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Boson Fields with Bounded Interaction Densities*

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Abstract. We consider interaction densities of the form $V(\phi(x))$, where $\phi(x)$ is a scalar boson field and $V(\alpha)$ is a bounded real continuous function. We define the cut-off interaction by $V_{\varepsilon,r} = \int_{|x|<r} V(\phi_\varepsilon(x))$, where $\phi_\varepsilon(x)$ is the momentum cut-off field. We prove that the scattering operator $S_{\varepsilon,r}(V)$ corresponding to the cut-off interaction exists, and we study the behavior of the scattering operator as well as the Heisenberg picture fields, as the cut-off is removed.

I. Introduction

In two earlier papers [2, 3] we studied self-interacting scalar Boson fields with interaction densities of the form $V(\phi(x))$, where $V(\alpha)$ is a bounded continuous real function. In Ref. [2] we proved that for the corresponding cut-off interaction the asymptotic limits of the fields existed. In Ref. [3] we proved that the Heisenberg picture fields existed as weak limits of the Heisenberg picture fields corresponding to the cut-off interactions. In Section 2 of this paper we prove that the Heisenberg picture fields are trivial in the sense that they are free fields. In Section 3 we prove that the scattering operator $S_{\varepsilon,r}(V)$ corresponding to the cut-off interaction exists, and we prove that the limit as ε tends to zero is 1 if r is small and fixed.

II. The Heisenberg Picture Fields

Let $\mathscr{F}$ by the Fock space of a free scalar boson field $\phi(x)$. The field operators are given in terms of the annihilation-creation operator a^* and a by

$$\phi(x) = 2^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}} \int_{R^3} e^{ipx}(a(p) + a^*(-p))\,dp\,. \tag{2.1}$$

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For annihilation-creation operator we use the Lorentz invariant commutation relations

$$[a(p), a^*(p')] = \omega(p)^{-1}\,\delta(p-p'), \tag{2.2}$$

where $\omega(p) = (m^2 + p^2)^{\frac{1}{2}}$. We assume that the mass m of the free field is strictly positive. Let $\mathscr{H} = L_2(R^3, \omega^{-1}(p)\,dp)$, then $\mathscr{H}$ carries in a natural way an irreducible representation of the inhomogeneous Lorentz group. Since $\mathscr{F}$ is the direct sum of the symmetric tensor products of $\mathscr{H}$, $\mathscr{F}$ also carries a representation of the inhomogeneous Lorentz group. For $h \in \mathscr{H}$ we set $a^*(h) = \int a^*(p)\,h(p)\,dp$, where a^* stands for a^* or a. Then $a^*(h)$ are closed operators with domain containing D_0, the domain of the free energy operator H_0. Moreover $a(\bar{h})$ and $a^*(h)$ are adjoint operators with the same domain of definition, hence $a^*(h) + a(\bar{h})$ are self-adjoint with domain containing D_0. The commutation relations for $a^*(h)$ may be written

$$[a(\bar{h}), a^*(g)] = (h, g)\,. \tag{2.3}$$

Let g be in $C_0^\infty(R^3)$, such that $g \geqq 0$, $g(x) = g(-x)$, $\int g(x)\,dx = 1$ and g has support in the open sphere of radius 1 and center at the origin in R^3. Let $g_\varepsilon(x) = \varepsilon^{-1} g(\varepsilon^{-1} x)$, then g_ε has support in the open sphere of radius ε, and g_ε tends to Dirac's δ-distribution as ε tends to zero. Define now the cut-off field operators by

$$\phi_\varepsilon(x) = \int g_\varepsilon(x-y)\,\phi(y)\,dy\,. \tag{2.4}$$

By what is said above the annihilation-creation operators, we see that $\phi_\varepsilon(x)$ is a self-adjoint operator with domain containing D_0, the domain of H_0.

