q}{n} \leqq t\right\}, \varrho\right)$$

then $0 \leqq \varrho_q \leqq \mathscr{E}_{q/n}(L_{q/n}^U, \varrho) \in V_U^+$ so $\varrho_q \in V_U^+$. The right hand side of Eq. (3.2) now becomes

$$\begin{aligned}
&= \sum_{p,q} P_{\varrho_q}\left\{\omega \in C_{2t-q/n}^n,\, x_j(\omega) \in U \quad \text{for the } n\text{th time where } \frac{p-1}{n} < t_j(\omega) \leqq \frac{p}{n} \leqq 2t - \frac{q}{n}\right\} \\
&= \sum_q P_{\varrho_q}\{C_{2t-q/n}^n \cap B_{2t-q/n}^{m,U}\} \\
&\geqq \sum_q P_{\varrho_q}\{C_{2t}^n \cap B_t^{m,U}\} \\
&\geqq \sum_q \left(1 - \frac{\varepsilon}{2}\right) \mathrm{tr}[\varrho_q]
\end{aligned}$$

$$= \left(1-\frac{\varepsilon}{2}\right)\sum_q P_\varrho\left\{\omega\in C^n_{q/n},\, x_i(\omega)\in U \quad \text{for the } n\text{th time where} \right.$$

$$\left. \frac{q-1}{n} < t_i(\omega) \leqq \frac{q}{n} \leqq t\right\}$$

$$\geqq \left(1-\frac{\varepsilon}{2}\right) P_\varrho\{C^n_{2t}\cap B^{m,U}_t\}$$

$$\geqq \left(1-\frac{\varepsilon}{2}\right)^2 \operatorname{tr}[\varrho] \geqq (1-\varepsilon)\operatorname{tr}[\varrho]\,.$$

As $\varepsilon>0$ is arbitrary we obtain $P_\varrho\{B^{2m,U}_\infty\}=\operatorname{tr}[\varrho]$. Now we know that $P_\varrho\{B^{1,U}_\infty\}=\operatorname{tr}[\varrho]$ for all $\varrho\in V^+_U$ so inductively we obtain $P_\varrho\{B^{\infty,U}_\infty\}=\operatorname{tr}[\varrho]$ for all $\varrho\in V^+_U$. The set

$$J=\{\varrho\in V^+ : P_\varrho\{B^{\infty,U}_\infty\}=\operatorname{tr}[\varrho]\}$$

is a norm closed invariant order ideal containing V^+_U, which is non-zero if U is not null. By irreducibility $J=V^+$ and the lemma is proved.

Lemma 7. *If the process $\mathscr{E}$ is singly recurrent at U for all $\varrho\in V^+_U$ then it is singly recurrent at W for all $\varrho\in V^+_U$.*

Proof. By Lemma 3 and the compactness of $\{\varrho\in V^+_U : \operatorname{tr}[\varrho]=1\}$ there is a constant $k>0$ such that $P_\varrho\{B^{1,W}_\infty\}\geqq k\operatorname{tr}[\varrho]$ for all $\varrho\in V^+_U$. Let k be the largest such constant and let $\varrho_0\in V^+_U$ be a state such that $P_{\varrho_0}\{B^{1,W}_\infty\}=k\operatorname{tr}[\varrho_0]=k$. Let $\varepsilon>0$ and let t be a sufficiently large integer so that $P_{\varrho_0}\{B^{l,W}_t\}>k-\varepsilon$. Using Lemma 6 let s be a sufficiently large integer that

$$P_{\varrho_0}\{\text{if}\quad t_i(\omega)\leqq t \quad\text{then}\quad x_i(\omega)\notin W, \quad\text{but}\quad x_j(\omega)\in U \quad \text{for some}$$
$$t<t_j(\omega)\leqq t+s\}>1-k-\varepsilon\,.$$

Using Lemma 2 let n be a sufficiently large integer that

$$P_{\varrho_0}\{\omega\in C^n_{s+t},\quad \text{if}\quad t_i(\omega)\leqq t\quad \text{then}\quad x_i(\omega)\notin W,\quad \text{but}\quad x_j(\omega)\in U$$
$$\text{for some}\quad t<t_j(\omega)\leqq t+s\}>1-k-\varepsilon\,.$$

Then

$$P_{\varrho_0}\{B^{1,W}_\infty\}\geqq P_{\varrho_0}\{B^{1,W}_t\}+\sum_{m=1}^{ns} P_{\varrho_m}\{B^{1,W}_\infty\} \tag{3.3}$$

where

$$\varrho_m=\mathscr{E}_{t+m/n}\left(\left\{\omega\in C^n_{t+m/n},\, x_i(\omega)\notin W \quad\text{for}\quad t_i(\omega)\leqq t,\, x_i(\omega)\notin U\right.\right.$$
$$\text{for}\quad t<t_i(\omega)\leqq\frac{m-1}{n},\, x_i(\omega)\in U\quad\text{for}$$
$$\left.\left.\text{some}\quad \frac{m-1}{n}<t_i(\omega)\leqq\frac{m}{n}\right\},\, \varrho_0\right)$$

$$\leqq \mathscr{E}_{t+m/n}(L^U_{t+m/n},\varrho_0)\in V^+_U\,.$$

Since $\varrho_m \in V_U^+$, $P_{\varrho_m}\{B_\infty^{1,W}\} \geqq k \operatorname{tr}[\varrho_m]$ and Eq. (3.3) becomes

$$\begin{aligned}
k &\geqq k - \varepsilon + k \sum_{m=1}^{ns} k \operatorname{tr}[\varrho_m] \\
&= k - \varepsilon + k \sum_{m=1}^{ns} P_{\varrho_0}\Big\{\omega \in C_{t+m/n}^n, x_i(\omega) \notin W \quad \text{for} \quad t_i(\omega) \leqq t, \\
&\qquad x_i(\omega) \notin U \quad \text{for} \quad t < t_i(\omega) \leqq \frac{m-1}{n}, \\
&\qquad x_i(\omega) \in U \quad \text{for some} \quad \frac{m-1}{n} < t_i(\omega) \leqq \frac{m}{n}\Big\} \\
&\geqq k - \varepsilon + k P_{\varrho_0}\{\omega \in C_{t+s}^n, x_i(\omega) \notin W \quad \text{for} \quad t_i(\omega) \leqq t \quad \text{but} \\
&\qquad x_i(\omega) \in U \quad \text{for some} \quad t < t_i(\omega) \leqq t + s\} \\
&> k - \varepsilon + k(1 - k - \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$ gives $0 \geqq k(1-k)$, and as $k > 0$ we have $k = 1$, which is the required result.

