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Lorentz Covariance of the $\lambda(\varphi^4)_2$ Quantum Field Theory

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Abstract. We prove that the $\lambda(\varphi^4)_2$ quantum field theory model is Lorentz covariant, and that the corresponding theory of bounded observables satisfies all the Haag-Kastler axioms. For each Poincaré transformation $\{a, \Lambda\}$ and each bounded region $\boldsymbol{B}$ of Minkowski space we construct a unitary operator U which correctly transforms the field bilinear forms: $U\varphi(x, t)\, U^* = \varphi(\{a, \Lambda\}\,(x, t))$, for $(x, t) \in \boldsymbol{B}$. We also consider the von Neumann algebra $\mathfrak{A}(\boldsymbol{B})$ of local observables, consisting of bounded functions of the field operators $\varphi(f) = \int \varphi(x, t)\, f(x, t)\, dx\, dt$, $\operatorname{supp} f \subset \boldsymbol{B}$. We define a $*$-isomorphism $\sigma_{\{a,\Lambda\}}: \mathfrak{A}(\boldsymbol{B}) \to \mathfrak{A}(\{a, \Lambda\}\,\boldsymbol{B})$ by setting $\sigma_{\{a,\Lambda\}}(A) = UAU^*$. The mapping $\{a, \Lambda\} \to \sigma_{\{a,\Lambda\}}$ is a representation of the Poincaré group by $*$-automorphisms of the normed algebra $\cup_{\boldsymbol{B}} \mathfrak{A}(\boldsymbol{B})$ of local observables.

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1. Introduction

1.1. Discussion

We study the quantum field φ of the $\lambda(\varphi^4)_2$ quantum field theory. This field φ satisfies the nonlinear equation

$$\varphi_{tt} - \varphi_{xx} + m^2\varphi + 4\lambda\varphi^3 = 0\,, \qquad \lambda > 0\,, \tag{1.1.1}$$

where φ^3 is suitably defined. The construction of a field φ satisfying (1.1.1) was carried out in [1–3]. The field $\varphi(x, t)$ is a densely defined bilinear form on Fock space, and the values of this form are continuous in x and t. If $f(x, t)$ is a real $\mathscr{C}^\infty$ function with compact support, then the bilinear form

$$\varphi(f) = \int \varphi(x, t)\, f(x, t)\, dx\, dt \tag{1.1.2}$$

uniquely determines a self adjoint operator on Fock space. The field (1.1.2) is local, so that $\varphi(f)$ commutes with $\varphi(g)$ if f and g have space-like separated supports. The field φ is space-time covariant. This means that for $a = (\alpha, \tau)$, there is an automorphism σ_a of the algebra of field operators such that

$$\sigma_a(\varphi(f)) = \varphi(f_a)\,, \tag{1.1.3}$$

where

$$(f_a)(x, t) = f(x - \alpha, t - \tau)\,.$$

The automorphism (1.1.3) is implemented locally by a unitary operator [3].

Another desired property of φ is covariance under Lorentz transformations, but this property was not established in [1–3]. Lorentz covariance requires the existence of an automorphism σ_Λ of the algebra of field operators, such that

$$\sigma_\Lambda(\varphi(f)) = \varphi(f_\Lambda)\,. \tag{1.1.5}$$

Here f_Λ is the function f transformed to the Lorentz frame moving at velocity $\tanh\beta$,

$$(f_\Lambda)(x, t) = f(x\cosh\beta - t\sinh\beta, t\cosh\beta - x\sinh\beta)\,. \tag{1.1.6}$$

The group of Lorentz transformations $\Lambda = \Lambda_\beta$ satisfies the multiplication law

$$\Lambda_{\beta_1}\Lambda_{\beta_2} = \Lambda_{\beta_1+\beta_2}\,. \tag{1.1.7}$$

Acting on space-time, the Lorentz transformation is defined by

$$\Lambda(x, t) = (x\cosh\beta + t\sinh\beta, t\cosh\beta + x\sinh\beta)\,. \tag{1.1.8}$$

In this paper we establish this Lorentz covariance. We prove the existence of the automorphism σ_Λ of (1.1.5), and we show that σ_Λ is implemented locally by a unitary operator. Combining the spacetime translation automorphism σ_a of (1.1.3) with the Lorentz automorphism (1.1.5), we obtain covariance under the Poincaré group. In other words the automorphism

$$\sigma_{\{a,\Lambda\}} = \sigma_a \sigma_\Lambda , \tag{1.1.9}$$

is a representation of the Poincaré group,

$$\sigma_{\{a,\Lambda\}} \sigma_{\{a',\Lambda'\}} = \sigma_{\{\Lambda a' + a, \Lambda\Lambda'\}} , \tag{1.1.10}$$

$$\sigma_{\{a,\Lambda\}}(\varphi(f)) = \varphi(f_{\{a,\Lambda\}}) . \tag{1.1.11}$$

In addition to the field operators themselves, it is convenient to study bounded functions of the fields. We let $\boldsymbol{B}$ be a bounded region of space-time, and $\mathfrak{A}(\boldsymbol{B})$ the von Neumann algebra generated by

$$\{e^{i\varphi(f)} : f = \bar{f} \in \mathscr{C}_0^\infty(\boldsymbol{B})\} . \tag{1.1.12}$$

The C^* algebra of observables $\mathfrak{A}$ is the norm closure of

$$\cup_{\boldsymbol{B}} \mathfrak{A}(\boldsymbol{B}) . \tag{1.1.13}$$

In [2–3] it was shown that the algebras $\mathfrak{A}(\boldsymbol{B})$ yield a theory of bounded observables satisfying all the Haag-Kastler axioms, with the possible exception of Lorentz covariance. Our results here on Lorentz covariance of the field ensure Lorentz covariance of the local algebras. Hence our results show that the algebras $\mathfrak{A}(\boldsymbol{B})$ give a complete Haag-Kastler theory in two dimensional space-time. The automorphism $\sigma_{\{a,\Lambda\}}$ is an isomorphism of $\mathfrak{A}(\boldsymbol{B})$ onto $\mathfrak{A}(\{a,\Lambda\}\,\boldsymbol{B})$.

We now explain the basic ideas which motivate our construction. One would expect that an operator of the form

$$M = M_0 + \lambda \int :\varphi^4(x):\, x g(x)\, dx, \quad g(x) \geqq 0 , \tag{1.1.14}$$

where $g(x) = 1$ on a sufficiently large interval, is the infinitesimal generator of Lorentz boosts in a bounded space-time region $\boldsymbol{B}$. The same physical ideas motivate the use in [1–3] of the locally correct Hamiltonian,

$$H = H_0 + \lambda \int :\varphi^4(x):\, g(x)\, dx . \tag{1.1.15}$$

Here M_0 and H_0 are the self adjoint generators of Lorentz rotations and time translation for the theory with no interaction, that is for the case $\lambda = 0$. One also expects that

$$[iH, M] = P^{\mathrm{loc}} , \tag{1.1.16}$$

where P^{loc} generates space translations in the region $\boldsymbol{B}$. We would say that P^{loc} is a locally correct momentum operator. For a locally correct momentum operator, we expect that

$$[iH, P^{\mathrm{loc}}] = R^{\mathrm{loc}}, \tag{1.1.17}$$

where R^{loc} is localized outside the bounded region $\boldsymbol{B}$. Thus for $\operatorname{supp} f \subset \boldsymbol{B}$,

$$[R^{\mathrm{loc}}, \varphi(f)] = 0. \tag{1.1.18}$$

These commutators lead to the equation

$$[iM, \varphi(f)] = -\varphi(xf_t + tf_x), \tag{1.1.19}$$

which in integrated form states that

$$e^{iM\beta}\varphi(f)\,e^{-iM\beta} = \varphi(f_{\Lambda_\beta}). \tag{1.1.20}$$

In other words, the putative Lorentz transformation

$$\varphi(f) \to \varphi(f_{\Lambda_\beta}) \tag{1.1.21}$$

is implemented by a unitary operator that depends only on β and the support of f.

In Section 2 we prove that it is sufficient to carry out the above argument for space-time regions $\boldsymbol{B}$ lying outside the light cone, in the region $x > 0$. For such a region $\boldsymbol{B}$, we can replace M of (1.1.14) by

$$\begin{aligned} M = \alpha H_0 + \int_0^\infty \frac{1}{2} : \{\pi(x)^2 + (\nabla\varphi)^2(x) + m^2\varphi(x)^2\} : x g_0(x)\,dx \\ + \lambda \int_0^\infty :\varphi^4(x):\; x g_1(x)\,dx, \end{aligned} \tag{1.1.22}$$

which formally generates Lorentz transformations in $\boldsymbol{B}$ provided that

$$\alpha + x g_0(x) = x = x g_1(x)$$

for x in a sufficiently large interval of the positive x axis.

In Section 2–6, we show that the formal ideas outlined above can be made mathematically rigorous. We prove in Section 3 that the expression M of (1.1.22) defines a symmetric operator M on Fock space, and in Sections 4–5 we prove that M is self adjoint. In Section 6 we prove that M generates Lorentz transformations in $\boldsymbol{B}$, by establishing the covariance (1.1.19–20).

In order to define and gain quantitative control over the operator M, and the other operators involved, we estimate the kernels of certain Wick ordered monomials in creation and annihilation operators. In [1–3], kernels are estimated with $\mathscr{L}^2$ norms. Such norms are insufficient for our problem, and we find it convenient to introduce $\mathscr{L}^1 - \mathscr{L}^\infty$ norms on the kernels. For these new norms we find estimates that yield Fock space operator inequalities. We use these inequalities, along with other inequalities, to define (1.1.22), to prove that M is self adjoint, and to establish (1.1.19–20) as a mathematically rigorous theorem. The results of Section 2 then ensure Lorentz covariance of the theory of the algebra of local observables $\mathfrak{A}$.

In this paper we do not consider Lorentz invariance of the vacuum state of [4], or other questions concerning the physical representation. The results of this paper combined with the results of [4] show, however, that $\sigma_{\{a,\Lambda\}}$ is locally implemented in the physical representation by a unitary operator.

An important feature of our proof is the use of operator inequalities of the form

$$H_0^2 N^{j-2} \leqq \text{const}(M+b)^j\,, \tag{1.1.23}$$

for $j \geqq 2$, proved in [10] and in sections 4–5. Formal perturbation theory suggests that (1.1.23) hold for the $\lambda(\phi^{2n})_2$ quantum field theory, although somewhat weaker estimates are proved in [10]. We remark that with higher order estimates established for the $\lambda(\phi^{2n})_2$ models, $n>2$, our method should ensure Lorentz covariance for these theories.

1.2. Notation

We work on Fock space $\mathscr{F}$, the Hilbert space completion of the direct sum

$$\bigoplus_{n=0}^{\infty} \mathscr{F}_n\,. \tag{1.2.1}$$

The vectors in the n particle space $\mathscr{F}_n$ are represented by the symmetric functions in $\mathscr{L}^2(\boldsymbol{R}^n)$. We use the standard notation for creation and annihilation forms on $\mathscr{F}$. A summary of mathematical properties of these bilinear forms and operators on Fock space is given in [4, Section 3.2]. We use the domain $\mathscr{D}$ of smooth vectors on Fock space,

$$\mathscr{D} = \{\psi : \psi \in \mathscr{F}, \psi^{(n)} \in \mathscr{S}(\boldsymbol{R}^n), \psi^{(n)} = 0 \quad \text{for } n \text{ sufficiently large}\}\,. \tag{1.2.2}$$

Here $\mathscr{S}(\boldsymbol{R}^n)$ is the Schwartz space of rapidly decreasing $\mathscr{C}^\infty$ functions with rapidly decreasing derivatives of all orders. The annihilation operator $a(k)$ is defined on the dense domain $\mathscr{D}$ by

$$(a(k)\,\psi)^{(n-1)}(k_1, \ldots, k_{n-1}) = n^{\frac{1}{2}}\,\psi^{(n)}(k, k_1, \ldots, k_{n-1})\,, \tag{1.2.3}$$

and the creation form $a(k)^*$ is defined on $\mathscr{D} \times \mathscr{D}$ as the adjoint of $a(k)$.

The time zero field $\varphi(x)$ and its canonical conjugate $\pi(x)$ are $\mathscr{S}(\boldsymbol{R}^1)$ valued forms on $\mathscr{D} \times \mathscr{D}$ defined by

$$\varphi(x) = (4\pi)^{-\frac{1}{2}} \int e^{-ikx} \{a(k)^* + a(-k)\} \mu(k)^{-\frac{1}{2}} dk , \tag{1.2.4}$$

and

$$\pi(x) = i(4\pi)^{-\frac{1}{2}} \int e^{-ikx} \{a(k)^* - a(-k)\} \mu(k)^{\frac{1}{2}} dk , \tag{1.2.5}$$

where $\mu(k) = (k^2 + m^2)^{\frac{1}{2}}$. The Wick powers of the time zero fields $:\varphi(x)^n:$, $:\pi(x)^n:$, $:(\nabla\varphi(x))^n:$ are also $\mathscr{S}(\boldsymbol{R}^1)$ valued bilinear forms on $\mathscr{D} \times \mathscr{D}$. The Wick powers of the field φ are defined by

$$\begin{aligned} :\varphi(x)^n: = (4\pi)^{-\frac{n}{2}} \sum_{j=0}^{n} \binom{n}{j} \int e^{-ix(k_1 + \cdots + k_n)} \\ \cdot \mu(k_1)^{-\frac{1}{2}} \ldots \mu(k_n)^{-\frac{1}{2}} a^*(k_1) \ldots a^*(k_j) \\ \cdot a(-k_{j+1}) \ldots a(-k_n) dk_1 \ldots dk_n . \end{aligned} \tag{1.2.6}$$

The Wick powers of π and $\nabla\varphi$ are similarly defined and the Wick dots : : are extended to polynomials in φ and π by linearity.

The free Hamiltonian H_0 is defined by

$$\begin{aligned} H_0 &= \tfrac{1}{2} \int :\pi^2 + (\nabla\varphi)^2 + m^2 \varphi^2 : dx \\ &= \int a(k)^* a(k) \mu(k) dk . \end{aligned} \tag{1.2.7}$$

The interaction energy density $T_I(x)$ for our theory is

$$T_I(x) = \lambda :\varphi(x)^4:, \qquad \lambda > 0 . \tag{1.2.8}$$

We also use the number operator

$$N = \int a(k)^* a(k) dk , \tag{1.2.9}$$

the fractional energy operators

$$N_\tau = \int a(k)^* a(k) \mu(k)^\tau dk , \tag{1.2.10}$$

and the momentum operator

$$P = \int a(k)^* a(k) k dk . \tag{1.2.11}$$

The operator N_τ and P are essentially self adjoint on $\mathscr{D}$.

The locally correct Hamiltonian

$$H(g) = H_0 + T_I(g), \qquad g(\cdot) \geqq 0, \qquad g \in \mathscr{C}_0^\infty , \tag{1.2.12}$$

is self adjoint and essentially self adjoint on $\mathscr{D}(H_0^2)$ [1–2].

Let $I = [a, b]$ be a bounded interval in $\boldsymbol{R}^1$. The *causal shadow* of I is defined as the region $\boldsymbol{B}_I \subset \boldsymbol{R}^2$,

$$\{(x, t) : a + |t| < x < b - |t|\} . \tag{1.2.13}$$

If $g(\cdot)$ of (1.2.12) equals one on I, then $H(g)$ is a correct $\lambda(\varphi^4)$ Hamiltonian for $\boldsymbol{B}_I$. In fact, for $(x,t)\in \boldsymbol{B}_I$,

$$\varphi(x,t)=e^{itH(g)}\varphi(x)e^{-itH(g)} \tag{1.2.14}$$

is a solution of (1.1.1) and is independent of $g(\cdot)$, see [3]. If $f=\bar{f}\in \mathscr{C}_0^\infty(\boldsymbol{B}_I)$, the bilinear form

$$\varphi(f)=\int \varphi(x,t)f(x,t)\,dx\,dt$$

uniquely determines a symmetric operators on the domain $\mathscr{D}(H(g))$, and $\varphi(f)$ is essentially self adjoint on this domain [3]. The bounded functions of these self adjoint operators $\varphi(f)$ generate the local von Neumann algebras $\mathfrak{A}(\boldsymbol{B})$ described in Section 1.1.

2. Lorentz Covariance

2.1. The Main Results

Let $\mathscr{G}$ be the restricted Poincaré group of transformations of two dimensional space-time. For $\{a,\Lambda_\beta\}\in\mathscr{G}$,

$$\{a,\Lambda_\beta\}(x,t)=(\alpha+x\cosh\beta+t\sinh\beta,\ \tau+t\cosh\beta+x\sinh\beta)\,. \tag{2.1.1}$$

On functions $f(x,t)$ we define

$$(f_{\{a,\Lambda\}})(x,t)=f(\{a,\Lambda\}^{-1}(x,t))\,. \tag{2.1.2}$$

We prove that the field φ of the $(\varphi^4)_2$ theory is Poincaré covariant in the following sense:

Theorem 2.1.1. *Let $\boldsymbol{B}\subset\boldsymbol{R}^2$ be a bounded set and let $\{a,\Lambda\}\in\mathscr{G}$. Then there exists a unitary operator U on $\mathscr{F}$ such that for all $f\in\mathscr{C}_0^\infty(\boldsymbol{B})$,*

$$U\varphi(f)U^*=\varphi(f_{\{a,\Lambda\}})\,. \tag{2.1.3}$$

This equality holds in the sense of self adjoint operators.

In the case of space time translations, $\Lambda=\boldsymbol{I}$, Theorem 2.1.1 is proved in [3]. In this case,

$$U(a;\boldsymbol{B})=e^{-iP\alpha}e^{i\tau H(g)}\,, \tag{2.1.4}$$

where $g(\cdot)=1$ on a sufficiently large set depending on a and $\boldsymbol{B}$.

We remark that it is sufficient to prove Theorem 2.1.1 for pure Lorentz transformations $\{0,\Lambda\}$, since

$$\{a,\Lambda\}=\{a,\boldsymbol{I}\}\,\{0,\Lambda\}\,, \tag{2.1.5}$$

and $\{a,\boldsymbol{I}\}$ is implemented by an appropriate unitary operator (2.1.4).

Using Theorem 2.1.1, we define a norm preserving map

$$\sigma_{\{a,\Lambda\}}:\mathfrak{A}(\boldsymbol{B})\to\mathfrak{A}(\{a,\Lambda\}\,\boldsymbol{B}) \tag{2.1.6}$$

by

$$\sigma_{\{a,\Lambda\}}(A) = UAU^*, \qquad A \in \mathfrak{A}(\boldsymbol{B}). \tag{2.1.7}$$

Since

$$U e^{i\varphi(f)} U^* = e^{i\varphi(f_\Lambda)}, \tag{2.1.8}$$

the mapping (2.1.6) can be defined on all generators of $\mathfrak{A}(\boldsymbol{B})$ and it yields generators of $\mathfrak{A}(\{a, \Lambda\}\boldsymbol{B})$. Furthermore, all the generators of $\mathfrak{A}(\{a, \Lambda\}\boldsymbol{B})$ are obtained in this manner. Since the mapping (2.1.8) is unitarily implemented, it extends to a $*$-isomorphism of the von Neumann algebras $\mathfrak{A}(\boldsymbol{B})$ and $\mathfrak{A}(\{a, \Lambda\}\boldsymbol{B})$. Furthermore if U and U_1 are two different unitary operators satisfying (2.1.3), then

$$U_1^* U \varphi(f) U^* U_1 = \varphi(f), \tag{2.1.9}$$

so that $\sigma_{\{a,\Lambda\}}$ of (2.1.6) is well defined, and independent of the particular choice of U satisfying (2.1.8). In this manner $\sigma_{\{a,\Lambda\}}$ is defined as a norm preserving $*$-automorphism if the algebra

$$\cup_{\boldsymbol{B}} \mathfrak{A}(\boldsymbol{B}), \tag{2.1.10}$$

and hence extends by continuity to a star isomorphism of the C^* algebra of local observables $\mathfrak{A}$. Finally the mapping

$$\{a, \Lambda\} \to \sigma_{\{a,\Lambda\}} \tag{2.1.11}$$

is a representation of $\mathscr{G}$. To prove this, we note that

$$\begin{aligned}
\sigma_{\{a_1,\Lambda_1\}}(\sigma_{\{a_2,\Lambda_2\}}(\exp(i\varphi(f)))) &= \sigma_{\{a_1,\Lambda_1\}}(\exp(i\varphi(f_{\{a_2,\Lambda_2\}}))) \\
&= \exp(i\varphi((f_{\{a_2,\Lambda_2\}})_{\{a_1,\Lambda_1\}})) \\
&= \sigma_{\{a_1,\Lambda_1\}\{a_2,\Lambda_2\}}(\exp(i\varphi(f))),
\end{aligned}$$

since

$$\begin{aligned}
(f_{\{a_2,\Lambda_2\}})_{\{a_1,\Lambda_1\}}(x,t) &= f_{\{a_2,\Lambda_2\}}(\{a_1,\Lambda_1\}^{-1}(x,t)) \\
&= f(\{a_2,\Lambda_2\}^{-1}\{a_1,\Lambda_1\}^{-1}(x,t)) \\
&= f((\{a_1,\Lambda_1\}\{a_2,\Lambda_2\})^{-1}(x,t)) \\
&= f_{\{a_1,\Lambda_1\}\{a_2,\Lambda_2\}}(x,t).
\end{aligned} \tag{2.1.12}$$

Hence we obtain

Corollary 2.1.2. *There is representation $\sigma_{\{a,\Lambda\}}$ of $\mathscr{G}$ by a group of $*$-automorphisms of $\mathfrak{A}$ such that*

$$\sigma_{\{a,\Lambda\}}(\mathfrak{A}(\boldsymbol{B})) = \mathfrak{A}(\{a,\Lambda\}\boldsymbol{B}).$$

This is one of the Haag-Kastler axioms for the algebra $\mathfrak{A}$ of local observables for the $\lambda(\varphi^4)_2$ model. We note that the automorphism $\sigma_{\{a,\Lambda\}}$ is implemented on $\mathfrak{A}(\boldsymbol{B})$ by the unitary operators U of Theorem 2.1.1.

Remark. We will construct operators $U = U(\{a, \Lambda\}; \boldsymbol{B})$ in Theorem 2.1.1 with the following continuity: For $\{a, \Lambda\}$ in a neighborhood of an element of $\mathscr{G}$, $U(\{a, \Lambda\}; \boldsymbol{B})$ can be chosen strongly continuous in $\{a, \Lambda\}$.

2.2. Reduction of the Problem

In this section we reduce the proof of Theorem 2.1.1 to the form that we analyze in the remainder of the paper.

Lemma 2.2.1. *To prove Theorem 2.1.1 it is sufficient to establish the following: Let Λ be a pure Lorentz transformation and let $\boldsymbol{B} \subset \boldsymbol{R}^2$ be a bounded subset of $\boldsymbol{R}^2$ with closure in $\{x > |t|\}$. Then there exists a unitary operator U_1 on $\mathscr{F}$ such that for all $f \in \mathscr{C}_0^\infty(\boldsymbol{B})$,*

$$U_1 \varphi(f) U_1^* = \varphi(f_\Lambda). \tag{2.2.1}$$

Proof. In Section 2.1 we observed that it is sufficient to prove Theorem 2.1.1 in the case of pure Lorentz transformations. Let Λ be a pure Lorentz transformation and $\boldsymbol{B} \subset \boldsymbol{R}^2$ a bounded region. Then a suitable space-time translation $\{a, \boldsymbol{I}\} \in \mathscr{G}$ exists such that

$$\{a, \boldsymbol{I}\} \boldsymbol{B}^- \subset \boldsymbol{S}_+ = \{(x, t) : x > |t|\}, \tag{2.2.2}$$

where $\boldsymbol{B}^-$ is the closure of $\boldsymbol{B}$. By hypotheses of the lemma, for $\operatorname{supp} f \subset \boldsymbol{B}$,

$$\operatorname{supp} f_{\{a, \boldsymbol{I}\}} \subset \{a, \boldsymbol{I}\} \boldsymbol{B} \subset \boldsymbol{S}_+,$$

and there exists a unitary U_1 such that

$$U_1 \varphi(f_{\{a, \boldsymbol{I}\}}) U_1^* = \varphi((f_{\{a, \boldsymbol{I}\}})_{\{0, \Lambda\}}) = \varphi(f_{\{\Lambda a, \Lambda\}}) \tag{2.2.3}$$

by (2.1.12). As discussed in Section 2.1, space-time covariance was established in [3] and the space-time translations are implemented in bounded regions by unitary operators in Fock space, namely (2.1.4). Let $U(a)$ and $U(-\Lambda a)$ be the unitary operators implementing space-time translation by a and $-\Lambda a$ in the convex hull of

$$\boldsymbol{B} \cup \Lambda \boldsymbol{B} \cup \{a, \boldsymbol{I}\} \boldsymbol{B} \cup \Lambda(\{a, \boldsymbol{I}\} \boldsymbol{B}).$$

Then

$$\begin{aligned} U(-\Lambda a) U_1 U(a) \varphi(f) (U(-\Lambda a) U_1 U(a))^* \\ &= U(-\Lambda a) U_1 U(a) \varphi(f) U(a)^* U_1^* U(-\Lambda a)^* \\ &= U(-\Lambda a) U_1 \varphi(f_{\{a, \boldsymbol{I}\}}) U_1^* U(-\Lambda a)^* \end{aligned}$$

by (2.2.3)

$$\begin{aligned} &= U(-\Lambda a) \varphi(f_{\{\Lambda a, \Lambda\}}) U(-\Lambda a)^* \\ &= \varphi(f_\Lambda). \end{aligned}$$

Thus the operator

$$U = U(-\Lambda a)\,U_1\,U(a)$$

is the operator U required by Theorem 2.2.1, and the proof of the lemma is complete.