Let $V(\alpha)$ be a bounded continuous real function. Then $V(\phi_\varepsilon(x))$ is a bounded self-adjoint operator, such that $\|V(\phi_\varepsilon(x))\| \leqq \|V\|_\infty = \sup_\alpha |V(\alpha)|$. Since $V(\phi_\varepsilon(x)) = U(-x)\,V(\phi_\varepsilon(0))\,U(x)$, where $U(x)$ is a strongly continuous unitary group, we see that $V(\phi_\varepsilon(x))$ is strongly continuous in x. Therefore we may define

$$V_{\varepsilon,r} = \int_{|x|<r} V(\phi_\varepsilon(x))\,dx\,,$$

where the integral is a strong integral. $V_{\varepsilon,r}$ is then a bounded self-adjoint operator, and we have the following ε-independent estimate for its norm.

$$\|V_{\varepsilon,r}\| \leqq \frac{4\pi}{3} r^3 \|V\|_\infty\,. \tag{2.5}$$

The cut-off energy operator is defined by

$$H_{\varepsilon,r} = H_0 + V_{\varepsilon,r}\,.$$

Since $V_{\varepsilon,r}$ is a bounded self-adjoint operator we get that $H_{\varepsilon,r}$ is a self-adjoint with the same domain D_0 as H_0.

Let h be real and in $L_2(R^3)$, we then define the free Heisenberg picture field and the Heisenberg picture field for the cut-off interaction by

$$\phi^t(h) = e^{-itH_0}\phi(h)e^{itH_0}$$

$$\phi_{\varepsilon,r,t}(h) = e^{-itH_{\varepsilon,r}}\phi(h)e^{itH_{\varepsilon,r}}$$

where $\phi(h) = \int \phi(x)\,h(x)dx$. Since e^{itH_0} and $e^{itH_{\varepsilon,r}}$ leave D_0 invariant we see that the operators defined above are self-adjoint operators with domain containing D_0.

Lemma 1. *Let $V(\alpha)$ be a real function which is the Fourier transform of an L_1-function $\hat{V}(s)$. Then*

$$V_{\varepsilon,r} = \int_{|x|\leqq r} V(\phi_\varepsilon(x))dx = \int_{|x|\leqq r}\int ds\,\hat{V}(s)e^{is\phi_\varepsilon(x)},$$

and $V_{\varepsilon,r}$ converge weakly to zero as ε tends to zero.

Proof. Let Ω be the Fock vacuum. Then Ω is in the domain of $e^{a^*(h)} = \sum_{n=0}^{\infty}\frac{1}{n!}(a^*(h))^n$ for h in $\mathscr{H}$. To see this we have only to compute the norm of $\sum_{n=0}^{\infty}\frac{1}{n!}(a^*(h))^n\Omega$, and the computation gives us $\|e^{a^*(h)}\Omega\|^2 = e^{\|h\|^2}$, which is finite. Moreover the set of vectors $e^{a^*(h)}\Omega$, with h real and in $\mathscr{H}$ spends a dense set in $\mathscr{F}$. This is easieast seen by taking the strong derivative of $e^{a^*(s_1h_1+\cdots+s_nh_n)}\Omega$ with respect to $s_1,\ldots,s_n$ at zero. By doing so we get $a^*(h_1)\ldots a^*(h_n)\Omega$, and these vectors we know spends a dense set in $\mathscr{F}$ for $h_1,\ldots,h_n$ real and in $\mathscr{H}$. Since the strong partial derivative is formed by taking a strong limit of linear combinations of vectors of the form $e^{a^*(s_1h_1+\cdots+s_mh_m)}\Omega$, we conclude that the vectors $e^{a^*(h)}\Omega$ with h real and in $\mathscr{H}$ spends a dense set.

The spectral theory gives us the identity

$$V_{\varepsilon,r} = \int_{|x|\leqq r} dx\int ds\,\hat{V}(s)e^{is\phi_\varepsilon(x)}.$$

Since $\hat{V}$ is in L_1, we get by Lebesgue's lemma on dominated convergence that it is enough to prove that $e^{is\phi_\varepsilon(x)}$ converge weakly to zero for almost all s and x. Using that $e^{is\phi_\varepsilon(x)} = U(-x)e^{is\phi_\varepsilon(0)}U(x)$, it is enough to prove that $e^{is\phi_\varepsilon(0)}$ tends weakly to zero for almost all s. By uniform boundedness it is therefore enough to prove that

$$(e^{a^*(h_1)}\Omega,\ e^{is\phi_\varepsilon(0)}e^{a^*(h_2)}\Omega) \tag{2.6}$$

converge to zero for almost all s when h_1 and h_2 are real and in $\mathscr{H}$.