Lemma 8. *Suppose U and W are not null. If $\mathscr{E}$ is singly recurrent at U for all $\varrho \in V_U^+$ then it is infinitely recurrent at W for all $\varrho \in V^+$.*

Proof. Let $\varrho \in V^+$ and for $\varepsilon > 0$ let t, n be large enough integers so that $P_\varrho\{C_t^n \cap B_t^{1,U}\} \geqq (1-\varepsilon) \operatorname{tr}[\varrho]$. Then defining

$$\begin{aligned}
\varrho_m &= \mathscr{E}_{m/n}\Big(\Big\{\omega \in C_{m/n}^n, x_i(\omega) \notin U \quad \text{for} \quad t_i(\omega) \leqq \frac{m-1}{n} \quad \text{but} \quad x_i(\omega) \in U \\
&\qquad \text{for some} \quad \frac{m-1}{n} < t_i(\omega) \leqq \frac{m}{n}\Big\}, \varrho\Big) \\
&\leqq \mathscr{E}_{m/n}(L_{m/n}^U, \varrho) \in V_U^+
\end{aligned}$$

we obtain

$$\begin{aligned}
P_\varrho\{B_\infty^{1,W}\} &\geqq \sum_{m=1}^{nt} P_{\varrho_m}\{B_\infty^{1,W}\} \\
&\geqq \sum_{m=1}^{nt} \operatorname{tr}[\varrho_m] \\
&\geqq P_\varrho\{C_t^n \cap B_t^{1,U}\} \\
&\geqq (1-\varepsilon) \operatorname{tr}[\varrho].
\end{aligned}$$

Letting $\varepsilon \to 0$ gives the equation $P_\varrho\{B_\infty^{1,W}\} = \operatorname{tr}[\varrho]$ for all $\varrho \in V^+$. By Lemma 6 it follows that $P_\varrho\{B_\infty^{\infty,W}\} = \operatorname{tr}[\varrho]$ for all $\varrho \in V^+$.

We now turn to the study of processes which do not satisfy the hypothesis of Lemma 6. We define an integer-valued function $N_t^E : X_\infty \to [0, \infty]$ for every Borel set $E \subseteq X$ and every $t < \infty$ by

$$N_t^E(\omega) = \{\text{number of } i \text{ for which} \quad x_i(\omega) \in E \quad \text{and} \quad t_i(\omega) \leqq t\} \tag{3.4}$$

while for $t=\infty$ we demand $t_i(\omega)<\infty$. For each state $\varrho \in V^+$ the expected number of occurences $N_t^E(\varrho)$ of events within E up to time t is given by

$$N_t^E(\varrho)=\int_{X_\infty} N_t^E(\omega)\,P_\varrho(d\omega) \tag{3.5}$$

so that

$$N_\infty^E(\varrho)=\sum_{n=1}^{\infty} P_\varrho\{x_n(\omega)\in E\}\,. \tag{3.6}$$

Lemma 9. *Suppose $\mathscr{E}$ is not singly recurrent at U for some $\varrho_0 \in V_U^+$. Then there exist $0<k<1$ and an integer m such that for all $\varrho \in V_U^+$ and all integers n*

$$P_\varrho\{B_\infty^{mn,U}\} \leqq k^n \operatorname{tr}[\varrho]\,. \tag{3.7}$$

Proof. Let $S_m \subseteqq V_U^+$ be the set of $\varrho \in V_U^+$ such that $\operatorname{tr}[\varrho]=1$ and $P_\varrho\{B_\infty^{m,U}\}=1$. Then $S_m \subseteqq S_{m-1}$ and if $\varrho_1 \in \bigcap_{m=1}^{\infty} S_m$ we have $\operatorname{tr}[\varrho_1]=1$ and $\varrho \in J$ where

$$J=\{\varrho \in V^+ : P_\varrho\{B_\infty^{\infty,U}\}=\operatorname{tr}[\varrho]\}\,.$$

But J is a norm closed invariant order ideal and $\varrho_0 \notin J$ so by irreducibility $J=0$. Therefore $\bigcap_{m=1}^{\infty} S_m=\phi$ and by compactness $S_m=\phi$ for some m. For that m there exists a constant $0<k<1$ such that for all $\varrho \in V_U^+$ Eq. (3.7) holds if $n=1$.

Suppose now that we have established that Eq. (3.7) holds for some n and all $\varrho \in V_U^+$. Then for any $\varepsilon>0$ there exist integers r and t such that

$$\begin{aligned} P_\varrho\{B_\infty^{m(n+1),U}\}-\varepsilon &\leqq P_\varrho\{B_t^{m(n+1),U}\cap C_t^r\} \\ &=\sum_{s=1}^{rt} P_{\varrho_s}\{C_{t-s/r}^r \cap B_{t-s/r}^{mn,U}\} \end{aligned} \tag{3.8}$$

where

$$\varrho_s=\mathscr{E}_{s/r}\left(\left\{\omega \in C_{s/r}^r,\ x_i(\omega)\in U \quad \text{for the } n\text{th time where} \right.\right.$$
$$\left.\left.\frac{s-1}{r}<t_i(\omega)\leqq \frac{s}{r}\right\},\ \varrho\right)\in V_U^+\,.$$

The right hand side of Eq. (3.8) is

$$\begin{aligned} &\leqq \sum_{s=1}^{rt} k^n \operatorname{tr}[\varrho_s] \\ &\leqq k^n P_\varrho\{B_\infty^{m,U}\} \\ &\leqq k^{n+1}\operatorname{tr}[\varrho]\,. \end{aligned}$$

Going to the limit as $\varepsilon \to 0$ gives the required result by induction.