Corollary 2.2.2. *To prove Theorem 2.1.1, it is sufficient to establish Lemma 2.2.1 for regions* $\boldsymbol{B}' \subset \boldsymbol{R}^2$ *such that*

$$\boldsymbol{B}' \cup \Lambda \boldsymbol{B}' \subset \boldsymbol{B}_I \tag{2.2.4}$$

for the causal shadow $\boldsymbol{B}_I$ *of a closed interval* I *of the right half line,* $\{x > 0\}$.

Proof. Given $\boldsymbol{B}$ and Λ, we can choose a space-time translation $\{a, \boldsymbol{I}\}$ such that

$$\{a, \boldsymbol{I}\}\,\boldsymbol{B} = \boldsymbol{B}'$$

satisfies (2.2.4) for an appropriate $\boldsymbol{B}_I$. We now follow the proof of Lemma 2.2.1.

We remark that the advantage of performing the Lorentz boost in the causal shadow of $I \subset \{x > 0\}$, is that a positive operator M can be used to generate the Lorentz transformation. This operator will be introduced and studied in the following sections. The positivity of M allows us to use known techniques [2] to study the self adjointness of M. In Theorem 6.1, we prove that there is a unitary operator

$$U_1 = e^{iM\beta}$$

satisfying the hypotheses of Lemma 2.2.1 and Corollary 2.2.2. Hence Theorem 6.1 completes the proof of Theorem 2.1.1.

3. First Order Estimates on the Local Energy Density

3.1. The Basic Inequalities

We estimate monomials in creation and annihilation operators in terms of the operators N_τ,

$$N_\tau = \int a(k)^* a(k)\,\mu(k)^\tau\,dk\,. \tag{3.1.1}$$

We note that for $\tau \geqq 1$,

$$N_\tau \leqq H_0^\tau\,, \quad \text{and} \quad N_\tau^2 \leqq H_0^{2\tau}\,. \tag{3.1.2}$$

For our first estimate we consider the bilinear form

$$W = \int a(k)^* w(k, p) a(p) \, dk \, dp \tag{3.1.3}$$

where the kernel $w(k, p)$ is measurable and $|w(k, p)|$ is symmetric. We define two $\mathscr{L}^1 - \mathscr{L}^\infty$ norms on the kernel w, which may be finite or infinite

$$M_1(\tau) = \sup_k \mu(k)^{-\tau} \int |w(k, p)| \, dp \,, \tag{3.1.4}$$

$$M_2(\tau) = \sup_k \mu(k)^{-2\tau} \int |w(k, p)| \, \mu(p)^\tau \, dp \,. \tag{3.1.5}$$

Lemma 3.1.1. *If for some $\tau, M_1(\tau) < \infty$, then W is a bilinear form on the domain $\mathscr{D}(N_\tau^{\frac{1}{2}}) \times \mathscr{D}(N_\tau^{\frac{1}{2}})$, and $N_\tau^{-\frac{1}{2}} W N_\tau^{-\frac{1}{2}}$ is a bounded operator on $\mathscr{F}$ with*

$$\|N_\tau^{-\frac{1}{2}} W N_\tau^{-\frac{1}{2}}\| \leqq M_1(\tau) \,. \tag{3.1.6}$$

Remarks. a) The operator $N_\tau^{-\frac{1}{2}}$ is defined on the orthogonal complement of the no particle vector. Since W equals zero on the no particle vector, we define $W N_\tau^{-\frac{1}{2}}$ to be zero on the no particle vector.

b) If $\tau \geqq 1$, then (3.1.2) ensures that

$$H_0^{-\tau/2} W H_0^{-\tau/2} \tag{3.1.7}$$

is a bounded operator with norm less than $M_1(\tau)$.

Proof. Since the bilinear form W commutes with the projection onto vectors with exactly n particles, it is sufficient to prove that for n particle vectors $\psi \in \mathscr{D}(N_\tau^{\frac{1}{2}})$, the following inequality of forms is valid:

$$|(\psi, W\psi)| \leqq M_1(\tau)(\psi, N_\tau \psi) \,. \tag{3.1.8}$$

The existence of a bounded operator satisfying (3.1.6) then follows by the Riesz representation theorem.

$$(\psi, W\psi) = n \int \overline{\psi}(p, k_2, \ldots, k_n) \, w(p, q) \, \psi(q, k_2, \ldots, k_n) \, dk \, dp \, dq$$

We use the Schwarz inequality in p and q,

$$|(\psi, W\psi)| \leqq n \int |\psi(p, k)^2 \, w(p, q)| \, dk \, dp \, dq$$

and by (3.1.4),

$$|(\psi, W\psi)| \leqq n M_1(\tau) \int |\psi(p, k)^2| \, \mu(p)^\tau \, dp \, dk = M_1(\tau)(\psi, N_\tau \psi) \,.$$

Lemma 3.1.2. *If for some $\tau, M_1(\tau)$ and $M_2(\tau)$ are finite, then W determines an operator on $\mathscr{D}(N_\tau)$ such that $W N_\tau^{-1}$ is bounded with*

$$\|W N_\tau^{-1}\| \leqq (M_1(\tau) + M_2(\tau)) = M_3(\tau) \,. \tag{3.1.9}$$

Remarks. Since $|w|$ is symmetric, $N_\tau^{-1} W$ is also bounded with a norm less than $M_3(\tau)$. If $\tau \geqq 1$, $WH_0^{-\tau}$ is bounded with a norm less than $M_3(\tau)$.

Proof. As in Lemma 3.1.1, it is sufficient to prove that for n particle vectors $\psi \in \mathscr{D}(N_\tau)$,

$$\|W\psi\| \leqq M_3(\tau) \, \|N_\tau \psi\| \, . \tag{3.1.10}$$

We define

$$\begin{aligned} C_{jl} = \int dk_1, \ldots, dk_n \{ \int dp_j \overline{w}(k_j, p_j) \overline{\psi}(k_1, \ldots, k_{j-1}, p_j, k_{j+1}, \ldots, k_n) \\ \cdot \int dp_l w(k_l, p_l) \psi(k_1, \ldots, k_{l-1}, p_l, k_{l+1}, \ldots, k_n) \} \, , \end{aligned} \tag{3.1.11}$$

and note that

$$\|W\psi\|^2 = \sum_{j,l=1}^{n} C_{jl} \, .$$

For $j = l = 1$, and $k = (k_2, \ldots, k_n)$, we have

$$\begin{aligned} C_{jj} &= \int dk dq (|\int dp \, w(q, p) \psi(p, k)|)^2 \\ &\leqq \int dk dq (\int dp \, |w(q, p) \psi(p, k)|)^2 \, . \end{aligned} \tag{3.1.12}$$

By the Schwarz inequality

$$C_{jj} \leqq \int dk dq (\int |w(q, p)| \, dp \int |\psi(r, k)^2 w(r, q)| \, dr)$$

by (3.1.4),

$$\leqq M_1(\tau) \int dk dq dr \, \mu(q)^\tau |\psi(r, k)^2 w(r, q)|$$

and by (3.1.5),

$$\leqq M_1(\tau) M_2(\tau) \int dk dp \, \mu(p)^{2\tau} |\psi(p, k)|^2 \, .$$

We also estimate C_{jl} for $j \neq l$. Suppressing all but the essential variables k_j, p_j, k_l and p_l, we have

$$|C_{jl}| \leqq \int dk_j dk_l dp_j dp_l |w(k_j, p_j) \psi(p_j, k_l) w(k_l, p_l) \psi(p_l, k_j)| \, . \tag{3.1.13}$$

By the Schwarz inequality in p, and (3.1.4),

$$\begin{aligned} |C_{jl}| \leqq M_1(\tau) \int dk_j dk_l \mu(k_j)^{\tau/2} \mu(k_l)^{\tau/2} (\int |w(k_j, p_j) \psi(p_j, k_l)^2| \, dp_j)^{\frac{1}{2}} \\ \cdot (\int |w(k_l, p_l) \psi(p_l, k_j)^2| \, dp_l)^{\frac{1}{2}} \end{aligned}$$

by the Schwarz inequality in k,

$$\leqq M_1(\tau) (\int dk_j dk_l dp_j \mu(k_l)^\tau |w(k_j, p_j) \psi(p_j, k_l)^2|)$$

and by (3.1.4)

$$\leqq M_1(\tau)^2 \int dk_l dp_j \mu(k_l)^\tau \mu(p_j)^\tau |\psi(p_j, k_l)^2| \, .$$

Hence by (3.1.11)–(3.1.13),

$$\|W\psi\|^2 \leqq M_1(\tau)(M_1(\tau)+(M_2(\tau))\int dk\left(\sum_{j,l=1}^{n}\mu(k_j)^\tau\mu(k_l)^\tau|\psi(k)|^2\right)$$
$$= M_1(M_1+M_2)(\psi, N_\tau^2\psi),$$

and (3.1.10) is proved.

We now let

$$W=\int a(k_1)^* \dots a(k_r)^* w(k_1,\dots,k_r;p_1,\dots,p_s)a(p_1)\dots a(p_s)\,dk\,dp \qquad (3.1.14)$$

where $w(k;p)$ is a measurable kernel. Let $\alpha \leqq r$, and define $E_C(k)$ by

$$E_C = \mu(k_1)\dots\mu(k_\alpha). \qquad (3.1.15)$$

Similarly, let $\beta \leqq s$ and define $E_A(p)$ by

$$E_A = \mu(p_1)\dots\mu(p_\beta). \qquad (3.1.16)$$

Let

$$\begin{aligned} M_4(\tau) &= \left\|\frac{w(k,p)}{E_C(k)^{\tau/2}E_A(p)^{\tau/2}}\right\|_{op} \\ &\leqq \left\|\frac{w(k,p)}{E_C(k)^{\tau/2}E_A(p)^{\tau/2}}\right\|_2 \end{aligned} \qquad (3.1.17)$$

where $\|v(k,p)\|_{op}$ denotes the operator norm of the kernel $v(k,p)$ as an integral operator from $\mathscr{L}^2(R^s)$ to $\mathscr{L}^2(R^r)$. The norm $\|\cdot\|_{op}$ is dominated by the Hilbert Schmidt norm $\|\cdot\|_2$.

We next give a lemma proved by Glimm [5, Theorem 2.4.3] for the Hilbert Schmidt norm. We present it here with a direct proof.

Lemma 3.1.3. *If $M_4(\tau)$ is finite for some α, β as above and for some τ, then W is a bilinear form on $\mathscr{D}(N_\tau^{\alpha/2}N^{\delta/2})\times\mathscr{D}(N_\tau^{\beta/2}N^{\varepsilon/2})$, where $\alpha+\delta=r$, $\beta+\varepsilon=s$. Also*

$$\hat{W}=N_\tau^{-\alpha/2}N^{-\delta/2}WN^{-\varepsilon/2}N_\tau^{-\beta/2} \qquad (3.1.18)$$

is a bounded operator and

$$\|\hat{W}\| \leqq M_4(\tau). \qquad (3.1.19)$$

Proof. Let Ω, ψ be vectors with a finite number of particles and wave functions in Schwartz space $\mathscr{S}$. Then, if $A_C(k)=a(k_1)\cdots a(k_r)$ and $A_A(p)=a(p_1)\dots a(p_s)$,

$$(\Omega, W\psi)=\int (A_C(k)\Omega, w(k,p)A_A(p)\psi)\,dk\,dp,$$

and by the Schwarz inequality,

$$\begin{aligned} |(\Omega, W\psi)|^2 &\leqq \left(\int \|A_C(k)\Omega\|\cdot|w(k,p)|\cdot\|A_A(p)\psi\|\,dk\,dp\right)^2 \\ &\leqq M_4(\tau)^2\int E_C(k)^\tau\|A_C(k)\Omega\|^2E_A(p)^\tau\|A_A(p)\psi\|^2\,dk\,dp \\ &\leqq M_4(\tau)^2\|N_\tau^{\alpha/2}N^{\delta/2}\Omega\|^2\cdot\|N_\tau^{\beta/2}N^{\varepsilon/2}\psi\|^2. \end{aligned}$$

The last inequality is proved as follows:

$$\int E_C(k)^\tau \|A_C(k)\Omega\|^2\, dk$$
$$= \sum_{n=0}^{\infty} \int (n+1)\dots(n+r)\mu(k_1)^\tau \dots \mu(k_\alpha)^\tau |\Omega^{(n+r)}(k_1,\dots k_{n+r})|^2\, dk$$
$$\leqq \sum_{n=0}^{\infty} \int \left(\sum_{j=1}^{n+r} \mu(k_j)^\tau\right)^\alpha (n+r)^\delta |\Omega^{(n+r)}(k)|^2\, dk$$
$$= \|N_\tau^{\alpha/2} N^{\delta/2}\Omega\|^2,$$

since $|\Omega^{(n+r)}|^2$ is symmetric and $\left(\sum_{j=1}^{n+r} \mu(k_j)^\tau\right)^\alpha \left(\sum_{j=1}^{n+r} \mu(k_j)^0\right)^\delta$ when expanded, has

$$r!\binom{n+r}{r} = (n+1)\dots(n+r)$$

terms with all variables distinct. The existence of the bounded operator $\hat{W}$ now follows by the Riesz representation theorem.

Remarks. a) We could choose any δ and ε, positive or negative, satisfying

$$\alpha+\beta+\delta+\varepsilon = r+s.$$

In that case (3.18)-(3.19) are replaced by

$$\|(I+N)^{-\delta/2} N_\tau^{-\alpha/2} W N_\tau^{-\beta/2} (I+N)^{-\varepsilon/2}\| \leqq \text{const}\, M_4(\tau), \tag{3.1.21}$$

where the construct depends on δ, ε.

b) If the kernel $w(k,p)$ is symmetric in the r creation variables or in the s annihilation variables, the norm $M_4(\tau)$ may be infinite even though $\hat{W}$ is bounded. For symmetric kernels and $\alpha \leqq r$ or $\beta \leqq s$, we can use a polynomial E_C of degree α. Let

$$E_C(\alpha,\tau) = E_C(\alpha,\tau;k_1,\dots,k_r)$$
$$= \left(\frac{(r-\alpha)!}{r!}\right) \sum{}' \mu(k_{i_1})^\tau \dots \mu(k_{i_\alpha})^\tau. \tag{3.1.22}$$

The sum $\sum'$ in (3.1.22) extends over the $\frac{r!}{(r-\alpha)!}$ monomials with no two of the $i_1,\dots,i_\alpha$ equal, and $1 \leqq i_j \leqq r$. In other words, $E_C(\alpha,\tau)$ is the average of those monomials in the expansion of

$$\left(\sum_{i=1}^{r} \mu(k_i)^\tau\right)^\alpha$$

that are products of energies of distinct variables. If $\tau=1$, $\alpha=r$, then $E_C(\alpha,\tau)=E_C$. Similarly let

$$E_A(\beta,\sigma) = \frac{(s-\beta)!}{s!} \sum{}' \mu(p_{i_1})^\sigma \dots \mu(p_{i_\beta})^\sigma.$$

By the same proof above, we obtain

Lemma 3.1.4. *If* $\alpha \leqq r$, $\beta \leqq s$ *and for some* τ, σ

$$M_5(\tau, \sigma) = \left\| \frac{w(k, p)}{E_C(\alpha, \tau) E_A(\beta, \sigma)} \right\|_{op} < \infty, \tag{3.1.23}$$

then W *is a bilinear form on* $D(N_\tau^{\alpha/2} N^{\delta/2}) \times D(N_\sigma^{\beta/2} N^{\varepsilon/2})$ *for any* δ, ε *such that* $\alpha + \beta + \delta + \varepsilon = r + s$. *Furthermore*

$$\hat{W} = (I + N)^{-\delta/2} N_\tau^{-\alpha/2} W N_\sigma^{-\beta/2} (N + I)^{-\varepsilon/2} \tag{3.1.24}$$

is a bounded operator with

$$\|\hat{W}\| \leqq \text{const}\, M_5(\tau, \sigma). \tag{3.1.25}$$

Remark. We note finally that $w(k_1, \ldots, k_r; p_1, \ldots, p_s)$ in (3.1.14) may be a bounded operator on Fock space. In that case, we replace $|w(k, p)|$ by $\|w(k, p)\|$ in (3.1.17) or (3.1.23). The Lemmas 3.1.3–3.1.4 are still valid in the case $\alpha + \delta = r$, $\beta + \varepsilon = s$. We require that $w(k, p)$ be measurable in the sense that for all $\Omega \in \mathscr{F}$, $(\Omega, w(k, p)\, \Omega)$ is measurable.

3.2. The Energy-Momentum Density

The energy-momentum density tensor[1] $T_{\mu\nu}(x, t)$ for the $\lambda(\varphi^4)_2$ theory is a bilinear form on Fock space. The energy momentum vector P_μ is formally related to $T_{\mu\nu}$ by

$$P_\mu = \int T_{0\mu}(x, t)\, dx, \tag{3.2.1}$$

and the generator M of Lorentz transformations is formally related to $T_{\mu\nu}$ by

$$M = \int T_{00}(x, 0)\, x\, dx. \tag{3.2.2}$$

The usual expression for the unrenormalized $T_{\mu\nu}$ is a Wick ordered polynomial in the time zero canonical fields φ and π. In this case the Hamiltonian $H = P_0$ defined by (3.2.1) is a bilinear form, but it is a semibounded operator only for free fields [6].

In this section we show that for the $\lambda(\varphi^4)_2$ theory the integration in (3.2.1) can be restricted to a bounded domain to yield a local energy or momentum operator on Fock space. The local version of (3.2.2) can be handled similarly. It is customary to write $T_{\mu\nu}$ as the sum of a free field part and an interaction part. Explicitly, we write the energy density as

$$T(x) \equiv T_{00}(x, 0) \equiv T_0(x) + T_I(x), \tag{3.2.3}$$

[1] For a general introduction to the properties of the energy-momentum tensor density, see for instance the book of R. Jost, "The general theory of quantized fields", Amer. Math. Soc. (1965), pp. 27–31.

where

$$T_0(x) = \tfrac{1}{2} :\pi(x)^2 + (\nabla\varphi(x))^2 + m^2\varphi(x)^2: , \tag{3.2.4}$$

and

$$T_{\mathrm{I}}(x) = \lambda :\varphi(x)^4: . \tag{3.2.5}$$

For the momentum density,

$$P(x) = T_{01}(x, 0) = \tfrac{1}{2} :\pi(x)\, \nabla\varphi(x) + \nabla\varphi(x)\pi(x): , \tag{3.2.6}$$

there is no interaction term.

In order to avoid problems caused by sharp spatial boundaries, we consider

$$T(g) = \int T(x) g(x)\, dx = T_0(g) + T_{\mathrm{I}}(g) \tag{3.2.7}$$

and

$$P(g) = \int P(x) g(x)\, dx , \tag{3.2.8}$$

where $g(\cdot)$ is a real function in $\mathscr{S}(\boldsymbol{R}^1)$, the space of smooth, rapidly decreasing functions. This is our local form of (3.2.1).

We establish here various properties of $T(g)$ and $P(g)$. We prove, among other things that $T(g)$ and $P(g)$ are symmetric operators on a reasonable domain. Since it is well known that the interaction term $T_{\mathrm{I}}(g)$ is a symmetric operator, our result pertains to $T_0(g)$ and $P(g)$. We expect that stronger results can be obtained. For instance we expect that for $g \geqq 0$, the operator $T(g)$ is bounded from below. Thus we expect that it is possible to construct a physically acceptable local energy that is affiliated with a bounded region of space.

We use different techniques to study the pure creation part of $T_0(g)$ and the particle number conserving part of $T_0(g)$. This is also true for $P(g)$. We estimate the particle number conserving parts with $\mathscr{L}^1 - \mathscr{L}^\infty$ norms and the pure creation parts with $\mathscr{L}^2$ norms. Thus we define two separate parts for each.

For the local free field energy we write

$$T_0(g) = T_0^{(1)}(g) + T_0^{(2)}(g) \tag{3.2.9}$$

where

$$T_0^{(1)}(g) = \frac{1}{4\pi} \int \tilde{g}(k_1 - k_2) \left\{ \frac{\mu(k_1)\mu(k_2) + k_1 k_2 + m^2}{\mu(k_1)^{\frac{1}{2}} \mu(k_2)^{\frac{1}{2}}} \right\} \cdot a^*(k_1) a(k_2)\, dk_1\, dk_2 \tag{3.2.10}$$

and

$$T_0^{(2)}(g) = \frac{1}{8\pi} \int \tilde{g}(k_1 - k_2) \left\{ \frac{-\mu(k_1)\mu(k_2) + k_1 k_2 + m^2}{\mu(k_1)^{\frac{1}{2}} \mu(k_2)^{\frac{1}{2}}} \right\} \cdot \{a^*(k_1) a^*(-k_2) + a(-k_1) a(k_2)\}\, dk_1\, dk_2 . \tag{3.2.11}$$

Similarly, for the local momentum we write

$$P(g) = P^{(1)}(g) + P^{(2)}(g) ,$$

where

$$P^{(1)}(g) = \frac{1}{4\pi} \int \tilde{g}(k_1 - k_2) \left\{ \frac{k_1 \mu(k_2) + k_2 \mu(k_1)}{\mu(k_1)^{\frac{1}{2}} \mu(k_2)^{\frac{1}{2}}} \right\} \cdot a^*(k_1) a(k_2) dk_1 dk_2 \tag{3.2.12}$$

and

$$P^{(2)}(g) = \frac{1}{8\pi} \int \tilde{g}(k_1 - k_2) \left\{ \frac{k_1 \mu(k_2) - k_2 \mu(k_1)}{\mu(k_1)^{\frac{1}{2}} \mu(k_2)^{\frac{1}{2}}} \right\} \cdot \{ -a^*(k_1) a^*(-k_2) + a(-k_1) a(k_2) \} \, dk_1 dk_2 \,. \tag{3.2.13}$$

Here $\tilde{g}(p) = \int e^{-ipx} g(x) \, dx$.

Theorem 3.2.1. *The bilinear forms $T_0(g)$ and $P(g)$ define symmetric operators on $\mathscr{D}(H_0)$. The following operators are all bounded*

$$T_0(g)(H_0 + I)^{-1}, \qquad P(g)(H_0 + I)^{-1}, \tag{3.2.14}$$

$$(I + H_0)^{-\frac{1}{2}} T_0(g)(I + H_0)^{-\frac{1}{2}}, \quad (I + H_0)^{-\frac{1}{2}} P(g)(I + H_0)^{-\frac{1}{2}}, \tag{3.2.15}$$

$$T_0^{(2)}(g)(I + N)^{-1}, \quad \text{and} \quad P^{(2)}(g)(I + N)^{-1}. \tag{3.2.16}$$

This theorem is an immediate consequence of the following two lemmas and of Lemmas 3.1.1–3.1.3.