From the definition of $\phi_\varepsilon(0)$ we see that $\phi_\varepsilon(0) = a(h_\varepsilon) + a^*(h_\varepsilon)$, where $h_\varepsilon = 2^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\hat{g}_\varepsilon$. The commutation relations then gives us

$$e^{is\phi_\varepsilon(0)} = e^{-\frac{1}{2}s^2\|h_\varepsilon\|^2} e^{isa^*(h_\varepsilon)} e^{isa(h_\varepsilon)}.$$

Hence (2.6) is equal to

$$\begin{aligned} &e^{-\frac{1}{2}s^2\|h_\varepsilon\|^2}(e^{-isa(h_\varepsilon)} e^{a^*(h_1)}\Omega, e^{isa(h_\varepsilon)} e^{a^*(h)}\Omega) \\ &\quad = e^{-\frac{1}{2}s^2\|h_\varepsilon\|^2} e^{is\overline{(h_\varepsilon, h_1)} + is(h_\varepsilon, h_2)}(e^{a^*(h_1)}\Omega, e^{a^*(h_2)}\Omega) \\ &\quad = e^{-\frac{1}{2}s^2\|h_\varepsilon\|^2} e^{is\overline{(h_\varepsilon, h_1)} + is(h_\varepsilon, h_2)} e^{(h_1, h_2)}. \end{aligned}$$

Since h_ε as well as h_1 and h_2 are all real, we get that $\overline{(h_\varepsilon, h_1)}$ and (h_ε, h_2) are both real. Therefore the absolute value of (2.6) is bounded by $e^{\frac{1}{2}s^2|h_\varepsilon|^2} e^{(h_1, h_2)}$. It is easy to see that $|h_\varepsilon|^2$ tends to infinity as tends to zero, and this gives us that (2.6) tends to zero for almost all s. This proves the lemma.

Lemma 2. *Let $V(\alpha)$ be a real continuous function which tends to zero at infinity. Then*

$$V_{\varepsilon,r} = \int_{|x| \leqq r} V(\phi_\varepsilon(x))\,dx$$

converge weakly to zero as ε tends to zero.

Proof. It is well known that a real continuous function which tends to zero at infinity may be uniformly approximated by the Fourier transform of L_1-functions. Hence for any $\delta > 0$ we can find a $\tilde{V}(\alpha)$ which is the Fourier transform of an L_1-function such that $\|V - \tilde{V}\|_\infty < \delta$. Let $\tilde{V}_{\varepsilon,r} = \int_{|x|<r} \tilde{V}(\phi_\varepsilon(x))\,dx$. From the spectral theory of self-adjoint operators we know that $\|\tilde{V}(\phi_\varepsilon(x)) - V(\phi_\varepsilon(x))\| \leqq \|V - \tilde{V}\|_\infty < \delta$. This gives us that $\|\tilde{V}_{\varepsilon,r} - V_{\varepsilon,r}\| \leqq \frac{4}{3}\pi r^3\delta$. Let ψ_1 and ψ_2 be vectors of unite length in $\mathscr{F}$. Then

$$|(\psi_1, V_{\varepsilon,r}\psi_2)| \leqq \tfrac{4}{3}\pi r^3\delta + |(\psi_1, \tilde{V}_{\varepsilon,r}\psi_2)|.$$

By Lemma 1 the last term on the right hand side tends to zero as ε tends to zero. This gives us that

$$\overline{\lim_{\varepsilon\to 0}}|(\psi_1, V_{\varepsilon,r}\psi_2)| < \tfrac{4}{3}\pi r^3\delta.$$

Since δ is arbitrary the lemma is proved.

Theorem 1. *Let $V(\alpha)$ be a continuous real function which tends to zero at infinity. Then $V_{\varepsilon,r} = \int_{|x|\leqq r} V(\phi_\varepsilon(x))\,dx$ converge strongly to zero as ε tends to zero, for all values of r.*

Proof. Since any real continuous function $V(\alpha)$ which tends to zero at infinity, may be written as a difference $V(\alpha) = V^+(\alpha) - V^-(\alpha)$ of two positive continuous functions which both tend to zero at infinity, we see that $V_{\varepsilon,r} = V^+_{\varepsilon,r} - V^-_{\varepsilon,r}$, where $V^\pm_{\varepsilon,r} = \int_{|x|<r} V^\pm(\phi(x))\,dx$. Hence it is sufficient

