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Entropy Inequalities

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Abstract. Some inequalities and relations among entropies of reduced quantum mechanical density matrices are discussed and proved. While these are not as strong as those available for classical systems they are nonetheless powerful enough to establish the existence of the limiting mean entropy for translationally invariant states of quantum continuous systems.

I. Introduction

In this note we shall be concerned with inequalities satisfied by the entropies of reduced density matrices. We begin with some definitions and a statement of our main Theorem 1. Section II contains the proof of the main theorem when the dimension is finite. Section III contains some other inequalities that can be derived from Theorem 1 by application of certain transformations. Section IV contains the proof of the main theorem when the dimension is infinite. Section V deals with the application of our theorem to the existence of the mean entropy for translationally invariant states of a quantum continuous system.

Definition 1. *A density matrix, ϱ, on a Hilbert space, H, is a self adjoint non-negative trace class operator on H whose trace is unity.*

Definition 2. *If ϱ is a density matrix,*

$$S(\varrho) = -\operatorname{Tr} \varrho \ln \varrho \tag{1.1}$$

is the entropy associated with ϱ.

Since $0 \leqq \varrho \leqq 1$, we have $-e^{-1} \leqq \varrho \ln \varrho \leqq 0$ and $(\psi_j, (\varrho \ln \varrho)\, \psi_j) \leqq 0$ for any ψ_j. Hence

$$S = -\sum_j (\psi_j, \varrho \ln \varrho\, \psi_j) \tag{1.2}$$

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exists for any orthonormal basis $\{\psi_j\}$ and $0 \leqq S \leqq +\infty$. If $S < \infty$ for one basis $\{\psi_j\}$, then the nonnegative operator $-\varrho \ln \varrho$ is in the trace class and hence S is independent of the basis $\{\psi_j\}$ and is finite for all $\{\psi_j\}$. Otherwise S must be $+\infty$. Therefore (1.2) does not depend on the orthonormal basis $\{\psi_j\}$ and defines the right hand side of (1.1).

Definition 3. *If ϱ^{12} is a density matrix on $H^1 \otimes H^2$ then ϱ^1, the reduced density matrix, is a density matrix on H^1 defined by*

$$\varrho^1 = \mathrm{Tr}^2 \varrho^{12} . \tag{1.3}$$

Here Tr^2 means the partial trace defined by

$$(x, \varrho^1 y) = \sum_i (x \otimes e_i, \varrho^{12}[y \otimes e_i]) ,$$

where $\{e_i\}$ is any complete orthonormal basis in H^2 and $x, y \in H^1$.

Notation. If ϱ^{12} is a density matrix on $H^1 \otimes H^2$ then we will denote $S(\varrho^{12})$ by S^{12} and $S(\varrho^1)$ by S^1.

A theorem that is true classically [1] (meaning that all relevant density matrices commute) is the following:

$$S^{123} + S^2 \leqq S^{12} + S^{23} . \tag{1.4}$$

We believe that (1.4) is true quantum mechanically as has been conjectured by Lanford and Robinson [2], but have been unable to prove it. We can, however, prove the following which is as good for some applications.

Theorem 1. *Let ϱ^{123} be a density matrix on $H^1 \otimes H^2 \otimes H^3$. Then*

$$S^{123} \leqq S^{12} + S^{23} + \ln \mathrm{Tr}^2 (\varrho^2)^2 \leqq S^{12} + S^{23} . \tag{1.5}$$

Furthermore, if $\varrho^2 \otimes I^3$ commutes with ϱ^{23} then

$$S^{123} + S^2 \leqq S^{12} + S^{23} . \tag{1.6}$$

II. Proof of Theorem 1 for the Finite Dimensional Case

In this section we prove Theorem 1 when the dimension of $H^1 \otimes H^2 \otimes H^3$ is *finite*. We need two lemmas.

Lemma 1 *(Peierls-Bogolyubov inequality). If R and F are hermitian, $\mathrm{Tr}\, e^R = 1$ and $f \equiv \mathrm{Tr}\, F e^R$, then $\mathrm{Tr}\, e^{R+F} \geqq e^f$.*

Proof. The statement of Peierls' theorem given in Ruelle [3] is not quite the same as the above. To prove Lemma 1 we use Klein's inequality [4]

$$\mathrm{Tr}\{f(A) - f(B) - (A - B)\, f'(B)\} \geqq 0$$

which holds for convex f and hermitian A and B. Take $f(x)=e^x$, $A=R+F$, $B=R+fI$.