Lemma 3.2.2. *The kernels of $T_0^{(2)}(g)$ and the kernels of $P^{(2)}(g)$ are $\mathscr{L}^2$ functions.*

Proof. We first note that

$$\mu(k_1)\mu(k_2) - k_1 k_2 = \tfrac{1}{2}(k_1 - k_2)^2 - \tfrac{1}{2}(\mu(k_1) - \mu(k_2))^2 + m^2 \leqq \tfrac{1}{2}(k_1 - k_2)^2 + m^2,$$

so

$$\mu(k_1)\mu(k_2) - k_1 k_2 \leqq \text{const}\, \mu(k_1 - k_2)^2. \tag{3.2.17}$$

Using (3.2.17) we can bound the kernel of $T_0^{(2)}(g)$ in (3.2.11) by

$$\text{const} \left| \tilde{g}(k_1 - k_2) \left\{ \frac{\mu(k_1)\mu(k_2) - k_1 k_2 - m^2}{\mu(k_1)^{\frac{1}{2}} \mu(k_2)^{\frac{1}{2}}} \right\} \right| \leqq \text{const}\, |\tilde{g}(k_1 - k_2)| \, \mu(k_1 - k_2)^2 \mu(k_1)^{-\frac{1}{2}} \mu(k_2)^{-\frac{1}{2}} \tag{3.2.18}$$

which is square integrable since $\tilde{g}$ is rapidly decreasing. Similarly, we bound the kernel of $P^{(2)}(g)$ by using the following inequalities:

$$|k_1 \mu(k_2) - k_2 \mu(k_1)| \leqq 2\mu(k_1)\mu(k_2) \leqq 2\mu(k_1 - k_2)^2, \quad \text{if} \quad k_1 k_2 < 0, \tag{3.2.19}$$

and

$$|k_1 \mu(k_2) - k_2 \mu(k_1)| \leqq \mu(k_1)\mu(k_2) - k_1 k_2, \quad \text{if} \quad k_1 k_2 \geqq 0. \tag{3.2.20}$$

The inequality (3.2.19) is clear, while from

$$|k_1\,\mu(k_2)| \leqq \mu(k_1)\,\mu(k_2)\,,$$

and

$$|k_2\,\mu(k_1)| \geqq |k_1 k_2|\,,$$

we derive (3.2.20) when $|k_1\,\mu(k_2)| > |k_2\,\mu(k_1)|$, and by symmetry it is valid in general. Thus by (3.2.17) and (3.2.19–3.2.20),

$$|k_1\,\mu(k_2) - k_2\,\mu(k_1)| \leqq \text{const}\,\mu(k_1 - k_2)^2\,. \tag{3.2.21}$$

Hence the kernel of $P^{(2)}(g)$ in (3.2.13) is bounded by

$$\text{const}\left|\tilde{g}(k_1 - k_2)\left\{\frac{k_1\,\mu(k_2) - k_2\,\mu(k_1)}{\mu(k_1)^{\frac{1}{2}}\,\mu(k_2)^{\frac{1}{2}}}\right\}\right|$$
$$\leqq \text{const}\,|\tilde{g}(k_1 - k_2)|\,\mu(k_1 - k_2)^2 \mu(k_1)^{-\frac{1}{2}}\,\mu(k_2)^{-\frac{1}{2}}\,,$$

which is square integrable, as is (3.2.18).

Lemma 3.2.3. *The kernel of* $T_0^{(1)}(g)$ *and the kernel of* $P^{(1)}(g)$ *have finite* $M_1(\tau)$, $M_2(\tau)$ *defined in* (3.1.4)–(3.1.5) *for* $\tau \geqq 1$.

Proof. Both the kernel of $T_0^{(1)}(g)$ and the kernel of $P^{(1)}(g)$ are dominated by

$$\text{const}\,|\tilde{g}(k_1 - k_2)|\,\mu(k_1)^{\frac{1}{2}}\,\mu(k_2)^{\frac{1}{2}}\,.$$

Thus

$$M_1(1) \leqq \text{const}\,\sup_k \mu(k)^{-1} \int |\tilde{g}(k-p)|\,\mu(k)^{\frac{1}{2}}\,\mu(p)^{\frac{1}{2}}\,dp\,.$$

Since $\mu(p)^{\frac{1}{2}} \leqq \text{const}\,\mu(k-p)^{\frac{1}{2}}\,\mu(k)^{\frac{1}{2}}$ and $\tilde{g}$ is rapidly decreasing,

$$\begin{aligned} M_1(1) &\leqq \text{const}\,\sup_k \int |\tilde{g}(k-p)|\,\mu(k-p)^{\frac{1}{2}}\,dp \\ &\leqq \text{const}\,. \end{aligned} \tag{3.2.22}$$

Similarly, $M_2(\tau)$ is finite for $\tau \geqq 1$. This completes the proof of the lemma and the proof of Theorem 3.2.1.

For the remainder of this section we define a momentum cutoff operator $T_{0\kappa}$, and we establish properties of $T_{0\kappa}$ that will be useful later. It is convenient to assume that

$$g(\cdot) = h^2(\cdot)\,,\ h \geqq 0\,,\ h \in \mathscr{S}(\boldsymbol{R}^1)\,, \tag{3.2.23}$$

and we do this in the following. We will use the cutoff function

$$G_\kappa(k_1, k_2) = \frac{1}{2\pi} \int_{|p| \leqq \kappa} \overline{\tilde{h}(p - k_1)}\,\tilde{h}(p - k_2)\,dp \tag{3.2.24}$$

For $\kappa < \infty$, $G_\kappa(k_1, k_2) \in \mathscr{S}(\boldsymbol{R}^2)$, and

$$G_\infty(k_1, k_2) = \tilde{g}(k_1 - k_2)\,. \tag{3.2.25}$$

We define

$$T_{0\kappa}(g) = T_{0\kappa}^{(1)}(g) + T_{0\kappa}^{(2)}(g) \tag{3.2.26}$$

by replacing $\tilde{g}(k_1 - k_2)$ in the kernels of $T_0^{(i)}(g)$ defined in (3.2.10)–(3.2.11) by $G_\kappa(k_1, k_2)$. If $\kappa < \infty$, then the $T_{0\kappa}^{(i)}(g)$ have $\mathscr{L}^2$ kernels and so $T_{0\kappa}(g)$ is essentially self adjoint on $\mathscr{D}(N)$, since vectors with a finite number of particles are analytic vectors. We write

$$T_0(g) = T_{0\kappa}(g) + \delta T_{0\kappa}(g), \tag{3.2.27}$$

defining $\delta T_{0\kappa}$, and similarly we define $\delta T_{0\kappa}^{(i)}(g)$.

Theorem 3.2.4. *a) The bounded operators*

$$\delta T_{0\kappa}^{(1)}(I + H_0)^{-1} \quad \text{and} \quad (I + H_0)^{-\frac{1}{2}} \delta T_{0\kappa}^{(1)}(I + H_0)^{-\frac{1}{2}} \tag{3.2.28}$$

converge strongly to zero as $\kappa \to \infty$.

b) The kernel of $\delta T_{0\kappa}^{(2)}(g)$ *has an* $\mathscr{L}^2$ *norm that is* $O(\kappa^{-\varepsilon})$ *for any* $\varepsilon < \frac{1}{2}$. *Thus*

$$\|\delta T_{0\kappa}^{(2)}(g)(I + N)^{-1}\| \leqq O(\kappa^{-\varepsilon}), \ \varepsilon < \tfrac{1}{2}. \tag{3.2.29}$$

c) As $\kappa \to \infty$,

$$\|(I + H_0)^{-1} \delta T_{0\kappa}(g)(I + H_0)^{-1}\| \leqq O(\kappa^{-1}). \tag{3.2.30}$$

Proof. a) We note that the kernel of $\delta T_{0\kappa}^{(1)}$ has bounded norms (3.1.4)–(3.1.5) for $\tau = 1$, and these bounds are uniform for $\kappa \leqq \infty$. Thus the operators (3.2.28) are uniformly bounded, and it is sufficient to prove convergence on a total set of vectors, namely vectors in $\mathscr{D}(H_0)$ with exactly n particles. It is sufficient to prove the strong convergence of $\delta T_{0\kappa}^{(1)}$ on this domain. For ψ an n particle vector in $\mathscr{D}(H_0)$,

$$\begin{aligned}
&|(\delta T_{0\kappa}^{(1)}\psi)(k_1, \ldots, k_n)|^2 \\
&= \Bigg| \sum_{j=1}^{n} \int_{|p|>k} dp \int dq \, \bar{\tilde{h}}(p - k_j) \tilde{h}(p - q) \left\{ \frac{\mu(k_j)\mu(q) + k_j q + m^2}{\mu(k_j)^{\frac{1}{2}} \mu(q)^{\frac{1}{2}}} \right\} \\
&\quad \cdot \psi(k_1, \ldots, k_{j-1}, q, k_{j+1}, \ldots, k_n) \Bigg|^2 \\
&\leqq \text{const} \Bigg\{ \sum_{j=1}^{n} \int_{|p|>\kappa} dp \int dq \, |\tilde{h}(p - k_j) \tilde{h}(p - q)| \\
&\quad \cdot \mu(k_j)^{\frac{1}{2}} \mu(q)^{\frac{1}{2}} |\psi(k_1, \ldots, q, \ldots, k_n)| \Bigg\}^2 .
\end{aligned} \tag{3.2.31}$$

The right side of (3.2.31) is monotonically decreasing as $\kappa \to \infty$. Since

$$\mu(q)^{\frac{1}{2}} \leqq \text{const} \, \mu(p - k)^{\frac{1}{2}} \mu(p - q)^{\frac{1}{2}} \mu(k)^{\frac{1}{2}},$$

and since $\tilde{h}$ is rapidly decreasing,

$$w(k, q) = \int dp \, |\tilde{h}(p - k) \tilde{h}(p - q)| \, \mu(k)^{\frac{1}{2}} \mu(q)^{\frac{1}{2}}$$

is a kernel with finite norms (3.1.4)–(3.1.5) for $\tau=1$. But the right side of (3.2.31) has the form $(W|\psi|)^2$, where $|\psi|\in\mathscr{D}(H_0)$ since ψ is. Hence by Lemma 3.1.2, the function $W|\psi|$ is $\mathscr{L}^2$ so that (3.2.31) is uniformly bounded by an $\mathscr{L}^1$ function. By the dominated convergence theorem, the integral of (3.2.31) tends to zero as $\kappa\to\infty$, which completes the proof of strong convergence.

b) The kernel of $\delta T_{0\kappa}^{(2)}(g)$ is bounded by

$$w(k,p)=\text{const}\,\mu(k-p)^2\,\mu(k)^{-\frac{1}{2}}\mu(p)^{-\frac{1}{2}}\int_{|q|>\kappa}|\tilde{h}(q-k)\,\tilde{h}(q-p)|\,dq\,. \tag{3.2.32}$$

By (3.2.20) we write

$$\mu(k)^{-\varepsilon}\leqq\text{const}\,\mu(p)^{-\varepsilon}\mu(k-p)^{\varepsilon},$$

$$\mu(k-p)^{3+\varepsilon}\leqq\text{const}\,\mu(q-k)^{3+\varepsilon}\mu(q-p)^{3+\varepsilon},$$

$$\mu(k)^{-\frac{1}{2}+\varepsilon}\leqq\text{const}\,\mu(q)^{-\frac{1}{2}+\varepsilon}\mu(q-k)^{\frac{1}{2}-\varepsilon}.$$

Then

$$\begin{aligned}|w(k,p)|&\leqq\text{const}\,\mu(k-p)^{-1}\mu(p)^{-\frac{1}{2}-\varepsilon}\\&\quad\cdot\int_{|q|>\kappa}dq\,|\tilde{h}(q-k)\,\tilde{h}(q-p)|\,\mu(q-k)^{\frac{7}{2}}\mu(q-p)^{3+\varepsilon}\mu(q)^{-\frac{1}{2}+\varepsilon}\\&\leqq\text{const}\,\mu(\kappa)^{-\frac{1}{2}+\varepsilon}\mu(k-p)^{-1}\mu(p)^{-\frac{1}{2}-\varepsilon}\\&\quad\cdot\int_{|q|>\kappa}dq\,|\tilde{h}(q-k)\,\tilde{h}(q-p)|\,\mu(q-k)^{\frac{7}{2}}\mu(q-p)^{3+\varepsilon}.\end{aligned}$$

We can now use the Schwarz inequality in q and the rapid decrease of $\tilde{h}$ to bound the integral over q by a constant. Thus

$$|w(k,p)|\leqq\text{const}\,O(\kappa^{-\frac{1}{2}+\varepsilon})\,\mu(k-p)^{-1}\mu(p)^{-\frac{1}{2}-\varepsilon},$$

which is $\mathscr{L}^2$ for any $\varepsilon>0$, and has an $\mathscr{L}^2$ norm that is $O(\kappa^{-\frac{1}{2}+\varepsilon})$ for any $\varepsilon>0$. This proves statement b of the theorem.

c) The proof of this estimate is carried out by estimates on the kernels of $\delta T_{0\kappa}^{(1)}(g)$ and $\delta T_{0\kappa}^{(2)}(g)$. The estimate on the kernel of $\delta T_{0\kappa}^{(2)}(g)$ is similar to the above, but we estimate the $\mathscr{L}^2$ norm of $w(k,p)\,\mu(k)^{-\frac{1}{2}}\mu(p)^{-\frac{1}{2}}$ for the w of (3.2.32). We then get an $\mathscr{L}^2$ norm that is $O(\kappa^{-\frac{3}{2}+\varepsilon})$, and Lemma 3.1.3 in the case $\alpha=2,\beta=0,\tau=1$ for the creation part or $\alpha=0,\beta=2,\tau=1$ for the annihilation part yields

$$\|(I+H_0)^{-1}\,\delta T_{0\kappa}^{(2)}(g)(I+H_0)^{-1}\|\leqq O(\kappa^{-\frac{3}{2}+\varepsilon})\,. \tag{3.2.33}$$

The estimate on the kernel of $\delta T_{0\kappa}^{(1)}(g)$ will be made with the norm (3.1.4). We find that $M_1(\tau=2)$ is $O(\kappa^{-1})$, so that by Lemma 3.1.1, and remark b following it,

$$\|H_0^{-1}\,\delta T_{0\kappa}^{(1)}(g)\,H_0^{-1}\|\leqq O(\kappa^{-1})\,. \tag{3.2.34}$$

We now prove this estimate on the kernel of $\delta T_{0\kappa}^{(1)}$. The kernel of $\delta T_{0\kappa}^{(1)}(g)$ is dominated by

$$w_\kappa(k,q) = \text{const}\,\mu(k)^{\frac{1}{2}}\mu(q)^{\frac{1}{2}} \int\limits_{|p|>\kappa} |\tilde{h}(p-k)\,\tilde{h}(p-q)|\,dp\,,$$

and

$$\begin{aligned}\int w_\kappa(k,q)\,dq &\leqq \text{const}\,\mu(k)^2 \int\limits_{|p|>\kappa} |\tilde{h}(p-k)\,\tilde{h}(p-q)| \\ &\quad\cdot \mu(p-k)^{\frac{3}{2}}\,\mu(p-q)^{\frac{1}{2}}\,\mu(p)^{-1}\,dp\,dq \\ &\leqq \text{const}\,\mu(\kappa)^{-1}\,\mu(k)^2 \int\limits_{|p|>\kappa} |\tilde{h}(p-k)\,\tilde{h}(p-q)| \\ &\quad\cdot \mu(p-k)^{\frac{3}{2}}\,\mu(p-q)^{\frac{1}{2}}\,dp\,dq\end{aligned}$$

or by the Schwarz inequality

$$\int w_\kappa(k,q)\,dq \leqq \text{const}\,\mu(\kappa)^{-1}\,\mu(k)^2\,.$$

Thus

$$\sup_k \mu(k)^{-2} \int w_k(k,q)\,dq \leqq O(\kappa^{-1})\,,$$

which completes the proof of (3.2.34) and the proof of the theorem.

It is convenient to write $T_{0\kappa}(g)$ and $\delta T_{0\kappa}(g)$ in another form. We define the following operators with $\mathscr{L}^2$ kernels on the domain $\mathscr{D}(N^{\frac{1}{2}})$.

$$B_1(p) = \frac{1}{2\pi} \int \tilde{h}(p-k)\,\mu(k)^{\frac{1}{2}}\,a(k)\,dk\,, \tag{3.2.35}$$

$$B_2(p) = \frac{1}{2\pi} \int \tilde{h}(p-k)\,k\,\mu(k)^{-\frac{1}{2}}\,a(k)\,dk\,, \tag{3.2.36}$$

$$B_3(p) = \frac{1}{2\pi} \int \tilde{h}(p-k)\,m\,\mu(k)^{-\frac{1}{2}}\,a(k)\,dk\,. \tag{3.2.37}$$

Then for $g = h^2$, and $\kappa \leqq \infty$, on the domain $\mathscr{D}(H_0)$,

$$T_{0\kappa}^{(1)}(g) = \frac{1}{2} \int\limits_{-\kappa}^{\kappa} \sum_{i=1}^{3} B_i(p)^*\,B_i(p)\,dp\,, \tag{3.2.38}$$

and

$$\delta T_{0\kappa}^{(1)}(g) = \frac{1}{2} \int\limits_{|p|\geqq\kappa} \sum_{i=1}^{3} B_i(p)^*\,B_i(p)\,dp\,. \tag{3.2.39}$$

We also define

$$A_i(p) = \frac{1}{2}\left\{B_i(p) + B_i(-p)^*\right\}, \qquad i = 1, 2, 3, \tag{3.2.40}$$

which are also operators on $\mathscr{D}(N^{\frac{1}{2}})$. Note that

$$[A_i(p),\ A_i(p)^*]\,\mathscr{D}(N) = 0\,. \tag{3.2.41}$$

The operators $A_i(p)$ are related to the operator $T_{0\kappa}(g)$ without Wick ordering. For $\kappa < \infty$, define

$$\hat{T}_{0\kappa}(g) = \sum_{i=1}^{3} \int_{-\kappa}^{\kappa} A_i(p)^* A_i(p)\, dp \geqq 0 \,. \tag{3.2.42}$$

An easy calculation shows that

$$T_{0\kappa}(g) = \hat{T}_{0\kappa}(g) - (\Omega_0, \hat{T}_{0\kappa}(g)\, \Omega_0) \tag{3.2.43}$$

where Ω_0 is the no-particle vector. Since

$$(\Omega_0, \hat{T}_{0\kappa}(g)\, \Omega_0) = \int G_\kappa(p, p)\, \mu(p)\, dp \,, \tag{3.2.44}$$

where G_κ is defined in (3.2.24), we have for $\kappa < \infty$ that $T_{0\kappa}$ is bounded from below and

$$T_{0\kappa} + \int G_\kappa(p, p)\, \mu(p)\, dp \geqq 0 \,. \tag{3.2.45}$$

Theorem 3.2.5. *Let $\varepsilon > 0$ and g, g_1 be positive as above. Then there is a finite constant b such that on $\mathscr{D}(H_0^2) \times \mathscr{D}(H_0^2)$,*

$$\delta T_{0\kappa}^{(1)}(g) \geqq 0 \,, \quad \textit{for all} \quad 0 \leqq \kappa \leqq \infty \,, \tag{3.2.46}$$

$$\varepsilon N + T_0(g) + b \geqq 0 \,, \tag{3.2.47}$$

$$\varepsilon N + T_{\mathrm{I}}(g) + b \geqq 0 \,, \tag{3.2.48}$$

and

$$\varepsilon N + T_0(g) + T_{\mathrm{I}}(g_1) + b \geqq 0 \,. \tag{3.2.49}$$

Of course these inequalities are also valid with H_0 in place of N.

Proof. The positivity of $\delta T_{0\kappa}^{(1)}(g)$ is a consequence of the representation (3.2.38). In order to prove (3.2.47) we write

$$\varepsilon N + T_0(g) = \varepsilon N + \delta T_{0\kappa}^{(2)}(g) + \delta T_{0\kappa}^{(1)}(g) + T_{0\kappa}(g) \,.$$

Since $\delta T_{0\kappa}^{(1)}$ is positive by (3.2.46) and $T_{0\kappa}(g)$ is bounded from below by (3.2.45), we need only prove that $\varepsilon N + \delta T_{0\kappa}^{(2)}(g)$ is bounded from below. By Theorem 3.2.4b, the $\mathscr{L}^2$ norm of the kernel of $\delta T_{0\kappa}^{(2)}$ is $O(\kappa^{-\frac{1}{4}})$. Thus

$$\|(I + N)^{-\frac{1}{2}}\, T_{0\kappa}^{(2)}(g)(I + N)^{-\frac{1}{2}}\| \leqq O(\kappa^{-\frac{1}{4}}) \tag{3.2.50}$$

and for sufficiently large κ, (3.2.50) is less than ε. Hence $\varepsilon N + \delta T_{0\kappa}^{(2)}(g) + \varepsilon \geqq 0$, and (3.2.47) is established.

The bound (3.2.48) is the semiboundedness result of Nelson [7] and Glimm [8]. Although they proved that $\varepsilon H_0 + T_{\mathrm{I}}(g_1)$ is also bounded from below, the same proof can be carried through with N replacing H_0. Combining (3.2.47) with (3.2.48) yields (3.2.49).

4. Second Order Estimates

In Section 3 we proved mainly linear estimates on the operators $T(g)$ and $P(g)$, the local energy and momentum. In this section we prove estimates that are quadratic in $T(g)$. These estimates give us better control over $T(g)$ and are an essential ingredient for the self adjointness proof of the next section.

The main result of this section is a second order estimate on operators of the form

$$H_0 + T_0(g_0) + T_{\mathrm{I}}(g_1) \tag{4.1}$$

where g_0 and g_1 are spatial cutoffs satisfying (3.2.23). For $g_0 = 0$, such an estimate was proved in [1], see also [2, Proposition 1].

Theorem 4.1. *Let $c > 1$. Then there is a constant $b < \infty$ such that for all $0 \leqq \beta \leqq 1$,*

$$(H_0 + I)^2 + \beta^2 T_0(g_0)^2 + T_{\mathrm{I}}(g_1)^2 \leqq c(H_0 + \beta T_0(g_0) + T_{\mathrm{I}}(g_1) + b)^2, \tag{4.2}$$

as a bilinear form on $\mathscr{D}(H_0^2) \times \mathscr{D}(H_0^2)$.

We remark that each operator H_0, $T_0(g_0)$, and $T_{\mathrm{I}}(g_1)$ is defined on $\mathscr{D}(H_0^2)$.

Lemma 4.2. *Let $c > 1$ and $\varepsilon > 0$. Then there is a constant $b < \infty$ such that*

$$T_0(g) H_0 + H_0 T_0(g) \geqq -\varepsilon H_0^2 - b \tag{4.3}$$

and for all $0 \leqq \beta \leqq 1$,

$$(H_0 + I)^2 + \beta^2 T_0(g)^2 \leqq c(H_0 + \beta T_0(g) + b)^2 \tag{4.4}$$

as bilinear forms on $\mathscr{D}(H_0) \times \mathscr{D}(H_0)$.

Proof. We expand $(H_0 + \beta T_0(g) + b)^2$ to find

$$\begin{aligned}(H_0 + \beta T_0(g) + b)^2 = {} & (H_0 + I)^2 + \beta^2 T_0(g)^2 \\ & + 2(b-1)(H_0 + I + \beta_1 T_0(g) + \tfrac{1}{4}(b-1)) \\ & + \beta(H_0 T_0(g) + T_0(g) H_0) + \tfrac{1}{2}(b-1)^2,\end{aligned} \tag{4.5}$$

where $\beta_1 = \beta b (b-1)^{-1}$. For b sufficiently large, $H_0 + \beta_1 T_0(g) + b/4$ is positive, for the proof of Theorem 3.2.5 gives an estimate that is uniform for $0 \leqq \beta_1 \leqq 2$. Hence it is sufficient to prove (4.3) to establish (4.4), for if

$$H_0 T_0(g) + T_0(g) H_0 \geqq -4\varepsilon H_0^2 - \gamma, \tag{4.6}$$

we choose ε and b so that

$$4\varepsilon < 1 \quad \text{and} \quad \tfrac{1}{2} b^2 > \gamma + 1 .$$

We write

$$T_0 = T_{0\kappa} + \delta T_{0\kappa}^{(2)} + \delta T_{0\kappa}^{(1)}, \tag{4.7}$$

and we prove (4.6) separately for each term in (4.7).