to prove that $V_{\varepsilon,r}$ tends strongly to zero for $V(\alpha)$ positive. But if $V(\alpha)$ is positive then $V_{\varepsilon,r}$ is a positive operator with a unique square root $V_{\varepsilon,r}^{\frac{1}{2}}$. Since $\|V_{\varepsilon,r}^{\frac{1}{2}}\psi\|^2=(\psi, V_{\varepsilon,r}^{\frac{1}{2}}\psi)$ we get by Lemma 2 that $V_{\varepsilon,r}^{\frac{1}{2}}$ converge strongly to zero as ε tends to zero. From (2.5) it follows that $V_{\varepsilon,r}^{\frac{1}{2}}$ is a norm bounded uniformly in ε. Hence $V_{\varepsilon,r}=V_{\varepsilon,r}^{\frac{1}{2}}\cdot V_{\varepsilon,r}^{\frac{1}{2}}$ is the product of two strongly convergent and uniformly bounded operators. Therefore we conclude that $V_{\varepsilon,r}$ converge strongly to the product of the limits which is zero. This proves the theorem.

Since D_0 is the domain of H_0 as well as $H_{\varepsilon,r}$ we know that both $e^{itH_{\varepsilon,r}}$ and e^{itH_0} leaves D_0 invariant. D_0 is a Hilbert space with its natural norm $\|(H_0+1)\psi\|$. Since $V_{\varepsilon,r}$ is bounded this norm is equivalent to the norm $\|(H_{\varepsilon,r}+b)\psi\|$ for b large enough. Therefore besides being unitary groups on $\mathscr{F}$, $e^{itH_{\varepsilon,r}}$ and e^{itH_0} are semigroups on D_0. As semigroups on D_0 they are strongly continuous in t, and uniformly bounded in t. Moreover, $e^{itH_{\varepsilon,r}}$ as an operator on D_0 is uniformly bounded in t and ε. To see the strong continuity let ψ be in D_0, then

$$\begin{aligned}\|(H_0+1)(e^{itH_{\varepsilon,r}}\psi-\psi)\| &\leqq a\|(H_{\varepsilon,r}+b)(e^{itH_{\varepsilon,r}}\psi-\psi)\|\\ &\leqq a\|(e^{itH_{\varepsilon,r}}-1)(H_{\varepsilon,r}+b)\psi\|\end{aligned}$$

which tends to zero by the strong continuity of $e^{itH_{\varepsilon,r}}$ on $\mathscr{F}$. The strong continuity of e^{itH_0} and the uniform boundedness in t of $e^{itH_{\varepsilon,r}}$ and e^{itH_0} is proved in the same way. To see that $e^{itH_{\varepsilon,r}}$ is uniformly bounded also in ε let ψ be in D_0. Then

$$\begin{aligned}\|(H_0+1)e^{itH_{\varepsilon,r}}\psi\| &\leqq a\|(H_{\varepsilon,r}+b)e^{itH_{\varepsilon,r}}\psi\|\\ &=a\|(H_{\varepsilon,r}+b)\psi\|\leqq a'\|(H_0+b')\psi\|\,.\end{aligned}$$

Since $V_{\varepsilon,r}$ is bounded uniformly in ε we may choose a and b as well as a' and b' independent of ε, this shows that $e^{itH_{\varepsilon,r}}$ is uniformly bounded also in ε as an operator on D_0.

Lemma 3. *Let $V(\alpha)$ be a real continuous function which tends to zero at infinity. Then $e^{itH_{\varepsilon,r}}$ converge strongly to e^{itH_0} both as operators on $\mathscr{F}$ and as operators on D_0. Moreover both convergences are uniform on compact intervals in t.*

Proof. We have already seen that $e^{itH_{\varepsilon,r}}$ and e^{itH_0} are strongly continuous semigroups on both D_0 and $\mathscr{F}$, and they are uniformly bounded both on D_0 and $\mathscr{F}$ with respect to ε and t. Therefore by the Trotter-Kato semigroup theorem (see Ref. [6], Ch. XI, § 12) it is enough to prove that $(z-H_{\varepsilon,r})^{-1}$ converge strongly both on D_0 and $\mathscr{F}$ to $(z-H_0)^{-1}$ for at least one z. Using that $V_{\varepsilon,r}$ is bounded we get for z nonreal or sufficiently negative.

$$(z-H_{\varepsilon,r})^{-1}-(z-H_0)^{-1}=(z-H_{\varepsilon,r})^{-1}\,V_{\varepsilon,r}(z-H_0)^{-1}\,. \tag{2.7}$$