Lemma 10. *Suppose $\mathscr{E}$ is not singly recurrent at U for some $\varrho_0 \in V_U^+$. Then there exists a constant $\alpha_U < \infty$ such that $N_\infty^U(\varrho) \leqq \alpha_U \operatorname{tr}[\varrho]$ for all $\varrho \in V^+$.*

Proof. For all $\varrho \in V^+$ we have

$$N_\infty^U(\varrho) = \lim_{t\to\infty} \lim_{n\to\infty} \int_{C_t^n \cap B_t^{1,U}} N_\infty^U(\omega)\, P_\varrho(d\omega).$$

Also if we define

$$\varrho_s = \mathscr{E}_{s/n}\left(\left\{\omega \in C_{s/n}^n, x_i(\omega) \notin U \quad \text{for} \quad t_i(\omega) \leqq \frac{s-1}{n}, x_i(\omega) \notin U \right.\right.$$
$$\left.\left.\text{for some} \quad \frac{s-1}{n} < t_i(\omega) \leqq \frac{s}{n}\right\}, \varrho\right) \in V_U^+$$

and $\sigma = \sum_{s=1}^{nt} \varrho_s \in V_U^+$, then $\operatorname{tr}[\sigma] \leqq \operatorname{tr}[\varrho]$ and

$$\int_{C_t^n \cap B_t^{1,U}} N_\infty^U(\omega)\, P_\varrho(d\omega) = \sum_{s=1}^{nt} \left\{\operatorname{tr}[\varrho_s] + \int_{C_{t-s/n}^n} N_\infty^U(\omega)\, P_{\varrho_s}(d\omega)\right\}$$
$$\leqq \operatorname{tr}[\varrho] + \int_{X_\infty} N_\infty^U(\omega)\, P_\sigma(d\omega).$$

By Lemma 9 there exist constants k and m such that

$$P_\varrho\{N_\infty^U(\omega) \geqq rm\} \leqq k^r \operatorname{tr}[\sigma]$$

for all integers r. Therefore

$$\int_{X_\infty} N_\infty^U(\omega)\, P_\sigma(d\omega) \leqq \sum_{r=0}^{\infty} m k^r \operatorname{tr}[\sigma] = \frac{m}{1-k} \operatorname{tr}[\sigma]$$

and

$$\int_{X_\infty} N_\infty^U(\omega)\, P_\varrho(d\omega) \leqq \left(1 + \frac{m}{1-k}\right) \operatorname{tr}[\varrho].$$

We summarise all the results obtained so far.

Theorem 11. *Call a process $\mathscr{E}$ on X, V recurrent if every state $\varrho \in V^+$ is infinitely recurrent at every open set $U \subseteqq X$ which is not null. Call it transient if every open subset $U \subseteqq X$ with compact closure has a finite expected number of occurences for every state $\varrho \in V^+$. Then every simple irreducible process is either recurrent or transient.*

The problem of determining whether a given process is recurrent or transient is in general a difficult one, depending on detailed properties of H_0 and $\mathscr{J}$, but some general guide is contained in the following results.

We call a process on X, V *substantially finite-dimensional* if there exists a finite-dimensional projection P in $\mathscr{H}$ and a constant $\alpha > 0$ such that

(i) $T_t(P)\, P = P\, T_t(P)$ for all $t \geqq 0$;
(ii) $\operatorname{tr}[P\, T_t(P)] \geqq \alpha$ for all $t \geqq 0$.

By taking $P=1$ it is immediate that every finite-dimensional process is also substantially finite-dimensional.

Proposition 12. *Every substantially finite-dimensional, simple, irreducible process is recurrent.*

Proof. If U_n is an increasing sequence of open sets with compact closures whose union is X then $N_\infty^{U_n}(\omega)$ converges monotonely to $N_\infty^X(\omega)$ for all $\omega \in X_\infty$. Therefore by Lemma 3 for all non-zero $\varrho \in V^+$,

$$N_\infty^{U_n}(\varrho) \to N_\infty^X(\varrho) > 0 .$$

If P and α are as described above then by the compactness of the set of ϱ with $\operatorname{tr}[\varrho]=1$ in

$$J = \{\varrho \in V^+ : P\varrho = \varrho P = \varrho\}$$

there exist n and $\beta > 0$ such that for all $\varrho \in J$

$$N_\infty^{U_n}(\varrho) \geqq \beta \operatorname{tr}[\varrho] . \tag{3.9}$$

If now $\mathscr{E}$ is transient then application of the dominated convergence theorem to

$$\int_{X_\infty} N_\infty^{U_n}(\omega)\, P_\varrho(d\omega) = \int_{X_\infty} N_t^{U_n}(\omega)\, P_\varrho(d\omega) + \int_{X_\infty} N_\infty^{U_n}(\omega)\, P_{T_t\varrho}(d\omega)$$

yields

$$\lim_{t\to\infty} \int_{X_\infty} N_\infty^{U_n}(\omega)\, P_{T_t\varrho}(d\omega) = 0$$

for all $\varrho \in V^+$. Choosing in particular $\varrho = P$ and defining $\varrho' = P T_t(P) \in J$ where t is large enough gives

$$\int_{X_\infty} \mathrm{N}_\infty^{U_n}(\omega)\, P_{\varrho'}(d\omega) \leqq \int_{X_\infty} N_\infty^{U_n}(\omega)\, P_{T_t(\varrho)}(d\omega) < \beta \operatorname{tr}[\varrho'] .$$

This contradicts Eq. (3.9), so $\mathscr{E}$ cannot be transient.