Using (3.2.40)–(3.2.45) we write

$$\begin{aligned} H_0 T_{0\kappa} + T_{0\kappa} H_0 &= -2H_0 \int G_\kappa(k,k)\mu(k)\,dk + H_0 \hat{T}_{0\kappa} + \hat{T}_{0\kappa} H_0 \\ &\geqq -\varepsilon H_0^2 - \text{const} + H_0 \hat{T}_{0\kappa} + \hat{T}_{0\kappa} H_0 \\ &= -\varepsilon H_0^2 - \text{const} + 2\sum_{i=1}^{3} \int_{-\kappa}^{\kappa} A_i(p)^* H_0 A_i(p)\,dp \\ &\quad + \sum_{i=1}^{3} \int_{-\kappa}^{\kappa} \{[H_0, A_i(p)^*] A_i(p) + A_i(p)^* [A_i(p), H_0]\}\,dp\,. \end{aligned} \tag{4.8}$$

Note that the kernels occurring in $A_i(p), A_i(p)^*, [H_0, A_i(p)]$ and $[H_0, A_i(p)^*]$ all belong to $\mathscr{S}(\boldsymbol{R}^1)$ for fixed p. The $\mathscr{L}^2$ norms of these kernels are uniformly bounded on compact intervals in p. Thus each of these operators is defined on $\mathscr{D}(N^{\frac{1}{2}})$ and maps $\mathscr{D}(H_0)$ into $\mathscr{D}(H_0^{\frac{1}{2}})$. As a consequence, each term in (4.8) is defined. Since

$$\int_{-\kappa}^{\kappa} A_i(p)^* H_0 A_i(p)\,dp \geqq 0\,, \tag{4.9}$$

we need only bound the commutator terms. By the above remarks on the $\mathscr{L}^2$ nature of the kernels, the operator

$$(I+H_0)^{-\frac{1}{2}} \int_{-\kappa}^{\kappa} \{[H_0, A_i(p)^*] A_i(p) + A_i(p)^* [A_i(p), H_0]\}\,dp (I+H_0)^{-\frac{1}{2}}$$

is bounded for any $\kappa < \infty$, so that

$$\begin{aligned} \sum_{i=1}^{3} \int_{-\kappa}^{\kappa} \{[H_0, A_i(p)^*] A_i(p) + A_i(p)^* [A_i(p), H_0]\}\,dp \\ \geqq -\text{const}(H_0 + I) \\ \geqq -\varepsilon H_0^2 - \text{const}\,. \end{aligned} \tag{4.10}$$

Thus by (4.8)–(4.10)

$$H_0 T_{0\kappa} + T_{0\kappa} H_0 \geqq -2\varepsilon H_0^2 - \text{const}\,, \tag{4.11}$$

which is the contribution of $T_{0\kappa}$ to (4.6).

By Theorem 3.2.4b, the kernel of $\delta T_{0\kappa}^{(2)}(g)$ has an $\mathscr{L}^2$ norm that is $O(\kappa^{-\frac{1}{4}})$. Hence

$$\|(I+H_0)^{-1}(H_0 \delta T_{0\kappa}^{(2)} + \delta T_{0\kappa}^{(2)} H_0)(I+H_0)^{-1}\| \leqq O(\kappa^{-\frac{1}{4}})$$

and for sufficiently large κ,

$$H_0 \delta T_{0\kappa}^{(2)} + \delta T_{0\kappa}^{(2)} H_0 \geqq -\varepsilon(H_0^2 + I)\,,$$

which is the contribution of $\delta T_{0\kappa}^{(2)}$ to (4.6).

Finally, for $\delta T^{(1)}_{0\kappa}$ we write

$$H_0 \delta T^{(1)}_{0\kappa} + \delta T^{(1)}_{0\kappa} H_0 = 2H_0^{\frac{1}{2}} \delta T^{(1)}_{0\kappa} H_0^{\frac{1}{2}} + [H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, \delta T^{(1)}_{0\kappa}]] . \tag{4.12}$$

By Theorem 3.2.5, the first term on the right of (4.12) is positive, and we now study the double commutator. Since neither $\delta T^{(1)}_{0\kappa}$ nor $H_0^{\frac{1}{2}}$ changes the particle number, we restrict attention to vectors $\psi \in \mathscr{D}(H_0^2)$ with exactly n particles. Let $\delta t(k_1, k_2)$ be the kernel of $\delta T^{(1)}_{0\kappa}(g)$.

Then

$$\begin{aligned}(\psi, [H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, \delta T^{(1)}_{0\kappa}]]\, \psi) \\ = n \int \overline{\psi}(k_1, \ldots, k_n)\psi(p, k_2, \ldots, k_n)\delta t(k_1, p)\lambda(p, k_1, \ldots, k_n) dp\, dk_1 \ldots dk_n,\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\lambda(p, k_1, k_2, \ldots, k_n) \\ = \left\{\left(\sum_{i=1}^{n} \mu(k_i)\right)^{\frac{1}{2}} - \left(\mu(p) + \sum_{i=2}^{n} \mu(k_i)\right)^{\frac{1}{2}}\right\}^2 \\ = \left(\sum_{i=1}^{n} \mu(k_i)\right) \left\{\left(1 + \frac{\mu(p) - \mu(k_1)}{\left(\sum_{i=1}^{n} \mu(k_i)\right)^{\frac{1}{2}}}\right)^{\frac{1}{2}} - 1\right\}^2 .\end{aligned} \tag{4.14}$$

If $\mu(p) - \mu(k_1) \geqq 0$, we use the inequality for $x \geqq 0$

$$(1 + x)^{\frac{1}{2}} - 1 \leqq \tfrac{1}{2} x$$

to prove

$$\lambda(p, k_1, \ldots, k_n) \leqq \tfrac{1}{4} (\mu(p) - \mu(k_1))^2 . \tag{4.15}$$

Since $\lambda(p, k_1, \ldots, k_n) = \lambda(k_1, p, \ldots, k_n)$, the bound (4.15) is valid for all $p, k_1, \ldots, k_n$. Since

$$|\mu(p) - \mu(k_1)| \leqq \text{const}\, \mu(p - k_1) ,$$

we have

$$\lambda(p, k_1, \ldots, k_n) \leqq \text{const}\, \mu(p - k_1)^2 . \tag{4.16}$$

Suppressing the variables $k_2, \ldots, k_n$ in (4.13) we have by (4.16), the Schwarz inequality, and the symmetry of $|w(k, p)|$,

$$\begin{aligned}|(\psi, [H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, \delta T^{(1)}_{0\kappa}]]\, \psi)| \\ \leqq \text{const}\, n \int |\psi(k_1)^2\, \delta t(k_1, p)|\, \mu(k_1 - p)^2\, dp\, dk_1 .\end{aligned}$$

As in the proof of Theorem 3.2.4, the kernel $\delta t(k_1, p)$ is dominated by

$$\text{const}\, |\tilde{g}(k_1 - p)|\, \mu(k_1)^{\frac{1}{2}}\, \mu(p)^{\frac{1}{2}} .$$

Thus we have the $\mathscr{L}^1 - \mathscr{L}^\infty$ estimate

$$\int |\delta t(k_1, p)|\, \mu(k_1 - p)^2\, dp \leqq \text{const}\, \mu(k_1) ,$$

and so, by Lemma 3.1.1,

$$\begin{aligned}|(\psi,[H_0^{\frac{1}{2}},[H_0^{\frac{1}{2}},\delta T_{0\kappa}^{(1)}(g)]]\psi)| &\leqq \text{const}\, n \int |\psi(k_1)^2|\,\mu(k_1)\,dk_1 \\ &= \text{const}(\psi, H_0\psi) \\ &\leqq (\psi,(\varepsilon H_0^2+\text{const})\psi)\,.\end{aligned}$$

Thus for (4.12),

$$\begin{aligned}H_0\delta T_{0\kappa}^{(1)}+\delta T_{0\kappa}^{(1)}H_0 &\geqq 2H_0^{\frac{1}{2}}\delta T_{0\kappa}^{(1)}H_0^{\frac{1}{2}}-\varepsilon H_0^2-\text{const} \\ &\geqq -\varepsilon H_0^2-\text{const}\,.\end{aligned}$$

This establishes (4.6) as an inequality on $D(H_0^2)\times\mathscr{D}(H_0^2)$. It extends by closure to $\mathscr{D}(H_0)\times\mathscr{D}(H_0)$, and this completes the proof of the lemma.

We remark that these methods can be used to prove that

$$W(\tau,n)=(\text{ad}\,H_0^\tau)^n(T_0^{(1)}(g)),\qquad \tau\leqq 1,\ n=1,2,3,\ldots,$$

is an operator on $\mathscr{D}(H_0)$, and that $W(\tau,n)H_0^{-1}$ is bounded.

Lemma 4.3. *Let $\varepsilon>0$ and $\kappa<\infty$. Then there exists a constant $b<\infty$ such that on $\mathscr{D}(H_0^2)\times\mathscr{D}(H_0^2)$,*

$$T_{\mathrm{I}}T_{0\kappa}+T_{0\kappa}T_{\mathrm{I}}\geqq-\varepsilon(H_0^2+T_{\mathrm{I}}^2)-b\,. \tag{4.17}$$

Proof. Using (3.2.40)–(3.2.45), we have the identity

$$\begin{aligned}T_{\mathrm{I}}T_{0\kappa}+T_{0\kappa}T_{\mathrm{I}} &= -\text{const}\,T_{\mathrm{I}}+T_{\mathrm{I}}\hat{T}_{0\kappa}+\hat{T}_{0\kappa}T_{\mathrm{I}} \\ &= -\text{const}\,T_{\mathrm{I}}+\sum_{i=1}^{3}\int_{-\kappa}^{\kappa}\{A_i(p)^*T_{\mathrm{I}}A_i(p)+A_i(p)T_{\mathrm{I}}A_i(p)^*\}\,dp \\ &\quad+\frac{1}{2}\sum_{i=1}^{3}\int_{-\kappa}^{\kappa}\{[A_i(p),[A_i(p)^*,T_{\mathrm{I}}]] \\ &\quad+[A_i(p)^*,[A_i(p),T_{\mathrm{I}}]]\}\,dp\,,\end{aligned}\tag{4.18}$$

which follows from

$$\begin{aligned}B(AA^*+A^*A)+(AA^*+A^*A)B &= 2ABA^*+2A^*BA+[A,[A^*,B]] \\ &\quad+[A^*,[A,B]]\,.\end{aligned}$$

We give a lower bound on each term on the right side of (4.18). Clearly for any $\varepsilon_1>0$,

$$-\text{const}\,T_{\mathrm{I}}\geqq-\varepsilon_1T_{\mathrm{I}}^2-\text{const}\,.$$

Furthermore, by (3.2.48), for $\varepsilon_2>0$,

$$\begin{aligned}&A_i(p)^*T_{\mathrm{I}}A_i(p)+A_i(p)T_{\mathrm{I}}A_i(p)^* \\ &\quad\geqq-\text{const}\,A_i(p)^*A_i(p)-\varepsilon_2\{A_i(p)^*NA_i(p)+A_i(p)NA_i(p)^*\}\,.\end{aligned}\tag{4.19}$$

By the remarks following (4.10) on the $\mathscr{L}^2$ nature of the kernels occurring in $A_i(p)$, we have for $|p| \leqq \kappa < \infty$, and any $\varepsilon_3 > 0$,

$$-\operatorname{const} A_i(p)^* A_i(p) \geqq -\operatorname{const}(N+I) \geqq -\varepsilon_3 H_0^2 - \operatorname{const}, \tag{4.20}$$

and

$$\begin{aligned} -\varepsilon_2 \{A_i(p)^* N A_i(p) + A_i(p) N A_i(p)^*\} & \\ \geqq -\varepsilon_2 \operatorname{const}(N+I)^2 & \\ \geqq -\varepsilon_2 \operatorname{const}(H_0^2 + I)\,. & \end{aligned} \tag{4.21}$$

Thus we can choose $\varepsilon_1, \varepsilon_2, \varepsilon_3$ sufficiently small so that after summing (4.19)–(4.21) over i and integrating over $|p| \leqq \kappa$, we obtain for (4.18),

$$\begin{aligned} T_I T_{0\kappa} + T_{0\kappa} T_I \geqq & -\frac{1}{2}\varepsilon({H^2}_0 + T_I^2) - \operatorname{const} \\ & + \frac{1}{2}\sum_{i=1}^{3}\int_{-\kappa}^{\kappa} \{[A_i(p), [A_i(p)^*, T_{\mathrm{I}}]] \\ & + [A_i(p)^*, [A_i(p), T_{\mathrm{I}}]]\}\, dp\,. \end{aligned} \tag{4.22}$$

We note that $[A_i(p), [A_i(p)^*, T_{\mathrm{I}}]]$ and its adjoint are sums of second order monomials in creation and annihilation operators with $\mathscr{L}^2$ kernels that have uniformly bounded $\mathscr{L}^2$ norms for $|p| \leqq \kappa$. In this interval of p,

$$\begin{aligned} [A_i(p), [A_i(p)^*, T_{\mathrm{I}}]] + [A_i(p)^*, [A_i(p), T_{\mathrm{I}}]] & \\ \geqq -\operatorname{const}(N+I) & \\ \geqq -\varepsilon_1 N^2 - \operatorname{const}\,. & \end{aligned}$$

Thus by choosing ε_1 sufficiently small, we obtain from (4.22) the following:

$$T_{\mathrm{I}} T_{0\kappa} + T_{0\kappa} T_{\mathrm{I}} \geqq -\varepsilon(H_0^2 + T_{\mathrm{I}}^2) - \operatorname{const}, \tag{4.23}$$

which is the desired inequality (4.17) and completes the proof.

Lemma 4.4. *Given $\varepsilon > 0$ there exists a finite constant κ_0 such that for $\kappa > \kappa_0$*

$$T_{\mathrm{I}} \delta T_{0\kappa}^{(2)} + \delta T_{0\kappa}^{(2)} T_{\mathrm{I}} \geqq -\varepsilon(H_0^2 + T_{\mathrm{I}}^2 + I), \tag{4.24}$$

as bilinear forms on $\mathscr{D}(H_0^2) \times \mathscr{D}(H_0^2)$.

Proof. For any $\varepsilon > 0$

$$\begin{aligned} |(\psi, T_I \delta T_{0\kappa}^{(2)} \psi)| & \leqq \|T_{\mathrm{I}}\psi\|\ \|\delta T_{0\kappa}^{(2)}\psi\| \\ & \leqq \frac{1}{2}\varepsilon \|T_{\mathrm{I}}\psi\|^2 + \frac{1}{2\varepsilon}\|\delta T_{0\kappa}^{(2)}\psi\|^2\,. \end{aligned} \tag{4.25}$$

By Theorem 3.2.4b, $\delta T_{0\kappa}^{(2)}$ has an $\mathscr{L}^2$ kernel with norm $O(\kappa^{-\frac{1}{4}})$. Thus for fixed $\varepsilon > 0$,

$$\frac{1}{2\varepsilon}\|\delta T_{0\kappa}^{(2)}\psi\|^2 \leqq o(1)\|(N+I)\psi\|^2 = o(1)\|(H_0+I)\psi\|^2 \leqq \frac{1}{2}\varepsilon(\|H_0\psi\|^2 + \|\psi\|^2)$$

for $\kappa > \kappa_0 = \kappa_0(\varepsilon)$. Thus for $\kappa > \kappa_0$,

$$T_{\mathrm{I}}\delta T_{0\kappa}^{(2)} + \delta T_{0\kappa}^{(2)} T_{\mathrm{I}} \geqq -\varepsilon(H_0^2 + T_{\mathrm{I}}^2) - \varepsilon,$$

which completes the proof.

Lemma 4.5. *Given any $\varepsilon > 0$, there is a constant $\kappa_0 < \infty$ such that for $\kappa > \kappa_0$*

$$T_{\mathrm{I}}\delta T_{0\kappa}^{(1)} + \delta T_{0\kappa}^{(1)} T_{\mathrm{I}} \geqq -\varepsilon(H_0^2 + I), \tag{4.26}$$

as bilinear forms on $\mathscr{D}(H_0^2) \times \mathscr{D}(H_0^2)$.

Proof. We consider $\delta T_{0\kappa}^{(1)}$ as (3.2.39) and write

$$\begin{aligned} T_{\mathrm{I}}\delta T_{0\kappa}^{(1)} + \delta T_{0\kappa}^{(1)} T_{\mathrm{I}} = &\int_{|p|>\kappa} B_i(p)^* T_{\mathrm{I}} B_i(p)\, dp \\ &+ \frac{1}{2}\sum_{i=1}^{3}\int_{|p|>\kappa} \{[T_{\mathrm{I}}, B_i(p)^*] B_i(p) \\ &+ B_i(p)^* [B_i(p), T_{\mathrm{I}}]\}\, dp. \end{aligned} \tag{4.27}$$

The integrals over p in (4.27) are absolutely convergent as weak integrals of bilinear forms on $\mathscr{D}(H_0^2) \times \mathscr{D}(H_0^2)$. We note that for any $\varepsilon_1 > 0$,

$$\begin{aligned} &\sum_{i=1}^{3}\int_{|p|>\kappa} B_i(p)^* T_{\mathrm{I}} B_i(p)\, dp \\ &\quad \geqq -\varepsilon_1 \sum_{i=1}^{3}\int_{|p|>\kappa} B_i(p)^* N B_i(p)\, dp - b\,\delta T_{0\kappa}^{(1)}, \end{aligned} \tag{4.28}$$

using (3.2.48). By Theorem 3.2.4c,

$$-b\,\delta T_{0\kappa}^{(1)} \geqq -O(\kappa^{-1})(H_0+I)^2 \geqq -\varepsilon_2(H_0^2+I), \tag{4.29}$$

for κ sufficiently large. Since the right side of (4.28) commutes with the projection onto vectors with n particles, it is sufficient to bound it below on such vectors. By Theorem 3.2.1, or Lemma 3.2.3,

$$\begin{aligned} &\sum_{i=1}^{3}\int_{|p|>\kappa} (\psi, B_i(p)^* N B_i(p)\psi)\, dp \\ &\quad = 2(n-1)(\psi, \delta T_{0\kappa}^{(1)}\psi) \\ &\quad \leqq \mathrm{const}\,(n-1)(\psi, H_0\psi) \\ &\quad \leqq \mathrm{const}\,(\psi, H_0^2\psi). \end{aligned} \tag{4.30}$$

Inserting the bounds (4.29)–(4.30) into (4.28), we have for sufficiently small ε_1 and ε_2,

$$\sum_{i=1}^{3} \int_{|p|>\kappa} B_i(p)^* T_\mathrm{I} B_i(p)\, dp \geqq -\frac{1}{2}\varepsilon(H_0^2 + I). \tag{4.31}$$

We now use Lemma 3.1.4 to bound the commutator terms in (4.27). We write out

$$T_\mathrm{I} = \sum_{r=0}^{4} \binom{4}{r} T_{\mathrm{I}r}, \tag{4.32}$$

$$T_{\mathrm{I}r} = \int b(k_1, \dots, k_4)\, a^*(k_1) \dots a^*(k_r)\, a(-k_{r+1}) \dots a(-k_4)\, dk \tag{4.33}$$

and

$$b(k_1, \dots, k_4) = c\, \frac{\tilde{g}_1(k_1 + \dots + k_4)}{\mu(k_1)^{\frac{1}{2}} \dots \mu(k_4)^{\frac{1}{2}}}, \tag{4.34}$$

for a constant c. Let us write B_i of (3.2.35)–(3.2.37) as

$$B_i(p) = \int \tilde{h}(p-k)\, b_i(k)\, a(k)\, dk, \tag{4.35}$$

where we note

$$|b_i(k)| \leqq \mu(k)^{\frac{1}{2}}. \tag{4.36}$$

Let

$$\begin{aligned} W_{ir}(\kappa) &= \frac{1}{2}\binom{4}{r} \int_{|p|>\kappa} B_i(p)^* [B_i(p), T_{\mathrm{I}r}]\, dp \\ &= \int w_{ir}(k_1, \dots, k_4; \kappa)\, a^*(k_1) \dots a^*(k_r)\, a(-k_{r+1}) \dots a(-k_4)\, dk, \end{aligned} \tag{4.37}$$

where w_{ir} is the symmetrization in $k_1, \dots, k_r$ of

$$\begin{aligned} &\frac{1}{2}\binom{4}{r} rc\, b_i(k_1)(\mu(k_2) \dots \mu(k_4))^{-\frac{1}{2}} \\ &\quad\cdot \int_{|p|>\kappa} dp \int dq\, b_i(q)\mu(q)^{-\frac{1}{2}} \bar{\tilde{h}}(p-k_1)\tilde{h}(p-q)\tilde{g}_1(q+k_2+k_3+k_4). \end{aligned} \tag{4.38}$$

Thus using (4.31) we write for (4.27)

$$T_\mathrm{I}\delta T_{0\kappa}^{(1)} + \delta T_{0\kappa}^{(1)} T_\mathrm{I} \geqq -\frac{1}{2}\varepsilon(H_0^2 + I) + \sum_{i=1}^{3}\sum_{r=0}^{4} (W_{ir}(\kappa) + W_{ir}(\kappa)^*). \tag{4.39}$$

We will use Lemma 3.1.4 in the case of r creators, $(4-r)$ annihilators, $\alpha = \min(2, r)$, $\beta = \min(2, 4-r)$, $\tau = 1$ and $\sigma = 1$ to prove that

$$\|N^{-(2-\alpha)/2} H_0^{-\alpha/2} W_{ir}(\kappa) H_0^{-\beta/2} N^{-(2-\beta)/2}\| \leqq O(\kappa^{-\delta}), \quad \delta < \tfrac{1}{2}. \tag{4.40}$$

Assuming this result, we have for all i and r,

$$\|(H_0 + I)^{-1} W_{ir}(\kappa)(H_0 + I)^{-1}\| \leqq O(\kappa^{-\delta}), \quad \delta < \tfrac{1}{2}.$$

Exchanging α and β gives a similar bound for $W_{ir}(\kappa)^*$. Thus for sufficiently large κ, we conclude from (4.39) that,

$$T_I \delta T_{0\kappa}^{(1)} + \delta T_{0\kappa}^{(1)} T_I \geqq -\varepsilon(H_0^2 + I), \tag{4.41}$$

which is the desired bound (4.26).

We now estimate the kernel w_{ir} of (4.38). Note that by (4.36)

$$\begin{aligned}
&\left| \int_{|p|>\kappa} dp \int dq\, b_i(q) \mu(q)^{-\frac{1}{2}} \tilde{\bar{h}}(p-k_1) \tilde{h}(p-q) \tilde{g}_1(q+k_2+k_3+k_4) \right| \\
&\qquad \leqq \int_{|p|>\kappa} dp \int dq\, |\tilde{h}(p-k_1) \tilde{h}(p-q) \tilde{g}_1(q+k_2+k_3+k_4)| \\
&\qquad \leqq \int_{|p|>\kappa} dp\, |\tilde{h}(p-k_1)|\, h_1(p+k_2+k_3+k_4),
\end{aligned} \tag{4.42}$$

where

$$h_1(p) = \int |\tilde{h}(p-q) \tilde{g}_1(q)|\, dq$$

is a rapidly decreasing function. Since for $0 \leqq \varepsilon \leqq 1$,

$$1 \leqq (\text{const})\, \mu(p-k_1)^\varepsilon \mu(p)^{-\varepsilon} \mu(k_1)^\varepsilon,$$

we have by (4.42)

$$\begin{aligned}
&\left| \int_{|p|>\kappa} dp \int dq\, b_i(q) \mu(q)^{-\frac{1}{2}} \tilde{\bar{h}}(p-k_1) \tilde{h}(p-q) \tilde{g}_1(q+k_2+k_3+k_4) \right| \\
&\leqq \text{const}\, \mu(k_1)^\varepsilon \int_{|p|>\kappa} dp\, \mu(p)^{-\varepsilon} \mu(p-k_1)^\varepsilon |\tilde{h}(p-k_1)|\, h_1(p+k_2+k_3+k_4) \\
&\leqq \text{const}\, \mu(k_1)^\varepsilon \mu(\kappa)^{-\varepsilon} \int_{|p|>\kappa} \mu(p-k_1)^\varepsilon |\tilde{h}(p-k_1)|\, h_1(p+k_2+\cdots+k_4) \\
&\leqq \text{const}\, \mu(k_1)^\varepsilon \mu(\kappa)^{-\varepsilon} g_2(k_1+\cdots+k_4),
\end{aligned} \tag{4.43}$$

where

$$g_2(k) = \int dp\, \mu(p)^\varepsilon |\tilde{h}(p)|\, h_1(p+k), \tag{4.44}$$

is a rapidly decreasing function (independent of κ). Thus for w_{ir}, the symmetrization of (4.38), we have by (4.36) and (4.43)

$$\begin{aligned}
&|w_{ir}(k_1, \ldots, k_4; \kappa)| \\
&\quad \leqq \text{const}\, \mu(\kappa)^{-\varepsilon} \left(\sum_{j=1}^{4} \mu(k_j)^{1+\varepsilon} \right) (\mu(k_1) \ldots \mu(k_4))^{-\frac{1}{2}} g_2(k_1+\cdots+k_4),
\end{aligned} \tag{4.45}$$

and this bound is independent of i and r.