Since $(z-H_{\varepsilon,r})^{-1}$ is bounded uniformly in ε, we get from Theorem 1 that $(z-H_{\varepsilon,r})^{-1}-(z-H_0)^{-1}$ converge strongly to zero as an operator on $\mathscr{F}$. To see that it also converge strongly as an operator on D_0, we apply (H_0+1) from the left in (2.7). Since $V_{\varepsilon,r}$ is bounded uniformly in we see that $(H_0+1)(z-H_{\varepsilon,r})^{-1}$ is bounded uniformly in ε. So again it follows from Theorem 1 that $(z-H_{\varepsilon,r})^{-1}-(z-H_0)^{-1}$ converge strongly to zero as an operator on D_0.

Theorem 2. *Let $V(\alpha)$ be a continuous real function which tends to zero at infinity. Then for h in L_2 and ψ in D_0, we have that $\phi_{\varepsilon,r,t}(h)\psi$ converge strongly to $\phi^t(h)\psi$ as ε tends to zero. Moreover, if $V(\alpha)$ has a bounded and uniformly continuous derivative $V'(a)$ and h is in $L_1\cap L_2$, then $\phi_{\varepsilon,r,t}(h)-\phi^t(h)$ is a bounded operator which converge strongly to zero as ε tends to zero.*

Proof. Let ψ be in D_0. Since H_0 has a strictly positive mass m, we know that $\phi(h)$ is a bounded linear map from D_0 to $\mathscr{F}$. Lemma 3 then gives us that $e^{itH_{\varepsilon,r}}\psi$ converge strongly to $\phi(h)\psi$. For the moreover part we shall need the following lemma which is the corollary 2 of Ref. [3].

Lemma 4. *Let $V(\alpha)$ be a continuous bounded real function with a bounded and uniformly continuous derivative $V'(\alpha)$. Then for h in $L_1\cap L_2$, we have that*

$$\|\phi_{\varepsilon,r,t}(h)-\phi^t(h)\|\leqq C(|t|^3+1)\,\|V'\|_\infty\,\|h\|_1\,,$$

where C depends only on the mass m.

For the proof of this lemma we refer to Ref. [3]. From this lemma we get that $\phi_{\varepsilon,r,t}(h)-\phi^t(h)$ is bounded uniformly in ε. We have already proved that it converges strongly to zero on D_0. Using now the uniform boundedness and the fact that D_0 is dense in $\mathscr{F}$ we get that it converges strongly to zero on all of $\mathscr{F}$. This completes the proof of Theorem 2.

Remark. The assumption in Theorem 1 that $V(\alpha)$ should tend to zero at infinity was chosen mainly to get a convenient class of functions to work with, and we may prove Theorem 1 for a larger class of bounded continuous real functions. We see that the class of continuous functions which are zero at infinity, arise as the uniform closure of the Fourier transforms of L_1-functions. The L_1-functions were introduced in Lemma 1. But we see that Lemma 1 remains true if we assume that $V(\alpha)$ is the Fourier transform of a bounded measure μ such that $\{0\}$ has μ-measure zero. From this we see that it is enough to assume in Theorem 1 that $V(\alpha)$ is in the uniform closure of the Fourier transforms of bounded measures for which $\{0\}$ is a null set. The almost periodic functions which are orthogonal, in the sense of almost periodic functions, to the constant function, belongs for instance to this class of functions.

III. The Vacuum for the Cut-Off-Interaction

The discussion of the vacuum for the cut-off interaction in this section is mainly an adaption of the discussion of the vacuum for the space cut-off $\lambda\phi^4$ interaction in two space time dimensions by Glimm and Jaffé [1]. The fact that $V(\alpha)$ is a bounded function leads to some minor changes from Glimm and Jaffé's discussion.