An *equilibrium state* of a process $\mathscr{E}$ on X, V is by definition a state ϱ such that $T_t\varrho = \varrho$ for all $t \geqq 0$.

Theorem 13. *An equilibrium state of an irreducible process $\mathscr{E}$ has support equal to $\mathscr{H}$. A simple irreducible process with an equilibrium state must be recurrent. A finite-dimensional irreducible process possesses an equilibrium state, unique up to a constant multiple.*

Proof. The first statement follows from the fact that if ϱ_0 is an equilibrium state

$$J = \{\varrho \in V^+ : \varrho \leqq \alpha\varrho_0 \text{ some } \alpha\}$$

is an invariant order ideal, and so must be dense in V^+. For the second statement we observe as in Proposition 12 that if $\mathscr{E}$ is transient and U is an open set in X with compact closure then $N_\infty^U(\varrho_0) = N_\infty^U(T_t\varrho_0) \to 0$

as $t\to\infty$. Therefore $N_\infty^U(\varrho_0)=0$ for all such U and so $N_\infty^X(\varrho_0)=0$. But now $N_\infty^X(\varrho)=0$ for all $\varrho\in J$ and so for all $\varrho\in V^+$. By Lemma 3 this contradicts irreducibility.

If $\dim\mathscr{H}<\infty$ then

$$\Omega=\{\varrho\in V^+:\mathrm{tr}[\varrho]=1\}$$

is a compact convex set and T_t is a one-parameter semigroup of affine endomorphisms, so Ω possesses at least one T-fixed point by the Markov-Kakutani theorem. If there is more than one fixed point then by the Krein-Milman theorem there exist two distinct extreme fixed points say ϱ_1 and ϱ_2. Both ϱ_1 and ϱ_2 have support equal to $\mathscr{H}$ so there exists a non-zero constant α such that $\alpha\varrho_1\leqq\varrho_2\leqq\alpha^{-1}\varrho_1$. But by the definition of extreme points this implies $\varrho_1=\varrho_2$.

Note. We have not been able to establish whether an infinite dimensional irreducible process may have more than one equilibrium state.

The following theorem gives rather weak sufficient conditions for irreducibility of finite-dimensional processes and also describes the limiting behaviour in time of processes satisfying those conditions.

Theorem 14. *Let $\mathscr{E}$ be a finite-dimensional process covariant with respect to a connected group G. Suppose*

(i) *there is no proper subspace of $\mathscr{H}$ invariant under all $A_x, x\in X$ and all $U_g, g\in G$;*

(ii) *the lowest common multiple of the dimensions of the eigenvalues of the operators $A_x, x\in X$ is one.*

Then $\mathscr{E}$ is irreducible. There is a unique equilibrium state ϱ_0 with $\mathrm{tr}[\varrho_0]=1$ *and for all $\varrho\in V^+$ with* $\mathrm{tr}[\varrho]=1$, *$T_t(\varrho)$ converges to ϱ_0 as $t\to\infty$.*

Proof. We refer to [8] for terminology and general theorems concerning finite-dimensional associative algebras. Let $\mathscr{S}$ be the multiplicative semigroup in $\mathscr{L}(\mathscr{H})$ generated by 1 and $\{A_x:x\in X\}$. Let $\mathscr{A}$ be the linear span of $\mathscr{S}$ so that $\mathscr{A}$ is a linear algebra with identity, and $U_g\mathscr{A}U_g^{-1}=\mathscr{A}$ for all $g\in G$. Let $\mathscr{R}$ be the radical of $\mathscr{A}$ and let $\mathscr{K}=\mathscr{R}\mathscr{H}$. Then $\mathscr{K}\neq\mathscr{H}$ and since $\mathscr{R}$ is a G-invariant ideal in $\mathscr{A}$, $\mathscr{K}$ is invariant under all $A_x, x\in X$ and all $U_g, g\in G$. By condition (i) $\mathscr{K}=0$ so $\mathscr{R}=0$ and $\mathscr{A}$ is semisimple. Let $\mathscr{C}$ be the centre of $\mathscr{A}$. Then $U_g\mathscr{C}U_g^{-1}=\mathscr{C}$ for all $g\in G$ and G is connected. Therefore the action of G on C is trivial and any central projection $P\in\mathscr{C}$ must be equal to zero or one, again by condition (i). Therefore $\mathscr{A}$ is simple and by Wedderburn's theorem we can write $\mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_2$ and $\mathscr{A}=\mathscr{L}(\mathscr{H}_1)\otimes\mathbb{C}1$. By condition (ii) we now obtain $\dim\mathscr{H}_2=1$, so $\mathscr{A}=\mathscr{L}(\mathscr{H})$. Therefore for all non-zero $\xi\in\mathscr{H}$ there exist $B_1,\ldots,B_n\in\mathscr{S}$ such that $B_1\xi,\ldots,B_n\xi$ span $\mathscr{H}$. In particular there is no subspace of $\mathscr{H}$ invariant under all $A_x, x\in X$ so by Lemma 1, $\mathscr{E}$ is irreducible.

We next show that for all non-zero $\xi\in\mathscr{H}$ the mixed state $T_t(\xi\otimes\bar{\xi})$ has support equal to $\mathscr{H}$ for all sufficiently small t. Suppose none of the operators $B_1, \ldots, B_n$ above is a product of more than m operators A_x. Then for $\varrho = \xi \otimes \bar{\xi}$

$$T_{t_0}(\varrho) \geqq \sum_{r=0}^{n} \mathscr{E}_{t_0}\left(\left\{\frac{i-1}{r} t_0 < t_i(\omega) \leqq \frac{i}{r} t_0 \quad \text{for} \quad i=1, \ldots, r\right\}, \varrho\right)$$

$$= S_{t_0}(\varrho) + \sum_{r=0}^{n} \int_{x_t \in X} \int_{\frac{i-1}{r} t_0 < t_t \leqq \frac{i}{r} t_0} \xi_{t,x} \otimes \bar{\xi}_{t,x} dt_1 \ldots dt_r \, dx_1 \ldots dx_r$$

where

$$\xi_{t,x} = B_{t-t_r} A_{x_r} B_{t_r - t_{r-1}} A_{x_{r-1}} \ldots A_{x_1} B_{t_1} \xi$$

and $t = (t_1, \ldots, t_r)$, $x = (x_1, \ldots, x_r)$, by Eq. (4.13) of [3]. The result now follows from the joint continuity of $t, x \to \xi_{t,x}$ by taking t_0 small enough.