In applying Lemma 3.1.4 with $\alpha = \min(2, r)$ and $\beta = \min(2, 4-r)$, we have

$$2 \leqq \alpha + \beta \leqq 4.$$

Since $E_C(\alpha, 1)\, E_A(\beta, 1)$ is a homogeneous polynomial of degree $\alpha+\beta$ in the $\mu(k_i)$'s, the most favorable bounds occur with $\alpha+\beta=4$ and the least favorable bounds occur with $\alpha+\beta=2$. In any case

$$E \equiv \sup_{\substack{i \neq j \\ 1 \leqq i,j \leqq 4}} \mu(k_i)\,\mu(k_j) \leqq \text{const}\, E_C(\alpha,1)\, E_A(\beta,1)\,. \tag{4.46}$$

We note, as in [1, Section 2], that

$$\begin{aligned} \mu(k_i)^2 &\leqq \text{const}\, E\mu(k_1+\cdots+k_4) \\ &\leqq \text{const}\, E_C(\alpha,1)\, E_A(\beta,1)\, \mu(k_1+\cdots+k_4)\,. \end{aligned} \tag{4.47}$$

Thus by (4.45),

$$\begin{aligned} &\frac{|w_{ir}(k_1,\ldots,k_4;\kappa)|}{(E_C(\alpha,1)\,E_A(\alpha,1))^{\frac{1}{2}}} \\ &\qquad \leqq \text{const}\, \mu(\kappa)^{-\varepsilon}\left(\sum_{j=1}^{4}\mu(k_j)^{\varepsilon}\right)(\mu(k_1)\ldots\mu(k_4))^{-\frac{1}{2}} \\ &\qquad \cdot \mu(k_1+\cdots+k_4)^{\frac{1}{2}}\, g_2(k_1+\cdots+k_4)\,. \end{aligned} \tag{4.48}$$

Since $g_2(k)$ is rapidly decreasing, the right side of (4.48) is square integrable for $\varepsilon < \frac{1}{2}$. Thus

$$\left\| \frac{w_{ir}(k_1,\ldots,k_4;\kappa)}{E_C(\alpha,1)^{\frac{1}{2}}\, E_A(\beta,1)^{\frac{1}{2}}} \right\|_2 \leqq 0(\kappa^{-\varepsilon}), \qquad \varepsilon < \frac{1}{2},$$

and by Lemma 3.1.4, (4.40) is valid. This completes the proof of the lemma.

Proof of Theorem 4.1. We expand

$$\begin{aligned} &(H_0+\beta T_0(g_0)+T_I(g_1)+b)^2 \\ &= (H_0+\beta T_0(g_0)+\tfrac{1}{2}b)^2 + T_I(g_1)^2 + b(H_0+\beta T_0(g_0)+2T_I(g_1)+\tfrac{5}{8}b) \\ &\quad + \tfrac{1}{8}b^2 + T_I(g_1)\,(H_0+\beta T_0(g_0)) + (H_0+\beta T_0(g_0))\, T_I(g_1)\,. \end{aligned} \tag{4.49}$$

Given $\varepsilon>0$ and b sufficiently large, Lemma 4.2 ensures that the first term on the right of (4.49) is greater than

$$(1-\varepsilon)\,(H_0^2+\beta^2 T_0(g_0)^2)\,. \tag{4.50}$$

Furthermore, for b sufficiently large, the proof of Theorem 3.2.5 ensures that for $0 \leqq \beta \leqq 1$,

$$H_0+\beta T_0(g_0)+2T_I(g_1)+\tfrac{5}{8}b \geqq 0\,. \tag{4.51}$$

Hence to prove the theorem it is sufficient to prove that for b sufficiently large, the last three terms of (4.49) satisfy

$$\begin{aligned} &\tfrac{1}{8}b^2 + T_I(g_1)\,(H_0+\beta T_0(g_0)) \\ &\quad + (H_0+\beta T_0(g_0))\, T_I(g_1) \geqq -\varepsilon(H_0^2+T_I(g_1)^2)\,. \end{aligned} \tag{4.52}$$

We write $T_0 = T_{0\kappa} + \delta T_{0\kappa}^{(2)} + \delta T_{0\kappa}^{(1)}$. Then by Lemmas 4.3–4.5, for b sufficiently large,

$$\tfrac{1}{16} b^2 + T_{\mathrm{I}}(g_1)\, T_0(g_0) + T_0(g_0)\, T_{\mathrm{I}}(g_1) \geqq -\varepsilon (H_0^2 + T_{\mathrm{I}}(g_1)^2). \quad (4.53)$$

Hence we need only prove that for large b,

$$\tfrac{1}{16} b^2 + (T_{\mathrm{I}}(g_1)\, H_0 + H_0\, T_{\mathrm{I}}(g_1)) \geqq -\varepsilon H_0^2 . \quad (4.54)$$

We expand

$$T_{\mathrm{I}} H_0 + H_0 T_{\mathrm{I}} = 2 H_0^{\frac{1}{2}} T_{\mathrm{I}} H_0^{\frac{1}{2}} + [H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, T_{\mathrm{I}}]] .$$

Using (3.2.48),

$$T_{\mathrm{I}} H_0 + H_0 T_{\mathrm{I}} \geqq -\varepsilon H_0^2 - \mathrm{const} + [H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, T_{\mathrm{I}}]] ,$$

and by [1, Theorem 2.1]

$$[H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, T_{\mathrm{I}}]] \geqq -\varepsilon H_0^2 - \mathrm{const} ,$$

which proves (4.54). Alternatively, a proof of (4.54) could be obtained by writing

$$\begin{aligned} T_{\mathrm{I}} H_0 + H_0 T_{\mathrm{I}} &= 2 \int a^*(k)\, T_{\mathrm{I}} a(k)\, \mu(k)\, dk \\ &\quad + \int \{[T_{\mathrm{I}}, a^*(k)]\, a(k) + a^*(k) [T_{\mathrm{I}}, a(k)]\}\, \mu(k)\, dk , \end{aligned}$$

and using the methods of the proof of Lemma 4.5.

5. Self Adjointness and a Fourth Order Estimate

In this section we study the operator

$$M = \alpha H_0 + T_0(g_0) + T_{\mathrm{I}}(g_1) \quad (5.1)$$

where $\alpha > 0$ and g_0, g_1 are spatial cutoffs satisfying (3.2.23). We prove that M is self adjoint and is essentially self adjoint on many reasonable domains. We can then use the spectral theorem to define operators $M^n, n > 2$, and we prove an estimate for M^4. It is sufficient to prove self adjointness for the case $\alpha = 1$, since $\alpha^{-1} g_0$ and $\alpha^{-1} g_1$ also satisfy (3.2.23). In Section 6 we make special choices for α, g_0 and g_1.

The proof of self adjointness of M relies on the following lemma. It concerns an operator B, relatively bounded with respect to A, with an A-bound greater than one.

Lemma 5.1. *Let A be essentially self adjoint on the domain $\mathscr{D}(A)$ and let B be a symmetric operator on $\mathscr{D}(A)$. If there exists a constant a such that for all $\psi \in \mathscr{D}(A)$ and all $0 \leqq \beta \leqq 1$,*

$$\|B\psi\| \leqq a\, \|(A + \beta B)\, \psi\| , \quad (5.2)$$

then $A+B$ *is essentially self adjoint on* $\mathscr{D}(A)$, *and its closure has domain* $\mathscr{D}(A^-)$.

Proof. Let $0 \leqq \gamma \leqq 1$ and $a_1 > a$. Then $\gamma a_1^{-1} B$ is a Kato perturbation of A. For $\psi \in \mathscr{D}(A)$, $\beta = 0$,

$$\|\gamma a_1^{-1} B \psi\| \leqq \delta \|A\psi\|, \quad \delta < 1 .$$

Thus by [9, page 288], $A + \gamma a_1^{-1} B$ is essentially self adjoint on $\mathscr{D}(A)$ and the domain of its closure is $\mathscr{D}(A^-)$. Thus by (5.2) with $\beta = a_1^{-1}$, we have that $\gamma a_1^{-1} B$ is a Kato perturbation of $A + a_1^{-1} B$. Hence $A + a_1^{-1}(1+\gamma) B$ is essentially self adjoint on $\mathscr{D}(A)$ and its closure has domain $\mathscr{D}(A^-)$. Continuing in this manner, for any integer j satisfying $j a_1^{-1} \leqq 1$, we prove that $\gamma a_1^{-1} B$ is a Kato perturbation of the essentially self adjoint operator $A + j a_1^{-1} B$, so that $A + a_1^{-1}(j+\gamma) B$ is essentially self adjoint on $\mathscr{D}(A)$ and the domain of its closure is $\mathscr{D}(A^-)$. By choosing the largest such j, we have for some $0 \leqq \gamma < 1$,

$$a_1^{-1}(j+\gamma) = 1 , \tag{5.3}$$

and so we establish the essential self adjointness of $A+B$.

Corollary 5.2. *Let* A *and* B *be as in Lemma 5.1. Then* A *and* $A+B$ *have the same cores. If* A *is bounded from below, then* $A+B$ *is bounded from below.*

Proof. If B is a Kato perturbation of A, the corollary is valid. The proof of Lemma 5.1 exhibits $A+B$ as a finite number of successive Kato perturbations, and yields the corollary.

Theorem 5.3. *Let* $\alpha > 0$ *and let* $g_i = h_i^2$, $h_i \in \mathscr{S}(\mathbf{R}^1)$, $h_i \geqq 0$, *for* $i = 0, 1$. *Then*

$$M = \alpha H_0 + T_0(g_0) + T_1(g_1) \tag{5.4}$$

is self adjoint on $\mathscr{D}(H_0) \cap \mathscr{D}(T_1(g_1))$ *and is essentially self adjoint on* $\mathscr{C}^\infty(H_0)$.

Proof. We let $\alpha = 1$. Let

$$A = H_0 + T_1(g_1) + b ,$$

and

$$B = T_0(g_0) .$$

We choose b sufficiently large so that $A \geqq I$. From [1, Theorem 4.1], we known that A is self adjoint on $\mathscr{D}(H_0) \cap \mathscr{D}(T_1(g_1))$ and that A is essentially self adjoint on $\mathscr{C}^\infty(H_0)$. Let $\mathscr{D}(A) = \mathscr{C}^\infty(H_0)$. The inequality (5.2) is proved as follows: By Theorem 3.2.1, namely the boundedness of (3.2.14), $\|T_0(g_0)\psi\| \leqq \text{const}\, \|(H_0 + I)\psi\|$. By Theorem 4.1, if $c > 1$ and b is sufficiently large,

$$\|(H_0 + I)\psi\| \leqq c\,\|(H_0 + \beta T_0(g_0) + T_1(g_1) + b)\psi\| ,$$

for all β, $0 \leqq \beta \leqq 1$.

Thus for $\psi \in \mathscr{D}(A) = \mathscr{C}^\infty(H_0)$,

$$\|T_0(g_0)\,\psi\| \leqq \text{const}\,\|(H_0 + \beta T_0(g_0) + T_{\rm I}(g_1) + b)\,\psi\|\,,$$

which is (5.2). By Lemma 5.1, M is essentially self adjoint on $\mathscr{C}^\infty(H_0)$, and M is self adjoint on

$$\mathscr{D}(H_0) \cap \mathscr{D}(T_{\rm I}(g_1)) = \mathscr{D}(A^-)\,.$$

Corollary 5.4. *The operator M of* (5.3) *has the same cores as the operator* $\alpha H_0 + T_{\rm I}(g_1)$.

Proof. We use Corollary 5.2.

We now prove a fourth order inequality for M of (5.4). Such an inequality was proved for the case $g_0 = 0$ by Rosen [10].

Theorem 5.5. *Let M denote the self adjoint operator $\alpha H_0 + T_0(g_0) + T_{\rm I}(g_1)$ for α and g_i as above. Then $\mathscr{D}(M^2) \subset \mathscr{D}(H_0 N)$, and there are finite constants b, c such that as forms on $\mathscr{D}(M^2) \times \mathscr{D}(M^2)$,*

$$H_0^2 N^2 \leqq c(M + b)^4\,. \tag{5.5}$$

Proof. We will prove that $\mathscr{D}(NM) \subset \mathscr{D}(NH_0)$ and that there are constants b, c such that, for $\psi \in \mathscr{D}(NM)$,

$$\|NH_0\psi\| \leqq c\,\|(N+I)\,(M+b)\,\psi\|\,. \tag{5.6}$$

The inequality of Theorem 4.1 extends to $\mathscr{D}(M) \times \mathscr{D}(M)$ since by Theorem 5.3, $\mathscr{C}^\infty(H_0)$ is a core for M and the operators involved are closable. Hence $\mathscr{D}(M) \subset \mathscr{D}(H_0)$, so

$$\mathscr{D}(M^2) \subset \mathscr{D}(H_0 M) \subset \mathscr{D}(NM) \subset \mathscr{D}(NH_0)\,,$$

and by (5.6) for new constants c_1, c_2, b_1 and $\psi \in \mathscr{D}(M^2)$,

$$\begin{aligned} \|NH_0\psi\| &\leqq c_1\,\|(H_0 + I)\,(M+b)\,\psi\| \\ &\leqq c_2\,\|(M+b_1)^2\,\psi\|\,. \end{aligned} \tag{5.7}$$

As a first step to prove (5.6), we establish that $\mathscr{C}^\infty(H_0)$ is a core for $(N+I)(M+b)$, where b is sufficiently large so that $M+b$ is positive. It is sufficient to show that the range of $(N+I)\,(M+b) \restriction \mathscr{C}^\infty(H_0)$ is dense, for this operator has a continuous inverse. Hence the closure of its inverse is the inverse of its closure. Let $\mathscr{D}_0$ denote vectors in Fock space with a finite number of particles. By the proof of Theorem 4.1 of [1], we have that $\mathscr{C}^\infty(H_0) \cap \mathscr{D}_0$ is a core for $\alpha H_0 + T_{\rm I}(g_1)$. Hence by Lemma 5.4, it is a core for M, so that

$$\mathscr{D}_1 = (M+b)\,(\mathscr{C}^\infty(H_0) \cap \mathscr{D}_0)$$

is dense. However every vector in $\mathscr{D}_1$ is an analytic vector for N, and hence $\mathscr{D}_1$ is a core for N. We conclude that $(N+I)\mathscr{D}_1$ is dense; so $\mathscr{C}^\infty(H_0)$ is a core for $(N+I)(M+b)$.

It is sufficient to prove (5.6) for ψ belonging to a core for $(N+I)(M+b)$, so we show that as forms on $\mathscr{C}^\infty(H_0)\times\mathscr{C}^\infty(H_0)$

$$H_0^2 N^2 \leqq c(M+b)(N+I)^2(M+b)\,. \tag{5.8}$$

We note that it is sufficient to establish (5.8) for $\alpha=1$, since the constant α may be absorbed into g_0, g_1, b and c. We let $T=T_0+T_\mathrm{I}$ and note that (5.8) is equivalent to showing that the following operator is positive

$$\begin{aligned}
&H_0^2(N+I)^2 - c^{-1}H_0^2N^2 + T(N+I)^2\,T + T(N+\mathrm{I})^2\,H_0\\
&\quad + H_0(N+I)^2\,T + b(H_0+T)(N+I)^2 + b(N+I)^2(H_0+T)\\
&\quad + b^2(N+I)^2\\
&= H_0^2(N+I)^2 - c^{-1}H_0^2N^2 + T(N+I)^2\,T\\
&\quad + 2b(N+I)(H_0+T+\tfrac{1}{4}b)(N+I) + 2b[N,[N,T]] + T(N+I)^2\,H_0\\
&\quad + H_0(N+I)^2\,T + \tfrac{1}{2}b^2(N+I)^2\,.
\end{aligned} \tag{5.9}$$

Note that for sufficiently large b,

$$T(N+I)^2\,T + 2b(N+I)(H_0+T+\tfrac{1}{4}b)(N+I)$$

is a sum of positive terms. Also if $c>\frac{1}{2}$,

$$\tfrac{1}{2}H_0^2(N+I)^2 - c^{-1}H_0^2N^2 \geqq 0\,.$$

Thus (5.9) is positive for large b if

$$\tfrac{1}{8}b(N+I)^2 + [N,[N,T]] \geqq 0 \tag{5.10}$$

and

$$\tfrac{1}{2}H_0^2(N+I)^2 + T(N+I)^2\,H_0 + H_0(N+I)^2\,T + \tfrac{1}{4}b^2(N+I)^2 \geqq 0\,. \tag{5.11}$$

To establish (5.10), we note that $[N,T_0^{(1)}]=0$. Hence $[N,[N,T]]$ is a sum of Wick ordered monomials of degree two or four with $\mathscr{L}^2$ kernels. Thus

$$(I+N)^{-1}[N,[N,T]](I+N)^{-1}$$

is bounded and (5.10) is positive for large b. To prove (5.11), we note

$$\begin{aligned}
&T(N+I)^2\,H_0 + H_0(N+I)^2\,T\\
&\quad = (N+I)(TH_0+H_0T)(N+I) + [[T,N],(N+I)\,H_0]\\
&\quad = (N+I)(T_0H_0+H_0T_0)(N+I) + 2(N+I)\,H_0^{\frac{1}{2}}\,T_I\,H_0^{\frac{1}{2}}(N+I)\\
&\qquad + (N+I)[H_0^{\frac{1}{2}},[H_0^{\frac{1}{2}},T_I]](N+I) + [[T,N],N]\,H_0\\
&\qquad + (N+I)[[T,N],H_0]\,.
\end{aligned} \tag{5.12}$$

By Lemma 4.2, we have for the first term in (5.12)

$$\begin{aligned}(N+I)(T_0 H_0 + H_0 T_0)(N+I)& \\ \geqq -\tfrac{1}{7}\varepsilon H_0^2 (N+I)^2 - b_1 (N+I)^2&\end{aligned} \tag{5.13}$$

for any $\varepsilon > 0$ and for some $b_1 < \infty$. The second term in (5.12) is bounded below by using (3.2.48)

$$\begin{aligned}2(N+I) H_0^{\frac{1}{2}} T_{\mathrm{I}} H_0^{\frac{1}{2}} (N+I)& \\ \geqq -\varepsilon_1 H_0^2 (N+I)^2 - b_1 H_0 (N+I)^2& \\ \geqq -\tfrac{1}{7}\varepsilon H_0^2 (N+I)^2 - b(N+I)^2&\end{aligned} \tag{5.14}$$

for any $\varepsilon > 0$ and some $b = b(\varepsilon)$.

In [1, Theorem 2.1] it is shown that for any $\varepsilon > 0$ there is a b with

$$[H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, T_{\mathrm{I}}]] \geqq -\tfrac{1}{7}\varepsilon H_0^2 - b\,,$$

so we infer

$$(N+I)[H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, T_{\mathrm{I}}]](N+I) \geqq -\tfrac{1}{7}\varepsilon H_0^2 (N+I)^2 - b(N+I)^2 \tag{5.15}$$

Since $[T, N]$ contains second or fourth order Wick monomials with $\mathscr{L}^2$ kernels,

$$(I+N)^{-1}[[T, N] N](I+N)^{-1} = A$$

is a bounded operator. Thus for $\psi \in \mathscr{C}^\infty(H_0)$

$$\begin{aligned}|(\psi, [[T, N], N] H_0 \psi)| &= |(N+I)\psi, A(N+I) H_0 \psi)| \\ &\leqq \mathrm{const}\, \|(N+I)\psi\|\ \|(N+I) H_0 \psi\| \\ &\leqq \tfrac{1}{7}\varepsilon \|H_0 (N+I)\psi\|^2 + \mathrm{const}\, \|(N+I)\psi\|^2\,.\end{aligned} \tag{5.16}$$

Finally we analyze $(N+I)[[T, N], H_0]$. We write $T = T_0 + T_{\mathrm{I}}$, and inspect these two terms separately. Let

$$[T_0, N] = D_{\mathrm{C}} + D_{\mathrm{A}}$$

where D_{C} and D_{A} are respectively terms of the form (3.1.14) with $r = 2, s = 0$ and with $r = 0, s = 2$. Each term has an $\mathscr{L}^2$ kernel. Applying Lemma 3.1.3, we have that

$$H_0^{-1}[D_{\mathrm{C}}, H_0] \quad \text{and} \quad [D_{\mathrm{A}}, H_0] H_0^{-1}$$

are bounded forms on $\mathscr{C}^\infty(H_0) \times \mathscr{C}^\infty(H_0)$. Thus

$$\begin{aligned}|(\psi, (N+I)[[T_0, N], H_0]\psi)| &\leqq |(H_0(N+I)\psi, H_0^{-1}[D_{\mathrm{C}}, H_0]\psi)| \\ &\quad + |((N+I)\psi, [D_{\mathrm{A}}, H_0] H_0^{-1} H_0 \psi)| \\ &\leqq \mathrm{const}(\|H_0(N+I)\psi\|\ \|\psi\| + \|(N+I)\psi\|\ \|H_0\psi\|) \\ &\leqq \tfrac{1}{7}\varepsilon \|H_0(N+I)\psi\|^2 + \mathrm{const}\, \|(N+I)\psi\|^2\,.\end{aligned} \tag{5.17}$$

The remaining part of $(N+I)[[T,N],H_0]$ contains the contribution from $[T_{\mathrm{I}},N]$. Let

$$T_{\mathrm{I}}=T_{\mathrm{I}\kappa}+\delta T_{\mathrm{I}\kappa},$$

where $T_{\mathrm{I}\kappa}$ is defined as in (4.32)–(4.34), but the kernel (4.34) is multiplied by the characteristic function of $\{k_i:|k_i|\leqq\kappa, i=1,2,3,4\}$. Then $[[T_{\mathrm{I}\kappa},N],H_0]$ is composed of Wick monomials with $\mathscr{L}^2$ kernels. As in (5.16)

$$\begin{aligned}|(\psi,(N+I)[[T_{\mathrm{I}\kappa},N],H_0]\psi)|\leqq\tfrac{1}{7}\varepsilon\|H_0(N+I)\psi\|^2&\\+\mathrm{const}\,\|(N+I)\psi\|^2.&\end{aligned}\tag{5.18}$$

Using Lemma 3.1.4, we analyze the high energy tail, $\delta T_{\mathrm{I}\kappa}$. It is a sum of Wick monomials of degree four, and at least one variable k_i is greater than κ in magnitude. By Lemma 3.1.4, and (4.47),

$$(I+H_0)^{-1}[[\delta T_{\mathrm{I}\kappa},N],H_0](I+H_0)^{-1}=W$$

is a bounded operator, and an estimate of the kernels of $[[\delta T_{\mathrm{I}\kappa},N],H_0]$, as in Lemma 4.5 or [1] shows that

$$\|W\|\leqq O(\kappa^{-\tau}),\qquad \tau<\tfrac{1}{2}.$$

Thus for sufficiently large κ,

$$\begin{aligned}&|(\psi,(N+I)[[\delta T_{\mathrm{I}\kappa},N],H_0]\psi)|\\&\qquad\leqq O(\kappa^{-\tau})\,\|(N+I)(H_0+I)\psi\|\;\|(H_0+I)\psi\|\\&\qquad\leqq\tfrac{1}{7}\varepsilon(\|H_0(N+I)\psi\|^2+\|(N+I)\psi\|^2).\end{aligned}\tag{5.19}$$

The inequalities (5.13)–(5.19) dominate the various terms in (5.12). Added together, they show that (5.12) is bounded by

$$T(N+I)^2H_0+H_0(N+I)^2T\geqq-\varepsilon H_0^2(N+I)^2-\mathrm{const}(N+I)^2.$$

Thus (5.11) is valid for b sufficiently large and the proof of the theorem is complete.