Lemma 2 *(Golden-Thompson inequality* [5]*). Let A and B be hermitian. Then*

$$\mathrm{Tr}\, e^{A+B} \leqq \mathrm{Tr}\, e^A e^B .$$

Proof of Theorem 1. We first assume that ϱ^{123} is positive definite. Let $\varrho^{123}=\exp R^{123}$, $\varrho^{12}=\exp R^{12}$, etc. We will also denote $R^{12}\otimes I^3$ by R^{12}, etc. Then

$$\Delta = S^{123} - S^{12} - S^{23} = \mathrm{Tr}^{123} \varrho^{123} [-R^{123} + R^{12} + R^{23}] .$$

Using Lemma 1, $e^\Delta \leqq \mathrm{Tr}^{123} \exp[R^{123} - R^{123} + R^{12} + R^{23}]$.

Using Lemma 2, $e^\Delta \leqq \mathrm{Tr}^{123} \exp(R^{12}) \exp(R^{23}) = \mathrm{Tr}^{123} \varrho^{12} \varrho^{23} = \mathrm{Tr}^2 (\varrho^2)^2$.

Since the eigenvalues of ϱ^2 are in $[0,1]$, and $\sum_i \lambda_i^2 \leqq (\sum \lambda_i)^2$ for $0 \leqq \lambda_i \leqq 1$, we have $\mathrm{Tr}^2(\varrho^2)^2 \leqq 1$.

If R^2 commutes with R^{23}, then consider

$$\Delta = S^{123} + S^2 - S^{12} - S^{23} = \mathrm{Tr}^{123} \varrho^{123} [-R^{123} - R^2 + R^{12} + R^{23}] .$$

By Lemma 1, $e^\Delta \leqq \mathrm{Tr}^{123} \exp[R^{12} + R^{23} - R^2]$.

By Lemma 2,

$$e^\Delta \leqq \mathrm{Tr}^{123} \exp(R^{12}) \exp(R^{23} - R^2) = \mathrm{Tr}^{123} \varrho^{12} \varrho^{23} (\varrho^2)^{-1} = 1 .$$

The case of semidefinite ϱ^{123} follows by the continuity of $\varrho \to S(\varrho)$ for the finite dimensional case. (Note that the statement $-S^2 \leqq \ln \mathrm{Tr}^2 (\varrho^2)^2$ follows trivially from Lemma 1.)

As a corollary we have a well known theorem [6]:

Corollary. *If ϱ^{12} is a density matrix on $H^1 \otimes H^2$ then*

$$S^{12} \leqq S^1 + S^2 .$$

Proof. Interchange 2 and 3 in Theorem 1 and take H^3 to be one dimensional.

III. More Inequalities

The following definition and two lemmas are well known and are repeated here only for the sake of completeness. Matrices here need not be finite dimensional.

Definition 4. *A density matrix ϱ is said to be a pure state if ϱ is a projection operator onto a one-dimensional subspace, i.e. $\varrho x = y(y, x)$ for some fixed y with $|y| = 1$.*

Lemma 3. *Let ϱ^{12} be a pure state density matrix on $H^1 \otimes H^2$. Let $f(\cdot)$ be a real valued function whose domain contains the spectra of ϱ^1 and ϱ^2 and $f(0)=0$. Then*

$$\mathrm{Tr}^1 f(\varrho^1) = \mathrm{Tr}^2 f(\varrho^2).$$

In particular $S^1 = S^2$.

Proof. Let $\varrho^{12} x = (y, x)\, y$, $y = \sum \lambda_i y_{1i} \otimes y_{2i}$, $\lambda_i > 0$, where $\{y_{1i}\}$ and $\{y_{2i}\}$ can be taken to be orthonormal [7]. Let $P(y_{vi})$ be the projection on the one dimensional subspace of H^v containing y_{vi}. Then $\varrho^v = \sum \lambda_i^2 P(y_{vi})$. Hence ϱ^1 and ϱ^2 have the same eigenvalues and multiplicities except possibly for the eigenvalue 0. The lemma follows immediately.