Now let Ω be the compact convex set $\{\varrho \in V^+ : \mathrm{tr}[\varrho] = 1\}$ generating the affine hyperplane $H = \{\varrho \in V : \mathrm{tr}[\varrho] = 1\}$. The interior of Ω in H consists of all ϱ whose support is equal to $\mathscr{H}$, so the unique equilibrium state ϱ_0 of T is interior. Let m be the Minkowski functional of Ω in H with respect to the "origin" ϱ_0 so

$$m(\varrho) = \inf\{\lambda \in \mathbb{R}^+ : (\varrho - \varrho_0) \in \lambda(\Omega - \varrho_0)\}.$$

It is clear that $m(\varrho) < 1$ if and only if $\varrho \in \mathrm{int}\,\Omega$, $m(\varrho_0) = 0$ and $m(T_t \varrho) \leqq m(\varrho)$ for all $\varrho \in H$ and $t \geqq 0$. Also we showed above that $m(T_t \varrho) < 1$ for all pure states ϱ in Ω and all sufficiently small $t > 0$, depending on ϱ. Therefore $m(T_t \varrho) < 1$ for all $t > 0$ and all $\varrho \in \Omega$ and by compactness for any $t_0 > 0$ there is a constant $k < 1$ such that $m(T_{t_0} \varrho) \leqq k$ for all $\varrho \in \Omega$. Therefore $m(T_{nt_0} \varrho) \leqq k^n$ and $T_t \varrho \to \varrho_0$ as $t \to \infty$.

§ 4. Transition Processes

It seems difficult to say very much more about quantum stochastic processes without imposing further conditions of a special nature on the infinitesmal generators. In this section we study the properties of one special class of processes, which we call transition processes because of their relevance to the time evolution of simple quantum mechanical systems emitting photons while undergoing transitions from one energy level to another. We define transition processes by certain global properties and then prove certain results about their infinitesmal generators, though it will become evident that we could just as easily have proceeded in the other direction.

A quantum stochastic process $\mathscr{E}$ on X, V will be called a *transition process* if

(i) *there is a given separable locally compact Hausdorff space Y and a continuous map σ of X into $Y \times Y$;*

(ii) *for all* $\varrho \in V^+$

$$P_\varrho\{\text{for some} \quad i, \sigma x_i(\omega) = (y_1, y_2), \sigma x_{i+1}(\omega) = (y_3, y_4) \quad \text{and} \quad y_2 \neq y_3\} = 0 . \tag{4.1}$$

The first of these conditions was observed to hold for the hydrogen atom, and this indeed constituted the first major advance in atomic spectroscopy. The second cannot be observed directly since the emissions of an individual atom cannot be distinguished from those of the surrounding atoms. Explicitly for the hydrogen atom we take $G = SU(2)$ and S_2 as the unit sphere in Euclidean space. Y is the set of energy levels of the atom and $X = Y \times Y \times S_2$ so at each event in X one determines the energy levels before and after the emission as well as the direction of emission of the photon whose detection is the event. The Hamiltonian H_0 is here known and the transition operators (determining the probability per unit time of the emission of a photon in a given direction causing a jump between given energy levels) are calculated by perturbation methods using quantum electrodynamics [9]. It is interesting here to note that the energy of the electron can only be observed by transitions, so this observable is *not* repeatable in the sense of [1]. In spite of this the following theorem still gives a general proof that the energy observable is associated with a *projection*-valued measure.

For any Borel set $E \subseteqq Y$ we define W_E^+ as the norm closed order ideal generated by the set of all $\mathscr{E}_t(L_t^{\sigma^{-1}(Y \times E)}, \varrho)$ where $\varrho \in V^+$ and $t > 0$. Let P_E be the corresponding orthogonal projection.

Theorem 15. *Let $\mathscr{E}$ be a simple, recurrent, irreducible, transition process on X, Y, covariant with respect to the connected group G. Then P is a projection-valued measure with discrete support in Y. P commutes with all U_g, $g \in G$ and with the Hamiltonian H_0. If $x \in X$ and $\sigma x = (y, z)$ then A_x has support in $P_y \mathscr{H}$ and range in $P_z \mathscr{H}$.*

Proof. Let $A \in \mathscr{L}(\mathscr{H}), 0 \leqq A \leqq 1$ be the operator defined by the equation

$$\operatorname{tr}[A\varrho] = P_\varrho\{x_1(\omega) \in \sigma^{-1}(F \times Y)\} .$$

The set

$$J = \{\varrho \in V^+ : \operatorname{tr}[A\varrho] = \operatorname{tr}[\varrho]\}$$

is a norm-closed order ideal containing all states of the form $\mathscr{E}_t(L_t^{\sigma^{-1}(Y \times F)}, \varrho)$ because $\mathscr{E}$ is recurrent. Therefore $J \supseteqq W_F^+$ and every $\varrho \in W_F^+$ has support in the eigenspace of A corresponding to the eigenvalue one. Similarly if $E \cap F = \phi$ every $\varrho \in W_E^+$ has support in the eigenspace of A corresponding to the eigenvalue zero. Therefore P_E and P_F are orthogonal projections. It is immediate from their definitions that $P_\phi = 0$, that $P_E \leqq P_F$ if $E \subseteqq F$, and from recurrence that $P_Y = 1$.