6. Local Lorentz Transformations

In this section we study the operator

$$M=\alpha H_0+T_0(xg_0)+T_{\mathrm{I}}(xg_1).\tag{6.1}$$

We impose certain conditions on α, g_0 and g_1, and we then prove that M is an infinitesimal generator of local Lorentz transformations. We assume the relations (6.2)–(6.4).

$$\alpha>0,\ xg_i(x)=h_i(x)^2,\ h_i(x)\geqq0,\ h_i\in\mathscr{S}(\boldsymbol{R}^1).\tag{6.2}$$

On a neighborhood of an interval $I = [a, b]$, we assume

$$\alpha + x g_0(x) = x = x g_1(x) . \tag{6.3}$$

For all $x \in \boldsymbol{R}^1$, we assume

$$x g_1(x) = (\alpha + x g_0(x)) g_1(x) . \tag{6.4}$$

The conditions (6.2–6.4) are understood as follows. Condition (6.2) means that M of (6.1) is an operator of the type dealt with in Theorem 5.3. Therefore M is self adjoint and generates a one parameter group of unitary transformations $\exp(iM\beta)$.

The condition (6.3) ensures that M agrees locally with the formal Lorentz generator of (1.1.14) and (1.1.22), and thus M is formally a Lorentz generator for the space-time region

$$\boldsymbol{B}_I = \{(x, t) : a + |t| < x < b - |t|\} . \tag{6.5}$$

The condition (6.4) is satisfied if $\alpha + x g_0(x) = x$ is valid on the support of g_1. In other words the free part, $\alpha H_0 + T_0(x g_0)$, is locally correct on $\operatorname{supp} g_1$. This restriction is necessary for technical reasons, because our methods rely on the possibility of defining certain multiple commutators between $H(g_1)$ and M. The condition (6.4) makes the required commutators densely defined operators, rather than bilinear forms. After analyzing operators M satisfying (6.4), we show that this condition can be dropped. (See Theorem 6.16.)

We also note that (6.2) implies that I lies in the positive half line. Of course, we can also consider

$$\hat{M} = -\alpha H_0 + T_0(x\hat{g}_0) + T_I(x\hat{g}_1) ,$$

where $\hat{g}_i(x) = g_i(-x)$. Thus $\hat{M}$ is a locally correct generator for $\hat{\boldsymbol{B}}_I = \boldsymbol{B}_{-I}$. Applying Theorem 5.3, we conclude that $\hat{M}$ is self adjoint, and our proof of Theorem 6.1 is also valid for $\hat{M}$. Thus the essential limitation (6.2) is that we cannot use M to generate Lorentz transformations inside or on the light cone. In Sections 2.1–2.2 we have dealt with this problem, and we showed that it causes no difficulty. Using space-time covariance, we can construct Lorentz transformations in an arbitrary region.

Theorem 6.1. *Let M satisfy conditions* (6.2)–(6.4).

a) *If $f \in \mathscr{S}(\boldsymbol{R}^2)$, $\operatorname{supp} f \subset \boldsymbol{B}_I$ and $\operatorname{supp} f_{\Lambda_\beta} \subset \boldsymbol{B}_I$, then*

$$e^{iM\beta} \varphi(f) e^{-iM\beta} = \varphi(f_{\Lambda_\beta}) , \tag{6.6}$$

as an equality for self adjoint operators.

b) *If $(x, t) \in \boldsymbol{B}_I$ and $\Lambda_\beta(x, t) \in \boldsymbol{B}_I$, then*

$$e^{iM\beta} \varphi(x, t) e^{-iM\beta} = \varphi(\Lambda_\beta(x, t)) , \tag{6.7}$$

in the sense of bilinear forms on $\mathscr{D}(M)\times\mathscr{D}(M)$. *These forms are continuous in* (x, t).

Remarks. a) In [3, Theorem 3.3.5] it is proved that for real $f\in\mathscr{C}_0^\infty(\boldsymbol{R}^2)$, $\varphi(f)$ is a self adjoint operator, essentially self adjoint on a variety of explicit domains. It is for this self adjoint operator that (6.6) is valid.

b) Equation (6.6) entails the domain equality

$$e^{iM\beta}\mathscr{D}(\varphi(f))=\mathscr{D}(\varphi(f_{\Lambda_\beta}))\,. \tag{6.8}$$

c) We write $I=[a, b]$, and define the expanded or contracted interval I_s by

$$I_s=[a-s, b+s]\,. \tag{6.9}$$

The conditions (6.2)–(6.4) are easily satisfied since we can choose g_i so that for some ε, $0<\varepsilon<a/3$,

$$\operatorname{supp} g_1\subset I_{2\varepsilon},\qquad \operatorname{supp} g_0\subset I_{3\varepsilon} \tag{6.10}$$

and

$$\alpha+xg_0(x)=x,\qquad x\in I_{2\varepsilon}\,.$$

Hence (6.4) is valid. We can also let

$$g_1(x)=1,\qquad x\in I_\varepsilon\,;$$

so (6.3) holds on I_ε.

The Hamiltonian

$$H=H_0+T_{\rm I}(g_1) \tag{6.11}$$

is correct in the region $\boldsymbol{B}_I$. We shall work with this particular choice of the Hamiltonian.

Lemma 6.2. *For* M *in* (6.1)–(6.2) *and* H *in* (6.11)

$$\mathscr{D}(M^2)\subset\mathscr{D}(H),\qquad \mathscr{D}(H^2)\subset\mathscr{D}(M)\,, \tag{6.12}$$

$$\mathscr{D}(M)\subset\mathscr{D}((H+b)^{\frac{1}{2}}),\qquad \mathscr{D}(H)\subset\mathscr{D}((M+b)^{\frac{1}{2}})\,, \tag{6.13}$$

where b *is a constant sufficiently large so that* $H+b$ *and* $M+b$ *are positive.*

Proof. By Theorem 5.5,

$$\mathscr{D}(M^2)\subset\mathscr{D}(H_0N)\quad\text{and}\quad\mathscr{D}(H^2)\subset\mathscr{D}(H_0N)\,.$$

Also elementary estimates show that

$$\mathscr{D}(N^2)\subset\mathscr{D}(T_{\rm I}(xg_1))\cap\mathscr{D}(T_{\rm I}(g_1))\,,$$

and by Theorem 3.2.1, $\mathscr{D}(H_0)\subset\mathscr{D}(T_0(xg_0))$. Thus

$$\mathscr{D}(H_0N)\subset\mathscr{D}(M)\cap\mathscr{D}(H)\,.$$

This proves (6.12).

It was proved in [3, Lemma 2.2.4] that

$$\mathscr{D}(H_0) \subset \mathscr{D}((H+b)^{\frac{1}{2}}).$$

By Theorem 3.2.1, the same proof extends in a straightforward fashion to show that

$$\mathscr{D}(H_0) \subset \mathscr{D}((M+b)^{\frac{1}{2}}).$$

Since

$$\mathscr{D}(M) \cup \mathscr{D}(H) \subset \mathscr{D}(H_0),$$

the inclusions (6.13) hold.

We now introduce another local operator, defined for $f \in \mathscr{S}(\boldsymbol{R}^1)$:

$$\dot{P}(f) \equiv T_0(f) - m^2 \int :\varphi(x)^2: f(x)\, dx. \tag{6.14}$$

By Theorem 3.2.1 and the definition of $\int :\varphi(x)^2: f(x)\, dx$ in terms of Wick ordered monomials with $\mathscr{L}^2$ kernels,

$$\mathscr{D}(\dot{P}(f)) \supset \mathscr{D}(H_0). \tag{6.15}$$

For f real, $\dot{P}(f)$ is symmetric on $\mathscr{D}(H_0)$.

In the next theorem, let M, given in (6.1), satisfy (6.2) and (6.4), and let H be given by (6.11).

Theorem 6.3. a) *For* $l = 2, 3, 4$,

$$M : \mathscr{D}(H^l) \to \mathscr{D}(H^{l-2}). \tag{6.16}$$

b) *As operators on* $\mathscr{D}(H^3)$,

$$[iH, M] = P\left(\frac{d}{dx}(x g_0)\right) \tag{6.17}$$

and as operators on $\mathscr{D}(H^4)$,

$$[iH, [iH, M]] = \dot{P}\left(\frac{d^2}{dx^2}(x g_0)\right) - T_1\left(\frac{d}{dx} g_1\right). \tag{6.18}$$

c) *The roles of H and M can be interchanged in the following sense*:

$$H : \mathscr{D}(M^l) \to \mathscr{D}(M^{l-2}), \qquad l = 2, 3, 4. \tag{6.16'}$$

The equality (6.17) *holds on the domain* $\mathscr{D}(M^3)$, *and on* $\mathscr{D}(M^4)$,

$$\begin{aligned}[iM, [iM, H]] = {} & T_0\left(\left(\frac{d}{dx}(x g_0)\right)^2\right) + T_1\left(\frac{d}{dx}(x g_1)\right) \\ & - \dot{P}\left((\alpha + x g_0)\frac{d^2}{dx^2}(x g_0)\right).\end{aligned} \tag{6.18'}$$

Remark. If condition (6.3) also holds, then the double commutator (6.18) is formally localized outside a neighborhood of I. It is this localization, made precise in the following, that results in M generating Lorentz transformations in $\boldsymbol{B}_I$.

Proof. The case of (6.16) for $l=2$ is covered by Lemma 6.2, which also defines M as a bilinear form on $\mathscr{D}(H)\times\mathscr{D}(H)$. From this and the fact that $P, \dot{P}$, and T_{I} are operators defined on $\mathscr{D}(H_0N)\supset\mathscr{D}(H^2)$, it follows that the terms involved in (6.17) and (6.18) are defined as bilinear forms on $\mathscr{D}(H^2)\times\mathscr{D}(H^2)$. In Lemma 6.6 we will prove that (6.17)–(6.18) hold as bilinear forms on $\mathscr{D}(H^2)\times\mathscr{D}(H^2)$. Assuming this, we now prove parts a) and b) of the theorem.

Let $\chi, \psi\in\mathscr{D}(H^3)$. We have

$$(H\chi, M\psi)=(\chi, MH\psi)-i\left(\chi, P\left(\frac{d}{dx}(xg_0)\right)\psi\right). \tag{6.19}$$

Since, by Theorem 4.1 and Theorem 5.3

$$\|(H_0+I)\,\Omega\|\leqq \text{const}\,\|(H+b)\,\Omega\| \tag{6.20}$$

for all $\Omega\in\mathscr{D}(H)$, it follows from Theorem 5.5 that

$$\begin{aligned}\|M\Omega\| &\leqq \text{const}\,\|(H_0+I)\,\Omega\|+\text{const}\,\|N^2\Omega\|\\ &\leqq \text{const}\,\|(H+b)^2\,\Omega\|\,,\end{aligned} \tag{6.21}$$

for all $\Omega\in\mathscr{D}(H^2)$. Let $\Omega=H\psi$; (6.21) yields the inequality

$$|(\chi, MH\psi)|\leqq\{\text{const}\,\|(H+b)^3\,\psi\|\}\,\|\chi\|\,. \tag{6.22}$$

Since by Theorem 3.2.1 and (6.20)

$$\begin{aligned}\left|\left(\chi, P\left(\frac{d}{dx}(xg_0)\right)\psi\right)\right| &\leqq\{\text{const}\,\|(H_0+I)\,\psi\|\}\,\|\chi\|\\ &\leqq\{\text{const}\,\|(H+b)\,\psi\|\}\,\|\chi\|\,,\end{aligned}$$

we have by (6.19) and (6.22) that

$$|(H\chi, M\psi)|\leqq\{\text{const}\,\|(H+b)^3\,\psi\|\}\,\|\chi\|\,. \tag{6.23}$$

Hence $M\psi\in\mathscr{D}((H\upharpoonright\mathscr{D}(H^3))^*)=\mathscr{D}(H)$, since H is essentially self adjoint on $\mathscr{D}(H^3)$. This proves part a) for $l=3$. As a consequence, $i[H, M]$ is an operator on $\mathscr{D}(H^3)$ and by (6.19),

$$(\chi, i[H, M]\,\psi)=\left(\chi, P\left(\frac{d}{dx}(xg_0)\right)\psi\right)$$

for all $\chi, \psi\in\mathscr{D}(H^3)$. This proves (6.17), since the χ's are dense.

The proof of (6.16) for the case $l=4$ and the proof of (6.18) are similar. Let $\chi, \psi \in \mathscr{D}(H^4)$. From (6.16) with $l=2, 3$, and the assumption that (6.18) is valid as a bilinear form, we have

$$\begin{aligned}(H^2\chi, M\psi) = & -(\chi, MH^2\psi) + 2(H\chi, MH\psi) \\ & -(\chi, [iH, [iH, M]]\,\psi) \\ = & -(\chi, MH^2\psi) + 2(\chi, HMH\psi) \\ & -\left(\chi, \left\{\dot{P}\left(\frac{d^2}{dx^2}\,(xg_0)\right) - T_{\mathrm{I}}\left(\frac{d}{dx}\,g_1\right)\right\}\psi\right).\end{aligned} \tag{6.24}$$

By (6.23), (6.21) and the inequality

$$\begin{aligned}|(\chi, \{\dot{P} - T_{\mathrm{I}}\}\,\psi)| &\leqq \{\mathrm{const}(\|(H_0+I)\,\psi\| + \|(N^2+I)\,\psi\|)\}\,\|\chi\| \\ &\leqq \{\mathrm{const}(\|(H+b)\,\psi\| + \|(H+b)^2\,\psi\|)\}\,\|\chi\|,\end{aligned}$$

which follows from Theorem 5.5, we have from (6.24) the inequality

$$|(H^2\chi, M\psi)| \leqq \{\mathrm{const}\,\|(H+b)^4\,\psi\|\}\,\|\chi\|.$$

Hence $M\psi \in ((H^2 \restriction \mathscr{D}(H^4))^*) = \mathscr{D}(H^2)$, proving (6.16) for the case $l=4$. Thus $[iH, [iH, M]]$ is an operator defined on $\mathscr{D}(H^4)$, and we find from (6.24) that (6.18) holds.

The proof of parts a) and b) of the theorem is thus completed when we establish the equalities (6.17)–(6.18) in the sense of bilinear forms on $\mathscr{D}(H^3) \times \mathscr{D}(H^3)$ and $\mathscr{D}(H^4) \times \mathscr{D}(H^4)$ respectively.

The proof of part c) of the theorem is similar. For example, we replace the inequality (6.20) by

$$\|(H_0+I)\,\Omega\| \leqq \mathrm{const}\,\|(M+b)\,\Omega\| \tag{6.25}$$

for all $\Omega \in \mathscr{D}(M)$. This also follows from Theorem 4.1 and Theorem 5.3. By Theorem 5.5, we replace (6.21) with

$$\|H\Omega\| \leqq \mathrm{const}\,\|(M+b)^2\,\Omega\|.$$

To complete the proof of part c) of the theorem, we need to establish (6.17) as a bilinear form on $\mathscr{D}(M^3) \times \mathscr{D}(M^3)$ and (6.18′) as a form on $\mathscr{D}(M^4) \times \mathscr{D}(M^4)$.

Lemma 6.4. *As bilinear forms on* $\mathscr{D}(H_0) \times \mathscr{D}(H_0)$,

$$[iT_0(f), T_0(g)] = P\left(f\,\frac{dg}{dx} - g\,\frac{df}{dx}\right), \tag{6.26}$$

and

$$[iT_0(f), P(g)] = \dot{P}\left(f\,\frac{dg}{dx}\right) - T_0\left(g\,\frac{df}{dx}\right), \tag{6.27}$$

for $f, g \in \mathscr{S}(\mathbf{R}^1)$. These equalities also hold if $f \equiv 1$ or $g \equiv 1$. For instance

$$[iH_0, P(g)] = \dot{P}\left(\frac{dg}{dx}\right). \tag{6.27'}$$

Since $\mathscr{D}(H_0) \supset \mathscr{D}(H) \cup \mathscr{D}(M)$, the equalities hold as forms on $\mathscr{D}(H) \times \mathscr{D}(H)$ and on $\mathscr{D}(M) \times \mathscr{D}(M)$.

Proof. The operators T_0, P, and $\dot{P}$ involved in (6.26)–(6.27) are closable (symmetric), defined on $\mathscr{D}(H_0)$ and bounded as operators relative to $H_0 + I$. Hence (6.26)–(6.27) are defined as forms on $\mathscr{D}(H_0) \times \mathscr{D}(H_0)$ and it suffices to establish equality on a core for H_0, e.g. on

$$\mathscr{D} = \{\psi \in \mathscr{F} : \psi^{(n)} \in \mathscr{S}(\mathbf{R}^n), \psi^{(n)} = 0 \text{ for } n \text{ large}\}. \tag{6.28}$$

In momentum space, elementary calculations on $\mathscr{D} \times \mathscr{D}$ yield the equalities. For instance

$$\begin{aligned}
&[iH_0, T_0^{(1)}(g)] \\
&= \frac{i}{4\pi} \int dk\, dp\, \tilde{g}(k-p) \left\{\frac{\mu(k)\,\mu(p) + kp + m^2}{\mu(k)^{\frac{1}{2}}\,\mu(p)^{\frac{1}{2}}}\right\} [H_0, a^*(k)\, a(p)] \\
&= \frac{i}{4\pi} \int dk\, dp\, \tilde{g}(k-p)(\mu(k) - \mu(p)) \left\{\frac{\mu(k)\mu(p) + kp + m^2}{\mu(k)^{\frac{1}{2}}\,\mu(p)^{\frac{1}{2}}}\right\} a^*(k) a(p) \qquad (6.29) \\
&= \frac{1}{4\pi} \int dk\, dp (i(k-p)\, \tilde{g}(k-p)) \left\{\frac{k\mu(p) + p\mu(k)}{\mu(k)^{\frac{1}{2}}\,\mu(p)^{\frac{1}{2}}}\right\} a^*(k)\, a(p) \\
&= P^{(1)}\left(\frac{dg}{dx}\right).
\end{aligned}$$

By a more length calculation,

$$[iT_0^{(1)}(f), T_0^{(1)}(g)] + [iT_0^{(2)}(f), T_0^{(2)}(g)] = P^{(1)}(fg' - gf').$$

The remaining calculations are similar.

Lemma 6.5. *As bilinear forms on $\mathscr{D}(H_0 N) \times \mathscr{D}(H_0 N)$,*

$$[iT_{\mathrm{I}}(h), T_0(f)] = -4\lambda \int f(x)\, h(x) : \varphi(x)^3\, \pi(x) : dx \tag{6.30}$$

and

$$[iT_{\mathrm{I}}(h), P(f)] = -T_{\mathrm{I}}\left(\frac{d}{dx}(fh)\right). \tag{6.31}$$

These equalities also hold if $f \equiv 1$.

It follows by Theorem 5.5 that $\mathscr{D}(H_0 N) \times \mathscr{D}(H_0 N) \supset \mathscr{D}(H^2) \times \mathscr{D}(H^2) \cup \mathscr{D}(M^2) \times \mathscr{D}(M^2)$; so (6.30)–(6.31) hold on $\mathscr{D}(H^2) \times \mathscr{D}(H^2)$ and on $\mathscr{D}(M^2) \times \mathscr{D}(M^2)$.

Proof. The operators T_0, T_{I} and P involved in (6.30)–(6.31) are closable, defined on $\mathscr{D}(H_0 N)$, and are bounded as operators relative to

(H_0N+I). Furthermore it is easy to check by Lemma 3.1.3 that the right hand side of (6.30) is a bilinear form on $\mathscr{D}(H_0N)\times\mathscr{D}(H_0N)$, and that

$$(I+H_0N)^{-1}\int :\varphi^3(x)\,\pi(x):(fh)(x)\,dx(I+H_0N)^{-1}$$

is a bounded operator. Hence each term in (6.30)–(6.31) is a bilinear form on $\mathscr{D}(H_0N)\times\mathscr{D}(H_0N)$. It suffices to establish equality on $\mathscr{D}\times\mathscr{D}$, as in the proof of Lemma 6.4, since $\mathscr{D}$ is a core for H_0N. On the domain $\mathscr{D}\times\mathscr{D}$, the equalities (6.30)–(6.31) are seen to hold by direct computation in momentum space – as in the proof of the previous lemma.

Lemma 6.6. *The equalities* (6.17), (6.18) *and* (6.18′) *hold as bilinear forms on* $\mathscr{D}(H^2)\times\mathscr{D}(H^2)$ *and on* $\mathscr{D}(M^2)\times\mathscr{D}(M^2)$. *(We are assuming the conditions* (6.2) *and* (6.4).*)*

Proof. As bilinear forms on $\mathscr{D}(H^2)\times\mathscr{D}(H^2)$ or $\mathscr{D}(M^2)\times\mathscr{D}(M^2)$,

$$\begin{aligned}[iH,M]&=[iH_0,T_0(xg_0)]\\&\quad+\{[iH_0,T_I(xg_1)]+[iT_I(g_1),\alpha H_0]+[iT_I(g_1),T_0(xg_0)]\}\end{aligned}$$

To compute these commutators we apply Lemmas 6.4 and 6.5.

$$\begin{aligned}[iH,M]&=P\left(\frac{d}{dx}(xg_0)\right)\\&\quad+4\lambda\int\{xg_1(x)-\alpha g_1(x)-xg_0(x)\,g_1(x)\}:\varphi^3(x)\,\pi(x):dx\\&=P\left(\frac{d}{dx}(xg_0)\right),\end{aligned}$$

by the condition (6.4). Hence (6.17) holds on $\mathscr{D}(H^2)\times\mathscr{D}(H^2)$ and on $\mathscr{D}(M^2)\times\mathscr{D}(M^2)$.

Similarly, using Lemmas 6.4 and 6.5, we compute in the sense of bilinear forms on $\mathscr{D}(H^2)\times\mathscr{D}(H^2)$ or on $\mathscr{D}(M^2)\times\mathscr{D}(M^2)$,

$$\begin{aligned}\left[iH,P\left(\frac{d}{dx}(xg_0)\right)\right]&=\left[iH_0,P\left(\frac{d}{dx}(xg_0)\right)\right]+\left[iT_I(g_1),P\left(\frac{d}{dx}(xg_0)\right)\right]\\&=\dot P\left(\frac{d^2}{dx^2}(xg_0)\right)-T_I\left(\frac{d}{dx}\left(g_1\frac{d}{dx}(xg_0)\right)\right).\end{aligned}$$

By condition (6.4),

$$\frac{d}{dx}((x-\alpha-xg_0)\,g_1)=0$$

and $x-\alpha-xg_0(x)=0$ for $x\in\operatorname{supp}g_1$; hence

$$g_1=g_1\frac{d}{dx}(xg_0)\,.$$

This proves (6.18). Similarly

$$-\left[iM, P\left(\frac{d}{dx}(xg_0)\right)\right] = -\alpha\left[iH_0, P\left(\frac{d}{dx}(xg_0)\right)\right]$$

$$-\left[iT_0(xg_0), P\left(\frac{d}{dx}(xg_0)\right)\right] - \left[iT_1(xg_1), P\left(\frac{d}{dx}(xg_0)\right)\right]$$

$$= -\alpha\dot{P}\left(\frac{d^2}{dx^2}(xg_0)\right) - \dot{P}\left(xg_0\frac{d^2}{dx^2}(xg_0)\right)$$

$$+T_0\left(\left(\frac{d}{dx}(xg_0)\right)^2\right) + T_1\left(\frac{d}{dx}\left(xg_1\frac{d}{dx}(xg_0)\right)\right),$$

which simplifies to (6.18′) by condition (6.4).