Lemma 4. *Let ϱ^1 be a density matrix on H^1. Then there exists a Hilbert space H^2 and a pure state density matrix ϱ^{12} on $H^1 \otimes H^2$ such that*

$$\mathrm{Tr}^2 \varrho^{12} = \varrho^1.$$

Proof. Let $\varrho^1 = \sum \lambda_j P_j$, $\lambda_j > 0$, $P_j x = (y_j, x)\, y_j$ and $\{y_j\}$ be orthonormal (the spectral decomposition). Let $\dim H^2 \geqq \dim H^1$ and let $\{z_j\}$ be an arbitrary orthonormal system in H^2. Let ϱ^{12} be the projection operator on the one dimensional subspace of $H^1 \otimes H^2$ containing the vector $\sum (\lambda_j)^{1/2} y_j \otimes z_j$. Then $\mathrm{Tr}^2 \varrho^{12} = \varrho^1$.

An application of lemmas 3 and 4 is the following:

Theorem 2. *Let ϱ^{123} be a density matrix on $H^1 \otimes H^2 \otimes H^3$. Then*

(a) $S^2 \leqq S^{23} + S^{12} + \ln \mathrm{Tr}^{123} (\varrho^{123})^2 \leqq S^{23} + S^{12}$.

(b) $S^2 \leqq S^{23} + S^{13} + \ln \mathrm{Tr}^1 (\varrho^1)^2 \leqq S^{23} + S^{13}$.

(c) $S^2 \leqq S^1 + S^{12}$.

Proof. We regard ϱ^{123} as a reduction of a pure ϱ^{1234}, whence $S^{123} = S^4$, $S^{12} = S^{34}$, $S^{23} = S^{14}$. Theorem 1 has the alternative forms:

(a′) $S^4 \leqq S^{34} + S^{14} + \ln \mathrm{Tr}^{134} (\varrho^{134})^2$.

(b′) $S^4 \leqq S^{34} + S^{23} + \ln \mathrm{Tr}^2 (\varrho^2)^2$.

(a′) and (b′) are general. In (a′) substitute 2 for 4. In (b′) substitute 2 for 4 and 1 for 2. To derive (c), let H^1 be one dimensional in (b) and then substitute 1 for 3.

Remarks. (1) Any other proof of any one of Theorems 1, 2a and 2b will give an alternative proof of Theorems 1, 2a, 2b and 2c.

(2) If we combine Theorem 2c with the corollary of Theorem 1, we obtain the triangle inequality

$$|S^1 - S^2| \leqq S^{12} \leqq S^1 + S^2. \tag{3.1}$$

[The left hand side should be taken to be 0 if $S^1 = S^2 = +\infty$.]

(3) Another application of Lemmas 3 and 4 is that the conjecture (1.4) is equivalent to

$$S^1 + S^2 \leqq S^{13} + S^{23}\,. \tag{3.2}$$

(4) We had to appeal to Lemmas 3 and 4 to prove Theorem 2a. A direct proof of Theorem 2a might indicate how to prove (3.2).

IV. Proof of Theorem 1 for the Infinite Dimensional Case

Definition 5. *Let $\{\psi_i^A\}$, $\{\lambda_i^A\}$ and $\{\psi_j^B\}$, $\{\lambda_j^B\}$ be complete orthonormal sets of eigenvectors and corresponding eigenvalues of selfadjoint operators A and B. Assume that $\lambda_i^A \geqq 0$, $\lambda_j^B \geqq 0$ for all i and j. Then we define*

$$\operatorname{Tr} AB \equiv \sum \lambda_i^A \lambda_j^B |(\psi_i^A, \psi_j^B)|^2\,. \tag{4.1}$$

(The value $+\infty$ is allowed.)

Remark. There exist cases where $\operatorname{Tr} AB < \infty$ and AB is not in the trace class (i.e. $\operatorname{Tr}|AB| = \infty$). If AB is in the trace class, then Definition 5 coincides with the ordinary definition of the trace.

Remark. $\operatorname{Tr} AB = \operatorname{Tr} BA$.

Lemma 5. *The definition of* $\operatorname{Tr} AB$ *is independent of the choice of the complete orthonormal sets of eigenvectors.*

Proof. (a) First we consider the case where A and B are projections. Then $\operatorname{Tr} AB = \sum \lambda_j^B (\psi_j^B, A\psi_j^B) = \operatorname{Tr} BAB$ where the trace of a nonnegative operator BAB is defined as in Definition 2 and is independent of the complete orthonormal sets $\{\psi_i^A\}$ and $\{\psi_j^B\}$.