Let E_n be a countable disjoint family of Borel sets in Y with union E and let P_0 be the projection $P_0 = P_E - \sum_{n=1}^{\infty} P_{E_n}$. We show $P_0 = 0$. For all $\varrho \in V^+$ and $t > 0$

$$\operatorname{tr}[P_0 \mathscr{E}_t(L_t^{\sigma^{-1}(Y \times E)}, \varrho)] = \sum_{n=1}^{\infty} \operatorname{tr}[P_0 \mathscr{E}_t(L_t^{\sigma^{-1}(Y \times E_n)}, \varrho)] = 0$$

since $P_0 P_n = 0$ for all n. It follows that $\operatorname{tr}[P_0 \varrho] = 0$ for all $\varrho \in W_E^+$ so $P_0 P_E = 0$. But $P_0 \leqq P_E$, so $P_0 = 0$, and P is a projection-valued measure.

We next show that if $\sigma x = (y, z)$ and $A_x \neq 0$ then $P_{\{z\}} \neq 0$. Let U be any open neighborhood of z and let $U' = \{x : \sigma x \in Y \times U\}$. If $A_x \xi = 0$ and U'' is any open neighborhood of x contained in U' then $W_{U'}^+$ contains

$$\varrho = \int_{y \in U''} \int_{s=0}^{t} \xi_{y,s} \otimes \xi_{y,s}^- dy\, ds$$

where $\xi_{y,s} = B_{t-s} A_y B_s \xi$. By the strong continuity of $y, s \to \xi_{y,s}$ and taking limits in the usual way it follows that

$$(A_x \xi) \otimes (A_x \xi)^- \in W_{\{z\}}^+ . \tag{4.2}$$

If $Y' = \{z \in Y : W_{\{z\}}^+ \neq 0\}$ then Y' is countable since we are supposing that $\mathscr{H}$ is separable, and $P_{Y-Y'} = 0$ since $A_x = 0$ for all x with $\sigma x \in Y \times (Y - Y')$. We write $X' = \{x \in X : A_x \neq 0\}$ and observe that X' is open and invariant under the action of G.

Suppose $x \in X'$. Then $\{\sigma(xg) : g \in G\}$ is a connected subset of $Y \times Y'$. Since Y' is countable it is totally disconnected so the image is contained in $Y \times \{z\}$ for some $z \in Y'$. Therefore for all $z \in Y'$, $\{x \in X' : \sigma x \in Y \times \{z\}\}$ is G-invariant, and $W_{\{z\}}^+$ is G-invariant. Therefore $U_g P_y U_g^* = P_y$ for all $g \in G$ and $y \in Y'$, and P commutes with U.

Eq. (4.2) also establishes that if $\sigma x = (y, z)$ the range of A_x is contained in $P_z \mathscr{H}$. By the transition hypothesis $A_x \varrho A_x^* = 0$ for all $\varrho \in W_{Y-\{y\}}^+$, so the support of A_x is contained in $P_y \mathscr{H}$. Therefore

$$A_x^* A_x = A_x^* A_x P_y = P_y A_x^* A_x$$

and in fact each operator $A_x^* A_x$ commutes with the projection-valued measure P. Therefore the total interaction rate R commutes with P. Similar arguments show that $B_t P_y \mathscr{H} \subseteqq P_y \mathscr{H}$ for all $y \in Y$ and $t \geqq 0$. If U_t is the one-paramter unitary group generated by the Hamiltonian H_0, where $Z = iH_0 - \frac{1}{2} R$, then U_t may be constructed from B_t, Z directly using a sequence of integral equations [7, p. 495]. Therefore $U_t P_y \mathscr{H} \subseteqq P_y \mathscr{H}$ for all $y \in Y$ and $t \geqq 0$. It follows that P commutes with the spectral projections of H_0.

The study of certain transition processes can be reduced essentially to a problem about ordinary classical Markov chains. We call a process *semi-classical* if it satisfies the conditions of the following proposition.

Proposition 16. *Suppose $\mathscr{E}$ is an irreducible, simple, recurrent, transition process on X, V, covaiant with respect to a compact connected group G. Then the set W^+ of states $\varrho \in V^+$ which commute with P and U is invariant under T_t for all $t \geqq 0$. W^+ can be identified with the set of all bounded positive measures on a countable set I if and only if the restriction of the representation U to each subspace $P_y\mathscr{H}$ is multiplicity free.*

Proof. The fact that W^+ is invariant follows from Theorem 15 and Eq. (2.8). As a partially ordered space, $W = W^+ - W^+$ can be identified with the complete direct sum

$$W = \sum_{y \in Y} \oplus \{\varrho \in \mathscr{T}_s(P_y\mathscr{H}) : [U_g, \varrho] = 0 \quad \text{for all} \quad g \in G\}. \tag{4.3}$$

W^+ can be identified with the set of bounded measures on a countable set only if W is a vector lattice, which happens if and only if U is multiplicity free on each subspace $P_y\mathscr{H}$. If this occurs we write $P_y = \sum_n P_{y,n}$ where $P_{y,n}$ are projections commuting with U and on which U is irreducible. Then $P_{y,r}\mathscr{H}$ is finite-dimensional since G is compact. If $e_{y,n} = P_{y,n}/\mathrm{tr}[P_{y,n}]$ and I is the set of double indices y, n then

$$W^+ = \left\{ \sum_{\alpha \in I} \varphi_\alpha e_\alpha : \varphi_\alpha \geqq 0, \ \sum_{\alpha \in I} \varphi_\alpha < \infty \right\} \tag{4.4}$$

Now suppose that $\varrho = \Sigma\, \varphi_\alpha e_\alpha$ and

$$T_t(\varrho) = \sum_{\alpha \in I} \varphi_\alpha(t)\, e_\alpha\,. \tag{4.5}$$

The following theorem shows that the restriction of T_t to W^+ is a classical Markov chain.