Again let M and H be given by (6.1) and (6.11) and assume that (6.2) and (6.4) hold:

Theorem 6.7. *If $n \geqq 2$, $\mathscr{D}(H^n)$ is a core for M and $\mathscr{D}(M^n)$ is a core for H.*

Proof. $\mathscr{D}(H^2) \subset \mathscr{D}(M)$ by Lemma 6.2. We prove first that $\mathscr{D}(H^2)$ is a core for M. Since $\mathscr{D}(M^2)$ is a core for M, it suffices to show that

$$\mathscr{D}((M \upharpoonright \mathscr{D}(H^2))^-) \supset \mathscr{D}(M^2). \tag{6.32}$$

We use the smoothing operator, for $j = 1, 2, 3, \ldots$,

$$E_j = \left(1 + \frac{1}{j}(H+b)\right)^{-1}, \tag{6.33}$$

which has the properties

$$E_j : \mathscr{D}(H^l) \to \mathscr{D}(H^{l+1}), \tag{6.34}$$

$$\|E_j\| \leqq 1, \tag{6.35}$$

$$\text{st.}\lim_{j\to\infty} E_j = I, \tag{6.36}$$

and on $\mathscr{D}(H)$,

$$[E_j, H] = 0.$$

Let $\psi \in \mathscr{D}(M^2)$. Since $\mathscr{D}(M^2) \subset \mathscr{D}(H)$, $E_j\psi \in \mathscr{D}(H^2)$, by (6.34). Since $E_j\psi \to \psi$, the desired inclusion (6.32) would follow from

$$ME_j\psi \to M\psi. \tag{6.37}$$

We now prove (6.37) for all $\psi \in \mathscr{D}(M^2)$.

First we show that for $\Omega \in \mathscr{D}(H^2)$,

$$ME_j\Omega = E_jM\Omega - \frac{i}{j}E_jP\left(\frac{d}{dx}(xg_0)\right)E_j\Omega. \tag{6.38}$$

Each term in (6.38) is defined since $\mathscr{D}(H^2) \subset \mathscr{D}(M)$, and P is defined on $\mathscr{D}(H) \subset \mathscr{D}(H_0)$. We now compute $[E_j, M]$ on $\mathscr{D}(H^2)$. If $\Omega \in \mathscr{D}(H^2)$,

$$\begin{aligned}[E_j, M]\,\Omega &= E_j E_j^{-1} [E_j, M]\, E_j^{-1} E_j \Omega \\ &= E_j [M, E_j^{-1}]\, E_j \Omega \\ &= \frac{1}{j} E_j [M, H]\, E_j \Omega \\ &= \frac{i}{j} E_j P\left(\frac{d}{dx}(x g_0)\right) E_j \Omega\,,\end{aligned}$$

where we have used Theorem 6.3, part a) and (6.17). Hence we have established (6.38) on the domain $\mathscr{D}(H^2)$. Let $\psi \in \mathscr{D}(M^2)$, $\Omega \in \mathscr{D}(H^2)$. Since M is self adjoint on $\mathscr{D}(M)$,

$$(E_j M \Omega, \psi) = (M\Omega, E_j \psi) = (\Omega, M E_j \psi)$$

and

$$(M E_j \Omega, \psi) = (\Omega, E_j M \psi)\,.$$

Thus

$$\begin{aligned}(\Omega, [M, E_j]\, \psi) &= ([E_j, M]\, \Omega, \psi) \\ &= \left(\frac{i}{j} E_j P\left(\frac{d}{dx}(x g_0)\right) E_j \Omega, \psi\right) \\ &= \left(\Omega, -\frac{i}{j} E_j P\left(\frac{d}{dx}(x g_0)\right) E_j \psi\right).\end{aligned}$$

Since $\mathscr{D}(H^2)$ is dense,

$$M E_j \psi = E_j M \psi - \frac{i}{j} E_j P\left(\frac{d}{dx}(x g_0)\right) E_j \psi\,, \tag{6.39}$$

and (6.38) holds on $\mathscr{D}(M^2)$.

The convergence (6.37) now follows. By (6.36),

$$E_j M \psi \to M \psi\,,$$

and

$$\begin{aligned}\frac{1}{j}\left\| E_j P\left(\frac{d}{dx}(x g_0)\right) E_j \psi\right\| &\leqq \frac{1}{j}\left\| P\left(\frac{d}{dx}(x g_0)\right) E_j \psi\right\| \\ &\leqq \text{const}\,\frac{1}{j}\,\|(H_0 + I)\, E_j \psi\| \\ &\leqq \text{const}\,\frac{1}{j}\,\|(H + b)\, E_j \psi\| \\ &= \text{const}\,\frac{1}{j}\,\|E_j (H + b)\, \psi\| \\ &\leqq \text{const}\,\frac{1}{j}\,\|(H + b)\, \psi\| \\ &\to 0\,.\end{aligned}$$

We have used the fact that $\psi \in \mathscr{D}(M^2) \subset \mathscr{D}(H) \subset \mathscr{D}(H_0)$. Hence by (6.39),

$$ME_j\psi \to M\psi$$

which proves (6.37) and establishes that $\mathscr{D}(H^2)$ is a core for M.

The inequality (6.21) and the fact that $\mathscr{D}(H^n)$, for $n \geqq 2$, is a core for H^2 shows that

$$\mathscr{D}((M \restriction \mathscr{D}(H^n))^-) \supset \mathscr{D}(H^2)\,.$$

Since $\mathscr{D}(H^2)$ is a core, it follows that $\mathscr{D}(H^n)$ is also a core for M.

The proof that $\mathscr{D}(M^n)$ is a core for H is similar, and follows the above proof by interchanging H with M.

In the following, we assume that M and H are given by (6.1) and (6.11), and that (6.2), (6.3) and (6.4) hold.

Theorem 6.8. *Let $f \in \mathscr{S}(\boldsymbol{R}^2)$ have support in $\boldsymbol{B}_I$. Then $\varphi(f)$ is defined on $\mathscr{D}(M)$,*

$$\varphi(f) : \mathscr{D}(M^2) \to \mathscr{D}(M)\,,$$

and, as an operator equality on $\mathscr{D}(M^2)$,

$$[iM, \varphi(f)] = -\varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right). \tag{6.40}$$

Remark. In [3, Section 3] it is shown, for f real, that $\varphi(f)$ is essentially self adjoint on $\mathscr{D}(H^n)$ for any $n \geqq \frac{1}{2}$, and

$$\varphi(f) : \mathscr{D}((H+b)^n) \to \mathscr{D}((H+b)^{n-\frac{1}{2}}) \tag{6.41}$$

Proof. The terms in (6.40) are operators on $\mathscr{D}(H^3)$ since $\varphi(f)\,\mathscr{D}(H^3) \subset \mathscr{D}(H^2) \subset \mathscr{D}(M)$ and $M\mathscr{D}(H^3) \subset \mathscr{D}(H) \subset \mathscr{D}(\varphi(f))$ by (6.41) and Theorem 6.3. In Lemma 6.14 we will establish that (6.40) holds on the domain $\mathscr{D}(H^5)$. Assuming this, we now prove the theorem.

Let $\psi \in \mathscr{D}(M^2)$. By Lemma 6.2, $\mathscr{D}(M^2) \subset \mathscr{D}(H)$; by (6.41), $\psi \in \mathscr{D}(\varphi(f))$. First, we show that

$$\varphi(f)\,\psi \in \mathscr{D}(M)\,. \tag{6.42}$$

Note that $M\psi \in \mathscr{D}(M) \subset \mathscr{D}((H+b)^{\frac{1}{2}}) \subset \mathscr{D}(\varphi(f))$ by Lemma 6.2 and (6.41). Also

$$\psi \in \mathscr{D}\left(\varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)\right).$$

Hence, by the assumption that (6.40) holds on $\mathscr{D}(H^5)$, we have for all $\chi \in \mathscr{D}(H^5)$ that

$$(M\chi, \varphi(f)\,\psi) = (\chi, \varphi(f)\,M\psi) + i\left(\chi, \varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)\psi\right). \tag{6.43}$$

So

$$\varphi(f)\,\psi \in \mathscr{D}((M \upharpoonright \mathscr{D}(H^5))^*)\,.$$

By Theorem 6.7, $\mathscr{D}(H^5)$ is a core for M; so we have (6.42). Next we can use (6.42) to rewrite (6.43) as

$$(\chi, [M, \varphi(f)]\,\psi) = \left(\chi, i\varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)\psi\right).$$

Since $\mathscr{D}(H^5)$ is dense, this implies

$$[M, \varphi(f)]\,\psi = i\varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)\psi\,,$$

proving (6.40) on the stated domain.

The next six lemmas complete the proof of Theorem 6.8 by establishing (6.40) on the domain $\mathscr{D}(H^5)$. We then show that Theorem 6.1 follows from Theorem 6.8.

We introduce the self adjoint operator

$$M(t) = e^{-iHt} M e^{iHt}\,.$$

Since e^{iHt} leaves $\mathscr{D}(H^n)$ invariant, we have by Lemmas 6.2 and Theorem 6.3 that

$$\mathscr{D}(H^2) \subset \mathscr{D}(M(t))$$

and for $l = 2, 3, 4$

$$M(t) : \mathscr{D}(H^l) \rightarrow \mathscr{D}(H^{l-2})\,. \tag{6.44}$$

Let $f \in \mathscr{S}(\boldsymbol{R}^2)$ have support in $\boldsymbol{B}_I$. By (6.41) and (6.44), $\varphi(f)\,\mathscr{D}(H^3) \subset \mathscr{D}(H^2) \subset \mathscr{D}(M(t))$ and $M(t)\,\mathscr{D}(H^3) \subset \mathscr{D}(H) \subset \mathscr{D}(\varphi(f))$. More generally, we can replace $\varphi(f)$ by $e^{iHt}\varphi(f)e^{-iHt}$. Thus for $\psi \in \mathscr{D}(H^3)$ and $f \in \mathscr{S}(\boldsymbol{R}^3)$ with support in $\boldsymbol{B}_I$, we define the function

$$\begin{aligned} F(t) &= (\psi, [iM(t), \varphi(f)]\,\psi) \\ &= (\psi(t), [iM, e^{iHt}\varphi(f)\,e^{-iHt}]\,\psi(t))\,, \end{aligned} \tag{6.45}$$

where

$$\psi(t) = e^{iHt}\psi\,. \tag{6.46}$$

Let $I = [a, b]$, $I_r = [a - r, b + r]$ and let $\boldsymbol{B}_{I_r}$ be the causal shadow of I_r. We define

$$\begin{aligned} \boldsymbol{B}_s &= \boldsymbol{B}_{I_{-|s|}} \cap \{(x, t) : |t| < \tfrac{1}{2}\varepsilon\} \\ &= \{(x, t) : |t| < \tfrac{1}{2}\varepsilon,\ a + |s| + |t| < b - |s| - |t|\}\,. \end{aligned} \tag{6.47}$$

Hence the points of $\boldsymbol{B}_s$ have small times, and $\boldsymbol{B}_s$ translated by times less than $|s|$ lies in B_I.

Lemma 6.9. *If* $\psi \in \mathscr{D}(H^5)$, $F(t)$ *in* (6.45) *is twice continuously differentiable. If* f *has compact support in* $\boldsymbol{B}_s$, *then for* $|t| \leqq |s|$,

$$F''(t) = 0 . \tag{6.48}$$

Proof. We first prove the differentiability of $F(t)$. Let $\Delta_n(\varepsilon)$ be the difference quotient for the n^{th} derivative of e^{iHt} at $t=0$. For instance,

$$\Delta_1(\varepsilon) = \frac{1}{\varepsilon}(e^{iH\varepsilon} - I) .$$

For $\psi \in \mathscr{D}(H^n)$, and $m + j \leqq n$, as $\varepsilon \to 0$,

$$\|H^m\{\Delta_j(\varepsilon) - (iH)^j\}\,\psi\| = \|\{\Delta_j(\varepsilon) - (iH)^j\}\,H^m\psi\| \to 0 .$$

Hence, for $\psi \in \mathscr{D}(H^n)$, $Me^{iHt}\psi$ is $n-2$ times differentiable, since for $j \leqq n-2$,

$$\|Me^{iHt}\{\Delta_j(\varepsilon) - (iH)^j\}\,\psi\| \leqq \|\{\Delta_j(\varepsilon) - (iH)^j\}\,(H+b)^2\,\psi\| \to 0 .$$

The function $F(t)$ has the form

$$F(t) = i(Me^{iHt}\psi, e^{iHt}\varphi(f)\,\psi) - i(e^{iHt}\varphi(f)\,\psi, Me^{iHt}\psi) .$$

For $\psi \in \mathscr{D}(H^5)$, $\varphi(f)\,\psi \in \mathscr{D}(H^4)$ and $F(t)$ is three times continuously differentiable.

$$\begin{aligned}\frac{dF(t)}{dt} &= (MH\psi(t), e^{iHt}\varphi(f)\,\psi) - (M\psi(t), He^{iHt}\varphi(f)\,\psi) \\ &\quad - (e^{iHt}\varphi(f)\,\psi, HM\psi(t)) + (e^{iHt}\varphi(f)\,\psi, MH\,\psi(t)) .\end{aligned} \tag{6.49}$$

We now rearrange the terms in (6.49), using the domain relations of Theorem 6.3. a)

$$\begin{aligned}\frac{dF}{dt} &= (\psi, [H, M(t)]\,\varphi(f)\,\psi) - (\varphi(f)\,\psi, [H, M(t)]\,\psi) \\ &= -i\left(\psi, e^{-iHt}P\left(\frac{d}{dx}(xg_0)\right)e^{iHt}\varphi(f)\,\psi\right) \\ &\quad + i\left(\varphi(f)\,\psi, e^{-iHt}P\left(\frac{d}{dx}(xg_0)\right)e^{iHt}\psi\right)\end{aligned} \tag{6.50}$$

by (6.17). We differentiate (6.50) as above, and writing P for $P\left(\frac{d}{dx}(xg_0)\right)$ we obtain

$$\begin{aligned}\frac{d^2F(t)}{dt^2} &= -(\psi, e^{-iHt}[H, P]\,e^{iHt}\varphi(f)\,\psi) + (\varphi(f)\,\psi, e^{-iHt}[H, P]\,e^{iHt}\psi) \\ &= i\left(\psi(t), \left[\dot{P}\left(\frac{d^2}{dx^2}(xg_0)\right) - T_1\left(\frac{d}{dx}g_1\right), e^{iHt}\varphi(f)\,e^{-iHt}\right]\psi(t)\right).\end{aligned} \tag{6.51}$$

We note that each term in (6.51) is defined. For instance,

$$HPe^{iHt}\varphi(f)\psi$$

is defined since, for $\psi\in\mathscr{D}(H^5)$, $e^{iHt}\varphi(f)\psi\in\mathscr{D}(H^4)$, and by Theorem 6.3,

$$Pe^{iHt}\varphi(f)\psi=[iH,M]\,e^{iHt}\varphi(f)\psi\,.$$

Both HM and MH map $\mathscr{D}(H^4)$ into $\mathscr{D}(H)$, so $HPe^{iHt}\varphi(f)\psi$ is defined. Likewise the commutator $\left[H,P\left(\frac{d}{dx}(xg_0)\right)\right]$ is defined on $\mathscr{D}(H^4)$, and $e^{iHt}\varphi(f)\psi$ belongs to that domain.

Now, assuming that the support of f is contained in $\boldsymbol{B}_s$ and $|t|\leqq s$, we must show that $F''(t)=0$.

The proof is based on the locality of

$$S\equiv\dot{P}\left(\frac{d^2}{dx^2}(xg_0)\right)-T_1\left(\frac{d}{dx}g_1\right).$$

which is symmetric on $\mathscr{D}(H_0N)$. By (6.3)

$$\frac{d^2}{dx^2}(xg_0)=0=\frac{d}{dx}g_1$$

in a neighborhood of I. We prove that S commutes with the von Neumann algebra

$$W(I)=\{\exp(i\varphi(h_1)+i\pi(h_2)):h_i=\overline{h}_i\in\mathscr{S}(\boldsymbol{R}^1),\ \operatorname{supp}h_i\subset I\}''$$

generated by the spectral projections of the time zero fields $\int\varphi(x)\,h_1(x)\,dx$ and $\int\pi(x)\,h_2(x)\,dx$, $h_i\in\mathscr{S}(\boldsymbol{R}^1)$, $\operatorname{supp}h_i\subset I$. We show that

$$[S,W(I)]\,\mathscr{D}(H^2)=0\,. \tag{6.52}$$

To this end, we modify [1, Lemma 3.3] as follows: Let $\mathscr{D}$ be the domain (6.28) of well-behaved vectors. If $\chi_1,\chi_2\in\mathscr{D}$, a direct momentum space computation (e.g. as in the proof of Lemma 6.4) shows that

$$(S\chi_1,(\varphi(h_1)+\pi(h_2))^n\chi_2)=((\varphi(h_1)+\pi(h_2))^n\chi_1,S\chi_2)\,. \tag{6.53}$$

An easy computation yields

$$\|(\varphi(h_1)+\pi(h_2))^n\chi\|\leqq AB^n(n!)^{\frac{1}{2}}$$

for constants A and B depending on $\chi\in\mathscr{D}$. Thus, the $\chi\in\mathscr{D}$ are entire vectors for the operator $(\varphi(h_1)+\pi(h_2))$, and the sum

$$U\chi=\sum_{n=0}^{\infty}\frac{(i\varphi(h_1)+i\pi(h_2))^n}{n!}\chi=\exp(i\varphi(h_1)+i\pi(h_2))\,\chi \tag{6.54}$$

converges strongly. Now, we multiply (6.53) by $i^n(n!)^{-1}$ and sum over n using the convergence (6.54) to obtain

$$(S\chi_1, U\chi_2) = (U^*\chi_1, S\chi_2) = (\chi_1, US\chi_2)$$

for $\chi_i \in \mathscr{D}$. This equality extends to $\chi_i \in \mathscr{D}(H_0 N)$ since $\mathscr{D}$ is a core for $H_0 N$, S is defined on $\mathscr{D}(H_0 N)$ and

$$\|S\chi\| \leqq \text{const}\, \|(H_0 N + I)\chi\| .$$

Hence for $\chi \in \mathscr{D}(H_0 N)$, we have proved that $U\chi \in \mathscr{D}(S^*)$ and

$$S^* U\chi = US\chi .$$

We now prove that $U\chi \in \mathscr{D}(H_0 N)$ if $\chi \in \mathscr{D}(H_0 N)$, so that

$$SU\chi = US\chi , \tag{6.55}$$

since S is symmetric on $\mathscr{D}(H_0 N)$. We give $\mathscr{D}(H_0 N)$ a norm,

$$\|\chi\|_1 = \|(I + H_0 N)\chi\| ;$$

the corresponding scalar product makes $\mathscr{D}(H_0 N)$ a Hilbert space, $\mathscr{H}_1$. We now prove that $\varphi(h_1) + \pi(h_2) = B$ generates a one parameter group

$$U(\alpha) = \exp(i\alpha B) = \exp\{i\alpha(\varphi(h_1) + \pi(h_2))\}$$

on $\mathscr{H}_1$. This is equivalent to proving that

$$\hat{B} = (I + H_0 N) B (I + H_0 N)^{-1} \tag{6.56}$$

generates a one parameter group on Fock space. Since B is essentially self adjoint on $\mathscr{D}$, and on this domain

$$\begin{aligned}\hat{B} &= B + [NH_0, B](1 + H_0 N)^{-1} \\ &= B + [N, B] H_0 (I + H_0 N)^{-1} + N[H_0, B](I + H_0 N)^{-1} \\ &= B + \text{bounded operator},\end{aligned}$$

by Lemma 3.1.3, we infer that $\hat{B} \restriction \mathscr{D}$ is a bounded perturbation of an essentially self adjoint operator. Hence $(\hat{B} \restriction \mathscr{D})^-$ generates a one parameter group on Fock space, and

$$B \restriction (I + H_0 N)^{-1} \mathscr{D}$$

has a closure in $\mathscr{H}_1$ that generates a one parameter group on $\mathscr{H}_1$. Since the topology of $\mathscr{H}_1$ is stronger than that of $\mathscr{F}$, the closure of $B \restriction (I + H_0 N)^{-1} \mathscr{D}$ in $\mathscr{H}_1$ is a restriction of B^- in $\mathscr{F}$ and the one parameter group in $\mathscr{H}_1$ is a restriction of the one parameter group generated by B^- in $\mathscr{F}$. This establishes that

$$U:\mathscr{D}(H_0N)\to\mathscr{D}(H_0N) \tag{6.57}$$

and (6.55).

By passing to strong limits of linear combinations of such U's, we obtain (6.52) (on restricting to the domain $\mathscr{D}(H^2)\subset\mathscr{D}(H_0N)$ via Theorem 5.5). This makes precise the statement that S is localized outside I.

We note that for each t_1, $|t_1|\leqq|s|$, the spectral projections of

$$\int \varphi(x) f(x,t_1)\,dx$$

belong to $W(\mathring{I}_{-|s|})$, where $\mathring{I}_{-|s|}$ is the interior of $I_{-|s|}$,

$$\mathring{I}_{-|s|}=\{x:(x,t_1)\in\boldsymbol{B}_s\}=\{x:a+|s|<x<b-|s|\}\,.$$

The support of f is contained in $\boldsymbol{B}_s$; hence the spectral projections of

$$e^{iH(t+t_1)}\int\varphi(x)f(x,t_1)\,dx\,e^{-iH(t+t_1)} \tag{6.58}$$

belong to $W(\mathring{I}_{|t|-|s|})$; [1, 3, 11]. For $|t|\leqq|s|$, $\mathring{I}_{|t|-|s|}\subset I$; so the spectral projections of (6.58) belong to $W(I)$.

We now can use the locality property (6.52) of S. For $\chi\in\mathscr{D}(H^2)$, $\psi\in\mathscr{D}(H^3)$, we have

$$\psi\in\mathscr{D}\left(\int\varphi(x,0)f(x,t_1)\,dx\right),$$

and for $\varphi(f)=\int\varphi(x,t)f(x,t)\,dx\,dt$, by (6.41)

$$e^{iHt}\varphi(f)e^{-iHt}\psi\in\mathscr{D}(H^2)\,. \tag{6.59}$$

Thus from (6.52) and the localization of (6.58),

$$\begin{aligned}&(S\chi, e^{iH(t+t_1)}\int\varphi(x)f(x,t_1)\,dx\,e^{-iH(t+t_1)}\psi)\\&\quad=(e^{iH(t+t_1)}\int\varphi(x)f(x,t_1)\,dx\,e^{-iH(t+t_1)}\chi, S\psi)\,,\end{aligned}$$

for $|t|\leqq|s|$ and $\operatorname{supp} f\subset\boldsymbol{B}_s$. By [3, Theorem 3.2.3] we can integrate over t_1 to obtain

$$\begin{aligned}(S\chi, e^{iHt}\varphi(f)e^{-iHt}\psi)&=(e^{iHt}\varphi(f)e^{-iHt}\chi, S\psi)\\&=(\chi, Se^{iHt}\varphi(f)e^{-iHt}\psi)\end{aligned} \tag{6.60}$$

where the last inequality follows by (6.59) and the fact that S is a symmetric operator on $\mathscr{D}(H_0N)\supset\mathscr{D}(H^2)$. From (6.60) we infer that

$$S\psi\in\mathscr{D}\big((e^{iHt}\varphi(f)e^{-iHt}\restriction\mathscr{D}(H^2))^*\big)$$

and hence that

$$S\psi \in \mathcal{D}(e^{iHt}\varphi(f)\,e^{-iHt}),$$

since by [3, Theorem 3.3.5], $\mathcal{D}(H^2)$ is a core for $\varphi(f)$. Hence from (6.60) we conclude

$$e^{iHt}\varphi(f)\,e^{-iHt}S\psi = S e^{iHt}\varphi(f)\,e^{-iHt}\psi$$

for $|t| \leqq |s|$, supp $f \subset \boldsymbol{B}_s$, and $\psi \in \mathcal{D}(H^3)$.