(b) For general A and B, let $A = \sum_x x P^A(x)$ and $B = \sum_y y P^B(y)$ be the spectral decompositions of A and B. Then $\operatorname{Tr} AB = \sum_{x,y} xy \operatorname{Tr}(P^A(x) P^B(y))$ which is again independent of the complete orthonormal sets. Q.E.D.

In the above, we have used the fact that a sum of positive numbers is independent of the order of the summation irrespective of whether the sum is finite or infinite.

Lemma 6. *Let A^{12} be a bounded nonnegative selfadjoint operator on $H^1 \otimes H^2$ and B^2 be a selfadjoint operator on H^2 with purely discrete nonnegative spectrum. Then* $\operatorname{Tr} A^{12}(I^1 \otimes B^2) = \operatorname{Tr}^2 A^2 B^2$.

Proof. From Definition 5,

$$\operatorname{Tr} AB = \sum \lambda_j^B \|A^{1/2} \psi_j^B\|^2\,. \tag{4.2}$$

Hence

$$\begin{aligned}\operatorname{Tr} A^{12}(I^1 \otimes B^2) &= \sum_l \lambda_l^B \sum_k (\chi_k \otimes \varphi_l, A^{12} \chi_k \otimes \varphi_l) \\ &= \sum_l \lambda_l^B (\varphi_l, A^2 \varphi_l) \\ &= \operatorname{Tr}^2 A^2 B^2 .\end{aligned}$$

Here $\{\varphi_l\}$ and $\{\lambda_l^B\}$ are a complete orthonormal set of eigenvectors and corresponding eigenvalues of B and $\{\chi_k\}$ is any complete orthonormal set in H^1.

Corollary 6.1. *Let* $A = A^{12} \otimes I^3$, $B = I^1 \otimes B^{23}$ *where* A^{12} *and* B^{23} *are trace class nonnegative operators. Then* $\operatorname{Tr} AB = \operatorname{Tr}^2 A^2 B^2$.

Corollary 6.2. $\operatorname{Tr} \varrho^{ab}(I^a \otimes \ln \varrho^b) = \operatorname{Tr}^b \varrho^b \ln \varrho^b$.

Let $A \geqq 0$, $B = B_1 + B_2$, $B_1 B_2 = B_2 B_1 = 0$, $B_1 \geqq 0$, $B_2 \geqq 0$. Then $\|B^{1/2}\psi\|^2 = \|B_1^{1/2}\psi\|^2 + \|B_2^{1/2}\psi\|^2$ and hence we have, from (4.2),

$$\operatorname{Tr} AB \geqq \operatorname{Tr} AB_1 . \tag{4.3}$$

If ψ is not in the domain of C, we define $\|C\psi\| = \infty$. If $B = \sum x P(x)$ is a spectral decomposition of $1 \geqq B \geqq 0$, then we define $(-\ln B)^{1/2}$ by $\sum_{x \neq 0} (-\ln x)^{1/2} P(x)$ on those vectors ψ which satisfy $P(0)\psi = 0$ and $-\sum (\ln x) \|P(x)\psi\|^2 < \infty$.

Lemma 7. *Let* $\{\psi_i\}$ *be an orthonormal set of vectors,* A *and* B *be non-negative operators with purely discrete spectrum majorized by* 1. *Then*

$$\sum_i \exp\{-\|(-\ln A)^{1/2}\psi_i\|^2 - \|(-\ln B)^{1/2}\psi_i\|^2 \leqq \operatorname{Tr} AB . \tag{4.4}$$

Proof. (a) First, assume that A and B are of finite rank. Then only a finite number of ψ_i can be in the domain of $(-\ln A)^{1/2}$ and the sum over i reduces to a finite sum. Hence we can discuss the inequality on a finite dimensional space for this case. We may assume that all ψ_i are in the domain of $(-\ln B)^{1/2}$ and $(-\ln A)^{1/2}$. If $A = \sum x P^A(x)$ and $B = \sum x P^B(x)$, we first prove the inequality for $A_\varepsilon = A + \varepsilon P^A(0)$ and $B_\varepsilon = B + \varepsilon P^B(0)$ where $\varepsilon > 0$. From Ref. [3] and the Golden-Thompson inequality, we have

$$\begin{aligned}&\sum_i \exp\{(\psi_i, (\ln A_\varepsilon)\psi_i) + (\psi_i, (\ln B_\varepsilon)\psi_i)\} \\ &\leqq \operatorname{Tr} \exp[\ln A_\varepsilon + \ln B_\varepsilon] \\ &\leqq \operatorname{Tr} A_\varepsilon B_\varepsilon .\end{aligned}$$

By taking the limit $\varepsilon \to 0$, we obtain (4.4) for this case.