Let $\mathscr{H}_l$ be the subspace of $\mathscr{H}$ consisting of functions which are constant on each cube of length l in R^3 and with center at the lattice points (ln_1, ln_2, ln_3) where $n_1 = 0, \pm 1, \pm 2, \ldots$. Let $\tilde{\mathscr{H}}_l$ be the orthogonal complement of $\mathscr{H}_l$, and let $\mathscr{F}_l$ and $\tilde{\mathscr{F}}_l$ be the Fock spaces with $\mathscr{H}_l$ and $\tilde{\mathscr{H}}_l$ as one particle spaces. $\mathscr{F}_l$ and $\tilde{\mathscr{F}}_l$ are then in a natural way identified with subspaces of $\mathscr{F}$. Let P_l be the orthogonal projection onto $\mathscr{F}_l$. The direct sum decomposition $\mathscr{H} = \mathscr{H}_l \oplus \tilde{\mathscr{H}}_l$ gives us the tensor product decomposition $\mathscr{F} = \mathscr{F}_l \otimes \tilde{\mathscr{F}}_l$. Relative to this tensor produce decomposition, the identification of $\mathscr{F}_l$ with a subspace in $\mathscr{F}$ is given by $\mathscr{F}_l \otimes \tilde{\Omega}_l$, where $\tilde{\Omega}$ is the Fock vacuum in $\tilde{\mathscr{F}}_l$; and similar for $\tilde{\mathscr{F}}_l$. From the definition of $\phi_\varepsilon(x)$ we see that $\phi_\varepsilon(x) = a^*(h_x) + a(\bar{h}_x)$, where

$$h_x(p) = 2^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}} e^{-ipx} \hat{g}_\varepsilon(p).$$

We now define

$$\phi_{\varepsilon,l}(x) = a^*(P_l h_x) + a(P_l \bar{h}_x). \tag{4.1}$$

Since P_l commutes with complex conjugation we see that $\phi_{\varepsilon,l}(x)$ is self-adjoint. It follows from (4.1) that relative to the decomposition $\mathscr{F} = \mathscr{F}_l \otimes \tilde{\mathscr{F}}_l$, $\phi_{\varepsilon,l}(x)$ takes the form

$$\phi_{\varepsilon,l}(x) = \phi^{(1)}_{\varepsilon,l}(x) \otimes 1, \tag{4.2}$$

where $\phi^{(1)}_{\varepsilon,l}(x)$ is the restriction of $\phi_{\varepsilon,l}(x)$ to $\mathscr{F}_l$. We now define

$$V_{\varepsilon,r,l} = \int_{|x| \leqq r} V(\phi_{\varepsilon,l}(x))\, dx. \tag{4.3}$$

From (4.2) we get that

$$V_{\varepsilon,r,l} = V^{(1)}_{\varepsilon,r,l} \otimes 1 \tag{4.4}$$

where again $V^{(1)}_{\varepsilon,r,l}$ is the restriction of $V_{\varepsilon,r,l}$ to $\mathscr{F}_l$.

H_0 is uniquely defined by its action as multiplication by $\omega(p)$ in the one particle space $\mathscr{H}$. We define $H_{0,l}$ as the operator we get by substituting ω_l for where $\omega_l = P_l \omega P_l$, i.e. the average of ω over the cubes. We then see that $H_{0,l}$ commutes with P_l and relative to the decomposition $\mathscr{F} = \mathscr{F}_l \otimes \tilde{\mathscr{F}}_l$ $H_{0,l}$ has the form

$$H_{0,l} = H^{(1)}_{0,l} \otimes 1 + 1 \otimes H^{(2)}_{0,l} \tag{4.5}$$

where $H_{0,l}^{(1)}$ and $H_{0,l}^{(2)}$ are the restrictions of $H_{0,l}$ to $\mathscr{F}_l$ and to $\tilde{\mathscr{F}}_l$. We now define

$$H_{\varepsilon,r,l} = H_{0,l} + V_{\varepsilon,r,l},$$

and we see that $H_{\varepsilon,r,l}$ are self-adjoint on D_0.

Lemma 5. *Let $V(\alpha) = \int e^{i\alpha s} d\mu(s)$, where μ is a bounded measure, then for z nonreal or sufficiently negative $(z - H_{\varepsilon,r,l})^{-1}$ converge in norm to $(z - H_{\varepsilon,r})^{-1}$ as l tends to zero.*

Proof. Since $H_{\varepsilon,r}$ and $H_{\varepsilon,r,l}$ has the same domain of definition we get

$$(z - H_{\varepsilon,r,l})^{-1} - (z - H_{\varepsilon,r})^{-1} = (z - H_{\varepsilon,r,l})^{-1} (H_{\varepsilon,r,l}) (z - H_{\varepsilon,r})^{-1}.$$