We now apply this relation to (6.51). In that case $\psi(t) \in \mathcal{D}(H^5) \subset \mathcal{D}(H^3)$, so

$$F''(t) = 0, \quad \text{for} \quad |t| \leqq |s|\,.$$

Lemma 6.10. *Let $f \in \mathcal{S}(\boldsymbol{R}^2)$ have support in $\boldsymbol{B}_s$. Then on $\mathcal{D}(H^5)$ we have the operator equality*

$$[iM(s), \varphi(f)] = [iM, \varphi(f)] - s\left[iP\left(\frac{d}{dx}(xg_0)\right), \varphi(f)\right]. \tag{6.61}$$

Proof. Each of the six terms in (6.61) is an operator defined on $\mathcal{D}(H^5)$, since $\varphi(f): \mathcal{D}(H^l) \to \mathcal{D}(H^{l-1})$, $M(s): \mathcal{D}(H^l) \to \mathcal{D}(H^{l-2})$ for $l = 2, 3, 4$, and (by Theorem 6.3) $P\left(\frac{d}{dx}(xg_0)\right): \mathcal{D}(H^3) \to \mathcal{D}(H)$.

Let $\psi \in \mathcal{D}(H^5)$. Then

$$(\psi, [iM(s), \varphi(f)]\,\psi) = F(s)$$

for F defined in (6.45). By Lemma 6.9, F has two derivatives. Hence by Taylor's theorem with remainder,

$$F(s) = F(0) + sF'(0) + \frac{s^2}{2}F''(t)$$

for some t, $|t| \leqq |s|$. Furthermore, by Lemma 6.9,

$$F(s) = F(0) + sF'(0)\,.$$

By definition,

$$F(0) = (\psi, [iM, \varphi(f)]\,\psi)\,,$$

and by (6.50),

$$F'(0) = -i\left(\psi, \left[P\left(\frac{d}{dx}(xg_0)\right), \varphi(f)\right]\psi\right).$$

This proves the equality

$$(\psi, [iM(s), \varphi(f)]\,\psi) = (\psi, [iM, \varphi(f)]\,\psi) - s\left(\psi, \left[iP\left(\frac{d}{dx}(xg_0)\right), \varphi(f)\right]\psi\right),$$

proving (6.61) by polarization and the density of $\mathcal{D}(H^5)$.

The next step in the proof of Theorem 6.8 is to pass to the sharp time limit of Lemma 6.10. We want to choose a sequence of functions $f_n \in \mathscr{S}(\boldsymbol{R}^2)$ which pick out a time zero contribution in the limit.

Let

$$A(f, t) = \int \varphi(x) f(x, t)\, dx \tag{6.62}$$

and

$$B(f, t) = \int \pi(x) f(x, t)\, dx , \tag{6.63}$$

for φ and π the canonical time-zero fields. For real $f \in \mathscr{S}(\boldsymbol{R}^2)$, with compact support, $A(f, t)$ and $B(f, t)$ are essentially self adjoint on $\mathscr{D}((H+b)^{\frac{1}{2}})$.

Let $f \in \mathscr{C}_0^\infty(\boldsymbol{B}_I)$ and let $f_n(x, t) \in \mathscr{S}(\boldsymbol{R}^2)$ be a sequence of functions of the form

$$f(x, s)\, \delta_n(t)$$

with support in $\boldsymbol{B}_s$ and converging in the w^* topology of measures to $f(x, s)\, \delta(t)$ as $n \to \infty$. For $\psi \in \mathscr{D}(H^5)$, the vectors

$$M(s)\,\psi,\ M\psi,\ P\left(\frac{d}{dx}(x g_0)\right)\psi \in \mathscr{D}(H)$$

as in the proof of Lemma 6.10. Furthermore by [3, Eq. (3.2.8)–(3.2.9)] the bilinear form $\varphi(x, t)$ for $(x, t) \in \boldsymbol{B}_I$ determines a bounded operator

$$(H+b)^{-\frac{1}{2}} \varphi(x, t) (H+b)^{-\frac{1}{2}} \tag{6.64}$$

which is continuous in (x, t).

Lemma 6.11. *Let $f \in \mathscr{S}(\boldsymbol{R}^2)$ have support in $\boldsymbol{B}_I$. Then, in the sense of bilinear forms on $\mathscr{D}(H^5) \times \mathscr{D}(H^5)$,*

$$[iM(s), A(f, s)] = [iM, A(f, s)] - s[iP, A(f, s)] \tag{6.65}$$

where

$$P \equiv P\left(\frac{d}{dx}(x g_0)\right). \tag{6.66}$$

Proof. Choose a w^*-convergent sequence of measures $f_n \in \mathscr{S}(\boldsymbol{R}^2)$ as above. Consider, for example, the first term in (6.61) as a bilinear form on $\mathscr{D}(H^5) \times \mathscr{D}(H^5)$. Let $\psi, \chi \in \mathscr{D}(H^5)$,

$$\begin{aligned}
&(\chi, [iM(s), \varphi(f_n)]\, \psi) \\
&= \int (-iM(s)\,\chi, \varphi(x, t)\,\psi)\, f(x, s)\, \delta_n(t)\, dx\, dt \\
&\quad + \int (\varphi(x, t)\,\chi, iM(s)\,\psi)\, f(x, s)\, \delta_n(t)\, dx\, dt ,
\end{aligned} \tag{6.67}$$

where on the right hand side $\varphi(x, t)$ is considered as a bilinear form on $\mathscr{D}((H+b)^{\frac{1}{2}}) \times \mathscr{D}((H+b)^{\frac{1}{2}})$ continuous in (x, t) by (6.64). Thus, by the convergence of the f_n, the terms on the right hand side of (6.67) converge as $n \to \infty$ to

$$\int (-iM(s)\,\chi, \varphi(x)\,\psi)\, f(x, s)\, dx + \int (\varphi(x)\,\chi, iM(s)\,\psi)\, f(x, s)\, dx\,.$$

This is the left side of (6.65), evaluated on $\chi \times \psi$. The other terms of (6.65) are similarly obtained by passing to the same limit in (6.61).

In Lemmas 6.12–6.14 let $f \in \mathscr{C}_0^\infty(\boldsymbol{B}_I)$.

Lemma 6.12. *As an equality of bilinear forms on* $\mathscr{D}(H) \times \mathscr{D}(H)$,

$$[iP, A(f, s)] = A\left(\frac{\partial f}{\partial x}, s\right) \tag{6.68}$$

where P is defined in (6.66).

Proof. Let $\mathscr{D}$ be the domain (6.28) of smooth vectors. We prove (6.68) in the sense of bilinear forms on $\mathscr{D} \times \mathscr{D}$ by direct computation in momentum space (e.g. as in the proof of Lemma 6.4):

$$\left[iP\left(\frac{d}{dx}(xg_0)\right), A(f, s)\right] = A\left(\frac{\partial}{\partial x}\left(f\frac{d}{dx}(xg_0)\right), s\right),$$

which agrees with (6.68) because $xg_0 = x - \alpha$ on a neighborhood of I, while $f(x, t)$ vanishes for $x \notin I$.

Note that $\mathscr{D}$ is a core for H_0 and

$$|(P\psi, A(f, s)\,\psi)| \leqq \text{const}\, \|(H_0 + I)\,\psi\|^2\,,$$

for all $\psi \in \mathscr{D}(H_0)$. Hence the equality (6.68) extends from $\mathscr{D} \times \mathscr{D}$ to $\mathscr{D}(H_0) \times \mathscr{D}(H_0)$, since the operators involved are closable. Since $\mathscr{D}(H) \subset \mathscr{D}(H_0)$, the lemma is proved.

Lemma 6.13. *As an equality of bilinear forms on* $\mathscr{D}(H^2) \times \mathscr{D}(H^2)$

$$[iM, A(f, s)] = [iH, A(xf, s)] = B(xf, s)\,. \tag{6.69}$$

Proof. The proof is similar to the proof of Lemma 6.12.

Lemma 6.14. *As an operator equality on* $\mathscr{D}(H^5)$,

$$[iM, \varphi(f)] = -\varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right). \tag{6.70}$$

Proof. We first establish (6.70) as an equality of bilinear forms on $\mathscr{D}(H^5) \times \mathscr{D}(H^5)$. Let $\psi \in \mathscr{D}(H^5)$. By Lemmas (6.11)–(6.13),

$$(\psi, [iM(s), A(f, s)]\,\psi) = (\psi, B(xf, s)\,\psi) - s\left(\psi, A\left(\frac{\partial f}{\partial x}, s\right)\psi\right).$$

Substituting $e^{-iHs}\psi$ for ψ, we obtain

$$\begin{aligned}&(\psi, [iM, e^{iHs}A(f, s)\,e^{-iHs}]\,\psi)\\&= \left(\psi, e^{iHs}\left\{B(xf, s) - A\left(s\frac{\partial f}{\partial x}, s\right)\right\}e^{-iHs}\psi\right).\end{aligned} \tag{6.71}$$

But [3, Theorem 3.2.4] states that

$$\int e^{iHt}\pi(x)\,e^{-iHt}f(x, t)\,dx\,dt = \varphi\left(-\frac{\partial f}{\partial t}\right)$$

on $\mathscr{D}(H^2) \times \mathscr{D}(H^2)$. Using this, we integrate (6.71) over s to obtain

$$(\psi, [iM, \varphi(f)]\,\psi) = -\left(\psi, \varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)\psi\right). \tag{6.72}$$

Since $M\varphi(f)$, $\varphi(f)\,M$, and $\varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)$ are operators on $\mathscr{D}(H^5)$, the operator equality (6.70) follows via polarization and the density of $\mathscr{D}(H^5)$. This completes the proof of Lemma 6.14, and hence it completes the proof of Theorem 6.8.

We now proceed to use Theorem 6.8 to prove Theorem 6.1. We need a simple uniqueness result for partial differential equations. In the following proposition, we assume that $F(\beta, x, t)$ and $\frac{\partial}{\partial\beta}F(\beta, x, t)$ are continuous in (β, x, t), where the partial derivative exists for each (x, t).

Proposition 6.15. *Let $\boldsymbol{B} \subset \boldsymbol{R}^2$ and for all $f \in \mathscr{C}_0^\infty(\boldsymbol{B})$,*

$$\int \frac{\partial F}{\partial\beta}(\beta, x, t)\,f(x, t)\,dx\,dt = -\int F(\beta, x, t)\,\{xf_t + tf_x\}\,dx\,dt. \tag{6.73}$$

Then for all (β, x, t) such that $\Lambda_{\gamma\beta}(x, t) \in \boldsymbol{B}$ for $0 \leqq \gamma \leqq 1$,

$$\begin{aligned}F(\beta, x, t) &= F(0, \Lambda_\beta(x, t))\\&= F(0, x\cosh\beta + t\sinh\beta, x\sinh\beta + t\cosh\beta).\end{aligned} \tag{6.74}$$

Proof. Clearly (6.74) is a solution to (6.73). Thus we need only prove uniqueness, and it is sufficient to prove uniqueness for the case

$F(0, x, t) \equiv 0$. Write $A = \left\{x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}\right\}$ for convenience, and note that

$$\begin{aligned}
&\frac{d}{d\beta'} \int F(\beta', x, t)\, f(\Lambda_{\beta'}(x, t))\, dx\, dt \\
&= \int \left\{ \frac{\partial F}{\partial \beta'}(\beta', x, t)\, f(\Lambda_{\beta'}(x, t)) + F(\beta', x, t)\, A f(\Lambda_{\beta'}(x, t)) \right\} dx\, dt \qquad (6.75) \\
&= 0
\end{aligned}$$

by (6.73), provided $f(\Lambda_{\beta'}(x, t))$ has support in $\boldsymbol{B}$. Let

$$\boldsymbol{B}_1 = \bigcap_{0 \leqq \gamma \leqq 1} \Lambda_{\gamma\beta}^{-1} \boldsymbol{B}.$$

If $f \in \mathscr{C}_0^\infty(\boldsymbol{B}_1)$, then (6.75) holds for all β' between 0 and β. So,

$$\int F(\beta, x, t)\, f(\Lambda_\beta(x, t))\, dx\, dt = 0$$

for all $f \in \mathscr{C}_0^\infty(\boldsymbol{B}_1)$. Thus, in the sense of distributions,

$$F(\beta, x, t) = 0\,, \qquad (x, t) \in \boldsymbol{B}_1\,. \tag{6.76}$$

Since F is continuous, (6.76) holds everywhere in $\boldsymbol{B}_1$. This establishes uniqueness, and completes the proof of the proposition.

Proof of Theorem 6.1. Let $\psi \in \mathscr{D}(M^2)$, let

$$F(\beta, x, t) = (e^{-iM\beta}\psi, \varphi(x, t)\, e^{-iM\beta}\psi)$$

for all (β, x, t) in $\boldsymbol{R}^3$; and for $f \in \mathscr{S}(\boldsymbol{R}^2)$, let

$$\begin{aligned}
F(\beta, f) &= (e^{-iM\beta}\psi, \varphi(f)\, e^{-iM\beta}\psi) \\
&= \int F(\beta, x, t)\, f(x, t)\, dx\, dt\,.
\end{aligned}$$

By [3, Lemma 3.2.1], $\varphi(x, t)$ is a bilinear form defined on $\mathscr{D}((H+b)^{\frac{1}{2}}) \times \mathscr{D}((H+b)^{\frac{1}{2}})$, continuous in $(x, t) \in \boldsymbol{R}^2$. Since $\mathscr{D}(M) \subset \mathscr{D}((H+b)^{\frac{1}{2}})$ by Lemma 6.2, $F(\beta, x, t)$ is well defined and continuous in (x, t). Furthermore, $F(\beta, x, t)$ is continuously differentiable in β,

$$\begin{aligned}
\frac{\partial F}{\partial \beta}(\beta, x, t) = &-(e^{-iM\beta} iM\psi, \varphi(x, t)\, e^{-iM\beta}\psi) \\
&-(e^{-iM\beta}\psi, \varphi(x, t)\, e^{-iM\beta} iM\psi)\,,
\end{aligned}$$

which is also continuous in (x, t). One checks in the usual way that this is the derivative of F. For example,

$$\frac{d}{d\beta} \{(H+b)^{-\frac{1}{2}} \varphi(x, t) (H+b)^{-\frac{1}{2}}\}\, (H+b)^{\frac{1}{2}} e^{-iM\beta}\psi$$

exists in the strong topology since the operator $(H+b)^{-\frac{1}{2}}\varphi(x,t)(H+b)^{-\frac{1}{2}}$ is bounded, since $\|(H+b)^{\frac{1}{2}}\psi\| \leqq \text{const}\,\|(M+b)\,\psi\|$ by Lemma 6.2, and since $e^{-iM\beta}M\psi$ is strongly differentiable with derivative $-e^{-iM\beta}iM^2\psi$.

To show that the two definitions above for $F(\beta, f)$ agree, we use (6.41) and [3, Theorem 3.2.3]. By the usual argument,

$$\begin{aligned}\frac{\partial F}{\partial \beta}(\beta, f) &= (e^{-iM\beta}\psi, [iM, \varphi(f)]\, e^{-iM\beta}\psi) \\ &= \int \frac{\partial F}{\partial \beta}(\beta, x, t)\, f(x,t)\, dx\, dt\,.\end{aligned} \tag{6.77}$$

By Theorem 6.8, we have that

$$\begin{aligned}\frac{\partial F}{\partial \beta}(\beta, f) &= -\left(e^{-iM\beta}\psi, \varphi\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right) e^{-iM\beta}\psi\right) \\ &= -\int F(\beta, x, t)\left\{x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right\} f(x,t)\, dx\, dt\,,\end{aligned} \tag{6.78}$$

provided that

$$\operatorname{supp} f \subset \boldsymbol{B}_I\,. \tag{6.79}$$

Hence, we conclude from Proposition 6.15 that

$$F(\beta, x, t) = F(0, \Lambda_\beta(x,t))\,, \tag{6.80}$$

provided

$$\bigcup_{0 \leqq \gamma \leqq 1} \Lambda_{\gamma\beta}(x,t) \in \boldsymbol{B}_I\,. \tag{6.81}$$

That is, if (6.81) holds,

$$e^{iM\beta}\varphi(x,t)\, e^{-iM\beta} = \varphi(\Lambda_\beta(x,t)) \tag{6.82}$$

in the sense of bilinear forms on $\mathscr{D}(M^2) \times \mathscr{D}(M^2)$. This equality extends by closure to $\mathscr{D}(M) \times \mathscr{D}(M)$, since $\mathscr{D}(M) \subset \mathscr{D}((H+b)^{\frac{1}{2}})$ by Lemma 6.2, and

$$\begin{aligned}|(\psi, e^{iM\beta}\varphi(x,t)\, e^{-iM\beta}\psi)| &= |(\psi, \varphi(\Lambda_\beta(x,t))\,\psi)| \\ &\leqq \text{const}\,\|(H+b)^{\frac{1}{2}}\psi\|^2\,.\end{aligned}$$

Furthermore $\mathscr{D}(M^2)$ is a core for H, by Theorem 6.7, and hence a core for $(H+b)^{\frac{1}{2}}$. Thus (6.82) extends to $\mathscr{D}((H+b)^{\frac{1}{2}}) \times \mathscr{D}((H+b)^{\frac{1}{2}})$, and on this domain we also have continuity of the form in (x,t).

We note that it is necessary to assume that

$$\bigcup_{0 \leqq \gamma \leqq 1} \Lambda_{\gamma\beta}(x,t) \in \boldsymbol{B}_I\,.$$

However for the regions $\boldsymbol{B}_I$ of (6.5), this is a consequence of

$$(x,t)\in\boldsymbol{B}_I\,,\qquad \Lambda_\beta(x,t)\in\boldsymbol{B}_I\,.$$

This completes the proof of Theorem 6.1, part b).

Finally, we prove the operator equality

$$e^{iM\beta}\varphi(f)\,e^{-iM\beta}=\varphi(f_{\Lambda_\beta})\,, \tag{6.83}$$

if $f\in\mathscr{S}(\boldsymbol{R}^2)$, $\operatorname{supp}f\cup\operatorname{supp}f_{\Lambda_\beta}\subset\boldsymbol{B}_I$. By Lemma 6.2, $\varphi(f)$ and $\varphi(f_{\Lambda_\beta})$ are defined on $\mathscr{D}(M^2)$; by integrating (6.82) against $f(x,t)$, we prove that (6.83) holds on $\mathscr{D}(M^2)$. Furthermore, for $\psi\in\mathscr{D}(M^2)$,

$$\varphi(f)\,e^{-iM\beta}\psi=e^{-iM\beta}\varphi(f_{\Lambda_\beta})\,\psi\,. \tag{6.84}$$

Since

$$\|\varphi(f_{\Lambda_\beta})\,\psi\|\leqq\text{const}\,\|(H+b)^{\frac{1}{2}}\,\psi\|$$

and $\mathscr{D}(M^2)$ is a core for H by Theorem 6.7, the equality (6.83) extends by closure to $\mathscr{D}(H)$ and (6.84) holds for $\psi\in\mathscr{D}(H)$. Since $\mathscr{D}(H)$ is a core for $\varphi(f_{\Lambda_\beta})$ by [3, Theorem 3.3.5], we conclude that (6.83) extends by closure to $\mathscr{D}(\varphi(f_{\Lambda_\beta}))$ and (6.84) holds for $\psi\in\mathscr{D}(\varphi(f_{\Lambda_\beta}))$. Thus

$$e^{-iM\beta}:\mathscr{D}(\varphi(f_{\Lambda_\beta}))\to\mathscr{D}(\varphi(f))\,.$$

Similarly,

$$e^{iM\beta}:\mathscr{D}(\varphi(f))\to\mathscr{D}(\varphi\,(f_{\Lambda_\beta}))\,.$$

This proves (6.83) as an equality between selfadjoint operators, completing the proof of Theorem 6.1.

Theorem 6.1 is sufficient for the proof of Lorentz covariance of the $\lambda(\varphi^4)_2$ model of Section 2. We complete this section, however, with the observation that the condition (6.4) is not necessary.

Theorem 6.16. *If M satisfies only the conditions* (6.2) *and* (6.3), *the conclusions of Theorem 6.1 still hold.*

Proof. By (6.3) there is an $\varepsilon>0$ so that

$$\alpha+xg_0(x)=x=xg_1(x)$$

for $x\in I_{2\varepsilon}=[a-2\varepsilon,b+2\varepsilon]$. Let $\hat{g}_1$ be a $\mathscr{C}^\infty$ function so that $x\hat{g}_1=\hat{h}_1^2$ for $\hat{h}_1\geqq 0, \hat{h}\in\mathscr{S}, \hat{g}_1(x)=0$ for $x\notin I_{2\varepsilon}$, and $\hat{g}_1(x)=1$ for $x\in I_\varepsilon$. Then conditions (6.2)–(6.4) hold for the pair g_0 and $\hat{g}_1$ and

$$\delta g_1=g_1-\hat{g}_1$$

is non-zero only in the complement of I_ε. Let

$$\hat{M}=\alpha H_0+T_0(xg_0)+T(x\hat{g}_1)\,,$$
$$\delta M=M-\hat{M}=T_{\mathrm{I}}(x\delta g_1)\,.$$

By Theorem 5.3, both M and $\hat{M}$ are essentially self-adjoint on $\mathscr{D}(H_0^2)$. $\hat{M}$ satisfies the conditions of Theorem 6.1. It is known that δM is also essentially self-adjoint on this domain (cf. [1, Theorem 3.1]). By [1, Theorem 3.2], the spectral projections of δM commute with $\varphi(f)$, for $\operatorname{supp} f \subset \boldsymbol{B}_I$. Hence if E is a spectral projection of $\varphi(f)$,

$$\begin{aligned} e^{iM\beta} E e^{-iM\beta} &= \lim_{n\to\infty} (e^{i\hat{M}\beta/n} e^{i\delta M\beta/n})^n E (e^{-i\delta M\beta/n} e^{-i\hat{M}\beta/n})^n \\ &= e^{i\hat{M}\beta} E e^{-i\hat{M}\beta} \end{aligned}$$

where we use the fact that

$$\bigcup_{0\leqq\gamma\leqq 1} \operatorname{supp} f_{\Lambda_{\gamma\beta}} \subset \boldsymbol{B}_I$$

if

$$\operatorname{supp} f \cup \operatorname{supp} f_{\Lambda_\beta} \subset \boldsymbol{B}_I .$$

Thus M and $\hat{M}$ generate the same transformations on the spectral projections of $\varphi(f)$, if $\operatorname{supp} f \cup \operatorname{supp} f_{\Lambda_\beta} \subset \boldsymbol{B}_I$.

By Lemma 6.2, Theorem 5.3, and [3, Lemma 2.24 and Theorem 3.23],

$$\begin{aligned} &\mathscr{D}(H^2) \subset \mathscr{D}(M) \cap \mathscr{D}(\hat{M}) \\ &\mathscr{D}(M) \cup \mathscr{D}(\hat{M}) \subset \mathscr{D}(H_0) \subset \mathscr{D}((H+b)^{\frac{1}{2}}) \subset \mathscr{D}(\varphi(f)) . \end{aligned}$$

So

$$\begin{aligned} e^{-iM\beta} &: \mathscr{D}(H^2) \to \mathscr{D}(\varphi(f)) , \\ e^{-i\hat{M}\beta} &: \mathscr{D}(H^2) \to \mathscr{D}(\varphi(f)) . \end{aligned}$$

Since we can express $\varphi(f)$ as a strong limit of an integral over its spectral projections on its domain $\mathscr{D}(\varphi(f))$, we obtain, on $\mathscr{D}(H^2)$

$$\begin{aligned} e^{iM\beta} \varphi(f)\, e^{-iM\beta} &= e^{i\hat{M}\beta} \varphi(f)\, e^{-i\hat{M}\beta} \\ &= \varphi(f_{\Lambda_\beta}) , \end{aligned}$$

by Theorem 6.1. Since $\mathscr{D}(H^2)$ is a core for $\varphi(f_{\Lambda_\beta})$, this equality extends by closure to the domain $\mathscr{D}(\varphi(f_{\Lambda_\beta}))$. Thus, part a) of Theorem 6.1 holds for M satisfying (6.2)–(6.3). Part b) of Theorem 6.1 follows from this since the form $\varphi(x, t)$ is continuous in (x, t) on $\mathscr{D}(M) \times \mathscr{D}(M)$ and for $\psi \in \mathscr{D}(M)$,

$$(\psi, \varphi(f)\, \psi) = \int (\psi, \varphi(x, t)\, \psi)\, f(x, t)\, dx\, dt .$$

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