(b) Consider the case of a finite number of ψ_i. Let $A = \sum \lambda_n^A P_n^A$ and $B = \sum \lambda_m^B p_m^B$ be spectral decompositions of A and B. (λ_n^A and λ_m^B are distinct.) Let P_{n0}^A and P_{m0}^B be finite subprojections of P_n^A and P_m^B, whose ranges are spanned by $\{P_n^A \psi_i\}$ and $\{P_m^B \psi_i\}$, respectively. Let $A_N = \sum_{<N} \lambda_n^A P_{n0}^A$, $A' = \sum \lambda_n^A P_{n0}^A$, $B_M = \sum_{<M} \lambda_m^B P_{m0}^B$, $B' = \sum \lambda_m^B P_{m0}^B$. From (a), we have (4.4) for A_N and B_M. We also have

$$\|(-\ln A_N)^{1/2} \psi_i\|^2 \uparrow \|(-\ln A')^{1/2} \psi_i\|^2 = \|(-\ln A)^{1/2} \psi_i\|^2 ,$$

$$\|(-\ln B_M)^{1/2} \psi_i\|^2 \uparrow \|(-\ln B')^{1/2} \psi_i|^2 = \|(-\ln B)^{1/2} \psi_i\|^2 .$$

From $B_M \leqq B$, $A_N \leqq A$, we have $\operatorname{Tr} A_N B_M \leqq \operatorname{Tr} A_N B \leqq \operatorname{Tr} AB$ because of (4.3). Hence, by taking the limit $N \to \infty$ and $M \to \infty$, we have (4.4) for A and B.

(c) For the general case, we have (4.4) for any finite subset of ψ_i from (b). Since the sum over i is the supremum of finite sums, we have (4.4) also for the general case. Q.E.D.

Proof of (1.5). Let $\{\psi_i\}$, $\{\lambda_i\}$ be a complete orthonormal set of eigenvectors and corresponding eigenvalues of ϱ^{123}. Let

$$\alpha_i = \exp - \{\|(-\ln \varrho^{12})^{1/2} \psi_i\|^2 + \|(-\ln \varrho^{23})^{1/2} \psi_i\|^2\} .$$

From Lemma 7 and Corollary 6.1, we have

$$\textstyle\sum_I \alpha_i \leqq \operatorname{Tr} \varrho^{12} \varrho^{23} = \operatorname{Tr}^2 (\varrho^2)^2 \leqq 1$$

where the sum is over any subset, I, of indices i. Choose I so that $\sum_I \lambda_i > 0$ whence, from the concavity of the logarithm, we have

$$\Delta_I \equiv \textstyle\sum_I \lambda_i^I \ln(\alpha_i/\lambda_i^I) \leqq \ln \sum_I \alpha_i$$

where $\lambda_i^I = \lambda_i / (\sum_I \lambda_i)$. For a finite I, we have

$$\Delta_I (\textstyle\sum_I \lambda_i) = S_I^{123} + (\sum_I \lambda_i) \ln(\sum_I \lambda_i) - S_I^{12} - S_I^{23} ,$$

where

$$S_I^\mu = \textstyle\sum_I \lambda_i \|(-\ln \varrho^\mu)^{1/2} \psi_i\|^2 , \qquad \mu = 123, 12, 23 .$$

Hence

$$S_I^{123} + (\textstyle\sum_I \lambda_i) \ln(\sum_I \lambda_i) \leqq S_I^{12} + S_I^{23} + (\sum_I \lambda_i) \ln \operatorname{Tr}^2 (\varrho^2)^2 .$$

From Corollary 6.2, we have $S_I^\mu \uparrow S^\mu$ for $\mu = 12, 23$. We also have $S_I^{123} \uparrow S^{123}$ and $\sum_I \lambda_i \uparrow 1$. Hence we have

$$S^{123} \leqq S^{12} + S^{23} + \ln \operatorname{Tr}^2 (\varrho^2)^2 .$$

Q.E.D.