$H_{\varepsilon,r,l}$ is bounded below uniformly in l, therefore it is enough to prove that $(H_{\varepsilon,r,l} - H_{\varepsilon,r})(z - H_{\varepsilon,r})^{-1}$ tends to zero in norm. Since $V_{\varepsilon,r}$ is bounded this is the same as proving that $(H_{\varepsilon,r,l} - H_{\varepsilon,r})(z - H_0)^{-1}$ tends to zero in norm. That $\|(H_{0,l} - H_0)(z - H_0)^{-1}\|$ tends to zero follows from a direct computation with ω and ω_l (see Ref. 1). To see that $(V_{\varepsilon,r,l} - V_{\varepsilon,r})(z - H_0)^{-1}$ tends to zero in norm we have

$$\begin{aligned} &\|(V_{\varepsilon,r,l} - V_{\varepsilon,r})(z - H_0)^{-1}\| \\ &\quad \leqq \int_{|x|<r} dx \int d|\mu|(s) \, \|(e^{is\phi_{\varepsilon,l}(x)} - e^{is\phi_\varepsilon(x)})(z - H_0)^{-1}\| \\ &\quad = \int_{|x|<r} dx \int d|\mu|(s) \, \|(e^{is\phi_{\varepsilon,l}(x) - \phi_\varepsilon(x))} - 1)(z - H_0)^{-1}\|. \end{aligned}$$

However

$$e^{is(\phi_{\varepsilon,l}(x) - \phi_\varepsilon(x))} - 1 = \frac{e^{is(\phi_{\varepsilon,l}(x) - \phi_\varepsilon(x)} - 1}{\phi_{\varepsilon,l}(x) - \phi_\varepsilon(x)} (\phi_{\varepsilon,l}(x) - \phi_\varepsilon(x)).$$

Since $\dfrac{e^{i\alpha} - 1}{\alpha}$ is a uniformly bounded function on the real axis we therefore get

$$\begin{aligned} \|(V_{\varepsilon,r,l} - V_{\varepsilon,r})(z - H_0)^{-1}\| &\leqq C \int_{|x| \leqq r} dx \, \|(\phi_{\varepsilon,l}(x) - \phi_\varepsilon(x))(z - H_0)^{-1}\| \\ &\leqq \frac{2C}{m} \int_{|x| \leqq r} dx \, \|P_l h_x - h_x\| \, \|(H_0 + 1)(z - H_0)^{-1}\|, \end{aligned}$$

where we have used the well known estimate

$$\|a^\#(h)\psi\| \leqq \frac{1}{m} \|h\| \, \|(H_0 + 1)\psi\|.$$

Since $\|P_l h_x - h_x\|$ converge to zero the lemma is proved.

Corollary 1. *Let $V(\alpha)$ be in the uniform closure of the Fourier transform of bounded measures, then for z non-real or sufficiently negative $(z - H_{\varepsilon,r,l})^{-1}$ converge in norm to $(z - H_{\varepsilon,r})^{-1}$ as l tends to zero.*

Proof. Since $V(\alpha)$ may be uniformly approximated by a Fourier transform $\tilde{V}(\alpha)$ of a bounded measure, we get that $V_{\varepsilon,r,l}$ is approximated in norm by $\tilde{V}_{\varepsilon,r,l}$ uniformly in l. But this gives us that $(z-H_{\varepsilon,r,l})^{-1}$ is approximated in norm by $(z-\tilde{H}_{\varepsilon,r,l})^{-1}$ uniformly in l. The norm convergence of $(z-H_{\varepsilon,r,l})^{-1}$ then follows from the norm convergence of $(z-\tilde{H}_{\varepsilon,r,l})^{-1}$. This proves the corollary.

From (4.4) and (4.5) we get that relative to the decomposition $\mathscr{F}=\mathscr{F}_l \otimes \tilde{\mathscr{F}}_l$ we have

$$H_{\varepsilon,r,l}=H^{(1)}_{\varepsilon,r,l}\otimes 1+1\otimes H^{(2)}_{0,l} \tag{4.6}$$

where $H^{(1)}_{\varepsilon,r,l}$ is the restriction of $H_{\varepsilon,r,l}$ to $\mathscr{F}_l$. We now define a vacuum of a semi-bounded operator H as a normalized eigenvector of H with eigenvalue equal to the lower spectral bound of H. Since $H^{(1)}_{0,l}$ has a compact resolvent and $V_{\varepsilon,r,l}$ is bounded we see that $H^{(1)}_{\varepsilon,r,l}$ has a compact resolvent and therefore a vacuum $\Omega_{\varepsilon,r,l}$. Since $H^{(2)}_{0,l}$ is positive we get from (4.6) that $\Omega_{\varepsilon,r,l}$ is also a vacuum for $H_{\varepsilon,r,l}$. We now have the following theorem.