Theorem 17. *Let $\mathscr{E}$ be a semi-classical process. Then there exist constants $r_\alpha \geqq 0$, $\beta_{\gamma \to \alpha} \geqq 0$ and $0 < K < \infty$ such that for all $t \geqq 0$*

$$\sum_{\alpha \in I} \beta_{\gamma \to \alpha} = r_\gamma \leqq K \tag{4.6}$$

and

$$\frac{d}{dt}\varphi_\alpha(t) = -r_\alpha \varphi_\alpha(t) + \sum_{\gamma \in I} \beta_{\gamma \to \alpha} \varphi_\gamma(t)\,. \tag{4.7}$$

Proof. The operator R commutes with U and P so there exist constant $0 \leqq r_\alpha \leqq K$ such that $\mathrm{R}e_\alpha = r_\alpha e_\alpha$ for all $\alpha \in I$. If $\varrho \in W^+$ then $\mathscr{J}(X, \varrho) \in W^+$ so we define $\beta_{\gamma \to \alpha} \geqq 0$ by $\mathscr{J}(X, e_\gamma) = \sum_\alpha \beta_{\gamma \to \alpha} e_\alpha$ and immediately obtain Eq. (4.6). If P_α is the projection defined above then by Eq. (2.10) of [3]

$$\mathrm{tr}[P_\alpha T_t(\varrho)] = \mathrm{tr}[P_\alpha S_t(\varrho)] + t\, \mathrm{tr}[P_\alpha \mathscr{J}(X, \varrho)] + 0(t^2)\,. \tag{4.8}$$

Now if $\varrho = \sum_\alpha \varphi_\alpha(t_0)\, e_\alpha$ then $S_t(\varphi) = \sum_\alpha \varphi_\alpha(t_0)\, e^{-r_\alpha t} e_\alpha$ so $\mathrm{tr}[P_\alpha S_t(\varrho)]$ is differentiable with derivative $-r_\alpha \varphi_\alpha(t_0)$ at $t = 0$. The right hand side of Eq. (4.8) is now seen to be differentiable at $t=0$ giving

$$\left\{\frac{d}{dt}\varphi_\alpha(t+t_0)\right\}_{t=0} = -r_\alpha\varphi_\alpha(t_0) + \operatorname{tr}\left[P_\alpha \sum_{\gamma,\delta} \beta_{\gamma\to\delta}\varphi_\gamma(t_0)\, e_\gamma\right]$$

which on simplification becomes Eq. (4.7).

§ 5. Reversible Processes

We treat here another special class of processes, which can be extended from $V=\mathcal{T}_s(\mathcal{H})$ to $\mathcal{L}_s(\mathcal{H})$. Closely related work on duals of instruments was done in [2].

Given X we define X_t and X_∞ as before. A normal process $\mathcal{E}$ on $X, \mathcal{L}_s(\mathcal{H})$ is then defined as a function $\mathcal{E}_t(E, A)$ taking values in $\mathcal{L}_s(\mathcal{H})$ for each Borel set $E \subseteqq X_t$, each $A \in \mathcal{L}_s(\mathcal{H})$ and each $t \geqq 0$, and satisfying

(i) $E \to \mathcal{E}_t(E, A)$ is σ-additive in X_t for each $A \in \mathcal{L}_s(\mathcal{H})$, the sums being taken in the weak operator topology;

(ii) $A \to \mathcal{E}_t(E, A)$ is a positive normal linear mapping for each $E \subseteqq X_t$;

(iii) for each $E \subseteqq X_s$, $F \subseteqq X_t$ and $A \in \mathcal{L}_s(\mathcal{H})$

$$\mathcal{E}_{s+t}(E \times F, A) = \mathcal{E}_s(E, \mathcal{E}_t(F, A)).$$

If E is any subset of X_t we define E^θ as its image under the *time inversion*

$$(x_1, t_1) \ldots (x_n, t_n) \xrightarrow{\theta} (x_n, t-t_n) \ldots (x_1, t-t_1).$$

Given any process $\mathcal{E}$ on X, V we can define its *dual process* $\mathcal{E}^*$ on $X, \mathcal{L}_s(\mathcal{H})$ by the equation

$$\operatorname{tr}[\mathcal{E}_t(E, \varrho)\, A] = \operatorname{tr}[\varrho\, \mathcal{E}_t^*(E^\theta, A)] \tag{5.1}$$

valid for all $\varrho \in V$, $A \in \mathcal{L}_s(\mathcal{H})$ and $E \subseteqq X_t$. It is easy to verify that $\mathcal{E}^*$ is a normal process.

Proposition 18. *Let $\mathcal{E}$ be a process on X, V constructed from a Hamiltonian H_0 and a Borel map $A: X \to \mathcal{L}(\mathcal{H})$. If there is a constant K such that for all $\xi \in \mathcal{H}$*

$$\int_X \|A_x\xi\|^2 dx = \int_X \|A_x^*\xi\|^2 dx \leqq K\|\xi\|^2 \tag{5.2}$$

then $\mathcal{E}$ has a unique extension to a normal process $\bar{\mathcal{E}}$ on $X, \mathcal{L}_s(\mathcal{H})$ satisfying $\bar{\mathcal{E}}_t(X, 1) = 1$ for all $t \geqq 0$.