Proof of (1.6). If $S^2 = +\infty$, then $S^{12} + S^{23} = +\infty$ due to Theorem 2a and hence (1.6) holds. [Theorem 2a is a consequence of (1.5), which has been proved above, and Lemma 4.] We now assume that $S^2 < +\infty$.

Let $\{\psi_i\}$ and $\{\lambda_i\}$ be a complete othonormal set of eigenvectors and corresponding eigenvalues of ϱ^{123}. From Corollary 6.2, it follows that ψ_i is in the domain of $(-\ln \varrho^2)^{1/2}$ if $S^2 < \infty$ and $\lambda_i \neq 0$. For a finite set, I, of indices, we have

$$\begin{aligned}\textstyle\sum_I \exp[\|(-\ln \varrho^2)^{1/2}\psi_i\|^2 - \|(-\ln \varrho^{12})^{1/2}\psi_i\|^2 - \|(-\ln \varrho^{23})\,\psi_i\|^2] \\ \leqq \operatorname{Tr}(\varrho^{12}\otimes I^3)\,(I^1\otimes \{[\varrho^2)^{-1}\otimes I^3]\,\varrho^{23}\}) = \operatorname{Tr}^2 \varrho^2 = 1\,.\end{aligned} \tag{4.5}$$

The proof is exactly the same as Lemma 7 except that Lemma 6 should be used instead of Corollary 6.1.

From (4.5), we have

$$S_I^{123} + S_I^2 + (\textstyle\sum_I \lambda_i)\ln(\sum_I \lambda_i) \leqq S_I^{12} + S_I^{23}$$

and hence we have (1.6) by taking the supremum over I.

V. Application to Statistical Mechanics

In this section we prove the existence of the limiting mean entropy for translationally invariant states of quantum continuous systems [8, 9].

We shall restrict our attention to finite closed boxes with a fixed orientation in R^ν and their finite union $\bigcup_j L_j = \Lambda$. If $\Lambda_1 \cap \Lambda_2$ has a lower dimension (or is empty), we say that Λ_1 and Λ_2 are disjoint. For a finite number of mutually disjoint Λ_j, we denote $\cup \Lambda_j$ by $\bigvee_j \Lambda_j$.

Let $H(\Lambda)$ be a Hilbert space for each box Λ such that $H(\bigvee_j \Lambda_j) = \bigotimes_j H(\Lambda_j)$. Let $U_a(\Lambda)$ be a unitary mapping from $H(\Lambda)$ onto $H(\Lambda + a)$ such that $U_a(\Lambda + b)\,U_b(\Lambda) = U_{a+b}(\Lambda)$ and $U_a(\bigvee_j \Lambda_j) = \bigotimes_j U_a(\Lambda_j)$.

A state of a quantum continuous system for our present purpose is a set of density matrices $\varrho(\Lambda)$ for each Λ such that

$$\operatorname{Tr}_{\Lambda_1}\varrho(\Lambda_1 \vee \Lambda_2) = \varrho(\Lambda_2)\,. \tag{5.1}$$

It is translationally invariant if $\varrho(\Lambda + a) = U_a(\Lambda)\,\varrho(\Lambda)\,U_a(\Lambda)^*$ for each Λ.

The entropy $S(\Lambda)$ for each Λ is

$$S(\Lambda) = -\operatorname{Tr}_\Lambda \varrho(\Lambda)\ln \varrho(\Lambda)\,. \tag{5.2}$$

If ϱ is translationally invariant, $S(\Lambda + a) = S(\Lambda)$. Let $V(\Lambda)$ be the volume of Λ. Let C_a denote a cube of side length a.

Theorem 3. *If ϱ is a translationally invariant state, then the following limit exists:*

$$S(\varrho) = \lim_{a\to\infty} S(C_a)/V(C_a)\,. \tag{5.3}$$

Proof. Let $L_1 \sim NL_2$ (resp. $L_1 \propto NL_2$) denote the situation where L_1 is equal to (resp. contained in) a disjoint union of N translates of a box L_2.