Proof. Uniqueness is clear from the fact that $\bar{\mathcal{E}}$ is normal and V is dense in $\mathcal{L}_s(\mathcal{H})$ for the weak operator topology. We construct $\bar{\mathcal{E}}$ as the dual of a process $\tilde{\mathcal{E}} \subseteqq \mathcal{E}^*$. $\tilde{\mathcal{J}}$ is defined as the kernel

$$\tilde{\mathcal{J}}(E, \varrho) = \int_E A_x^* \varrho A_x dx \tag{5.3}$$

so for all $\varrho, \sigma \in V$

$$\operatorname{tr}[\tilde{\mathscr{J}}(E, \varrho)\, \sigma] = \operatorname{tr}[\varrho\, \mathscr{J}(E, \sigma)] \tag{5.4}$$

and in particular $R = \tilde{R}$. Incorporating a time reversal we define $\tilde{Z} = -iH_0 - \frac{1}{2}\tilde{R}$, so $\tilde{B}_t = B_t^*$. $\tilde{\mathscr{E}}$ is now constructed from $\tilde{\mathscr{J}}$ and $\tilde{Z}$ by the procedure of [3]. It follows from Eq. (4.6) of [3] that

$$\operatorname{tr}[\mathscr{E}_t(E, \varrho)\, \sigma] = \operatorname{tr}[\varrho\, \tilde{\mathscr{E}}_t(E^\theta, \sigma)] \tag{5.5}$$

for all $\varrho, \varrho \in V$ and all $E \subseteq X_t$. Therefore $\tilde{\mathscr{E}}^*$ is an extension of $\mathscr{E}$, which we call $\bar{\mathscr{E}}$. For all $\varrho \in V$ and $t \geqq 0$

$$\operatorname{tr}[\bar{\mathscr{E}}_t(X_t, 1)\, \varrho] = \operatorname{tr}[\tilde{\mathscr{E}}_t(X_t, \varrho)] = \operatorname{tr}[\varrho]$$

so $\bar{\mathscr{E}}_t(X_t, 1) = 1$.

The following theorem is a generalisation of a known result for Markov processes with doubly stochastic kernels.

Theorem 19. *Let $\mathscr{E}$ be an irreducible, infinite-dimensional process on X, V such that the operator A_x is normal for all $x \in X$. Then $\mathscr{E}$ cannot have an equilibrium state.*

Proof. Eq. (5.2) is satisfied so we have a normal extension $\bar{\mathscr{E}}$ with corresponding one-parameter semigroup $\bar{T}$. Suppose ϱ is an equilibrium state with largest eigenvalue $\lambda > 0$ and corresponding proper eigenspace $\mathscr{K}$. Then $A = \lambda 1 - \varrho$ is a non-zero positive operator invariant under $\bar{T}$ and with support contained in $\mathscr{K}^\perp$. If

$$J = \{\varrho \in V^+ : \varrho \leqq \alpha A \quad \text{some} \quad \alpha \geqq 0\}^-$$

then J is a proper, norm closed ideal in V^+ invariant under T_t all $t \geqq 0$. But this contradicts the irreducibility of $\mathscr{E}$.

For such processes the operator 1 can be interpreted as an unbounded equilibrium state.

We can illustrate the application of this and earlier theorems by a rather interesting example. The physical model is of an elementary particle moving freely in one-dimensional space with periodic boundary conditions, and interacting with a position-measuring instrument.

Let X be the circle with the Haar measure dx of total mass one, let $\mathscr{H} = \mathscr{L}^2(X)$ and let $V = \mathscr{T}_s(\mathscr{H})$. The group $G = \mathbb{R}$ has a natural representation by translations on X, and hence on $\mathscr{H}$ and V. Let the free Hamiltonian $H_0 = -\dfrac{\partial^2}{\partial \theta^2}$. In Theorem 4 of [2] we have shown how to construct a covariant instrument $\mathscr{J}$ on X, V from a certain family of normal bounded operators $A : X \to \mathscr{L}(\mathscr{H})$. There exist constants $K > 0$

and $L>0$ such that for all $\varrho \in V^+$

$$\operatorname{tr}[\mathcal{J}(X,\varrho)] = K \operatorname{tr}[\varrho]\,, \tag{5.6}$$

$$\operatorname{tr}[P\,\mathcal{J}(X,\varrho)] = \operatorname{tr}[P\varrho]\,, \tag{5.7}$$

$$\operatorname{tr}[P^2\mathcal{J}(X,\varrho)] = \operatorname{tr}[P^2\varrho] + L\operatorname{tr}[\varrho]\,, \tag{5.8}$$

where P is the unbounded momentum operator $-i\frac{\partial}{\partial\theta}$. The interaction rate $R=K\,1$ of the instrument is independent of the state ϱ and the constant L gives the amount the momentum distribution of the state is perturbed during the act of measurement. We construct the covariant process $H_0, \mathcal{J}$ in the standard manner. Observe that $B_t = e^{-\frac{1}{2}Kt}e^{iH_0t}$, so

$$\operatorname{tr}[S_t\varrho] = e^{-Kt}\operatorname{tr}[\varrho]\,. \tag{5.9}$$

By Lemma 1 and [2] it follows that $\mathscr{E}$ is irreducible, and by Theorem 19, $\mathscr{E}$ has no equilibrium state. However by Eq. (5.9) the waiting time for an event is finite and independent of the initial state, so the process is recurrent in a very strong sense. This feature of the example is explained by the fact that each time an event occurs the variance of the momentum increases by a fixed amount. Therefore the variance of the momentum of any state should diverge linearly to ∞ as $t\to\infty$. However one would expect that the position distribution of the state would converge to the uniform distribution on X as $t\to\infty$.

References

1. Davies, E. B., Lewis, J. T.: An operational approach to quantum probability. Commun. Math. Phys. **17**, 239–260 (1970).
2. — On the repeated measurement of continuous observables in quantum mechanics. To appear in J. Functional Analysis.
3. — Quantum stochastic processes. Commun. Math. Phys. **15**, 277–304 (1969).
4. Chung, K. L.: Markov chains with stationary transition probabilities. Berlin-Göttingen-Heidelberg: Springer 1960.
5. Phelps, R. R.: Lectures on Choquet's theorem, 1st edition. Princeton-London-Toronto: van Nostrand, 1966.
6. Edwards, C. M.: The operational approach to algebraic quantum theory 1. Commun. math. Phys. **16**, 207–230 (1970).
7. Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966.
8. Albert, A. A.: Structure of algebras. Am. Math. Soc. Coll. Publ. vol. XXIV, 1st edition (1961).
9. Neumann, J. von: Mathematical foundations of quantum mechanics, translated from the German edition by R. T. Beyer. Princeton: Princeton Univ. Press 1955.

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