(a) If $L_1 \sim NL_2$, we have, from the subadditivity (Corollary of Theorem 1),

$$S(L_1) \leqq NS(L_2). \tag{5.4}$$

(b) If $L_1 \propto L_2$ (two boxes are assumed to have the same orientation), then there exists α and β such that $L_1 = (L_2 + \alpha) \cap (L_2 + \beta)$. By Theorem 2(a), we have

$$2S(L_2) \geqq S(L_1). \tag{5.5}$$

(c) $S(L) = \infty$ for one box L if and only if $S(L') = \infty$ for all boxes L'. This is because, there exists N for any L and L' such that $L \propto NL'$ and hence $S(L') \geqq (2N)^{-1} S(L) = \infty$ by (5.4) and (5.5). Assume that $S(L) \neq \infty$ for all L in the following.

(d) Let Λ be a union of N mutually disjoint boxes L_j such that $V(L_j) \geqq v_0$ and $L_j \propto L$ for a fixed L.

From the subadditivity, we have

$$S(\Lambda) \leqq \sum S(L_j) \leqq 2NS(L).$$

We also have

$$V(\Lambda) = \sum V(L_j) \geqq Nv_0.$$

Hence

$$S(\Lambda)/V(\Lambda) \leqq 2S(L)/v_0. \tag{5.6}$$

(e) Let

$$\alpha_n(a) = S(C_{na})/V(C_{na}), \tag{5.7}$$

$$\alpha_\infty(a) = \inf_n \alpha_n(a). \tag{5.8}$$

From (5.4),

$$\alpha_{mn}(a) \leqq \alpha_n(a). \tag{5.9}$$

(f) Given $\varepsilon > 0$, there exists n such that

$$|\alpha_\infty(a) - \alpha_n(a)| < \varepsilon/3. \tag{5.10}$$

For this n, there exists $l \geqq na$ such that

$$2^{\nu+1} S(C_{na}) \sum_{k=1}^{\nu} \binom{\nu}{k} l^{-k} (na)^{k-\nu} < \varepsilon/3. \tag{5.11}$$

We then have the following estimates for $b > l$:

Let $m \geqq 1$ be an integer such that

$$1 \geqq |(na)^{-1} b - m| \geqq 1/2 .$$

Such an m exists. If $(na)^{-1} b > m$, then C_b is a disjoint union of a translate of C_{mna} and some Λ which, in turn, is a disjoint union of boxes L_j satisfying $L_j \propto C_{na}$ and $V(L_j) \geqq (na/2)^\nu$. If $(na)^{-1} b < m$, then C_{mna} is a disjoint union of C_b and some Λ which is a disjoint union of boxes L_j satisfying the same relations.

By (3.1), we have

$$|S(C_{mna}) - S(C_b)| \leqq S(\Lambda) . \tag{5.12}$$

From (5.6), we have

$$S(\Lambda)\, b^{-\nu} \leqq 2S(C_{na})\, (na/2)^{-\nu}\, (b^{-\nu} V(\Lambda)) .$$

Since

$$b^{-\nu} V(\Lambda) = |(mna/b)^\nu - 1| = \left| \sum_{k=1}^{\nu} \binom{\nu}{k} \left(\frac{mna}{b} - 1 \right)^k \right| \leqq \sum_{k=1}^{\nu} \binom{\nu}{k} (na/b)^k ,$$

we have from (5.12) and (5.11),

$$|S(C_{mna})\, b^{-\nu} - \alpha_1(b)| < \varepsilon/3 . \tag{5.13}$$

Next,

$$\begin{aligned} &|S(C_{mna})\, b^{-\nu} - \alpha_{mn}(a)| \\ &= \alpha_{mn}(a)\, V(\Lambda)\, b^{-\nu} \\ &\leqq \alpha_n(a)\, V(\Lambda)\, b^{-\nu} \\ &< \varepsilon/3 . \end{aligned} \tag{5.14}$$

Finally, from $\alpha_\infty(a) \leqq \alpha_{mn}(a) \leqq \alpha_n(a)$, we have

$$|\alpha_\infty(a) - \alpha_{mn}(a)| < \varepsilon/3 . \tag{5.15}$$

Collecting (5.13), (5.14) and (5.15) together, we have

$$|\alpha_\infty(a) - S(C_b)/V(C_b)| < \varepsilon . \tag{5.16}$$

Q.E.D.

Remark. From the above proof, it is clear that if Λ is restricted to a disjoint union of boxes L_j whose volume is larger than a fixed v_0, then

$$\lim_{\Lambda \to \infty} S(\Lambda)/V(\Lambda) = \alpha_\infty(a)$$

where $\Lambda \to \infty$ in the sense of Van Hove.

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