Theorem 3. *Let $V(\alpha)$ be in the uniform closure of the Fourier transform of bounded measures. Then both $H_{\varepsilon,r,l}$ and $H_{\varepsilon,r}$ have unique vacuums $\Omega_{\varepsilon,r,l}$ and $\Omega_{\varepsilon,r}$. Moreover with the phases determined by $(\Omega, \Omega_{\varepsilon,r,l})>0$ and $(\Omega, \Omega_{\varepsilon,r})>0$ where Ω is the Fock vaccum, we have that $\Omega_{\varepsilon,r,l}$ converge strongly to $\Omega_{\varepsilon,r}$ as l tends to zero.*

Proof. The proof of the corresponding thing in Glimm and Jaffé [1] goes in two steps. First they prove that any sequence $\Omega_{\varepsilon,r,l_n}$ has a subsequence $\Omega_{\varepsilon,r,l_n}$ which converge to a vacuum $\Omega_{\varepsilon,r}$ of $H_{\varepsilon,r}$. We see from their proof that once we have Corollary 1 and formula (4.6) then this part of their proof can be carried over. Their second step is to prove that $\Omega_{\varepsilon,r,l}$ and $\Omega_{\varepsilon,r}$ are unique. This part is done with the help of the theory of positive ergodic kernels and carries over point by point to our case; apart from some simplifications due to the fact that $V(\alpha)$ is bounded. For the details we refer to Ref. [1]. This proves Theorem 3.

We shall now be interested in what happens to the vacuum $\Omega_{\varepsilon,r}$ as ε tends to zero. In view of Theorem 1 we would expect it to converge to the Fock vacuum Ω, if it converges at all.

Theorem 4. *Let $V(\alpha)$ be a continuous real function which tends to zero at infinity. If $r^3 V|_\infty \leqq C$, where C is positive and depends only on the mass m; then with the phases determined as in Theorem 3, $\Omega_{\varepsilon,r}$ converge strongly to Ω as ε tends to zero.*

Proof. $r^3\|V\|_\infty \leqq C$ implies that $\|V_{\varepsilon,r}\| \leqq \frac{4\pi}{3}C$. Since the eigenvalue of H_0 corresponding to the eigenvector Ω, is separated from the rest of the spectrum by a distance equal to m, we know by the theory of regular perturbation that there is an interval $I \subset \langle 0, m \rangle$ such that the spectrum of the operators $H = H_0 + V$ with $\|V\| \leqq C'$, do not intersect I (see Theorem 4.10, Ch. V, Ref. [5]). Moreover C' depends only on m; and to the left of I, H has a single eigenvalue which depends analytic on V. If we now choose $C = \frac{3}{4\pi} C'$ we see that zero is a stable eigenvalue under the perturbation $H_{\varepsilon,r} = H_0 + V_{\varepsilon,r}$. Stable eigenvalues is here used in the sense of Kato (see § 1.4, Ch. VIII, Ref. [5]). Since $V_{\varepsilon,r}$ is uniformly bounded and tends strongly to zero as ε tends to zero by Theorem 1, we find that $H_{\varepsilon,r}$ converge to H_0 in the generalized strong sense of Kato and therefore the theory of asymptotic perturbation applies (Ch. VIII, Ref. [5]). Since zero is a stable eigenvalue we therefore get that the projection onto $\Omega_{\varepsilon,r}$ converge in norm to the projection onto Ω. Hence $\Omega_{\varepsilon,r}(\Omega_{\varepsilon,r}, \Omega)$ converge strongly to Ω, and since the phases are determined as in Theorem 3 this gives us that $\Omega_{\varepsilon,r}$ converge strongly to Ω. This proves the theorem.

IV. The Asymptotic Fields and the Scattering Operator for the Cut-Off Interaction

Throughout this section we shall assume that $V(\alpha)$ is differentiable with a bounded and uniformly continuous derivative $V'(\alpha)$. In Ref. [2], we discussed the asymptotic fields for the cut-off interaction, and we begin this section by stating some of the results obtained in Ref. [2]. The assumption on $V(\alpha)$ in Ref. [2] was that it was the Fourier transform of a bounded measure with a bounded first order moment. However, using the Lemma 1 of Ref. [3] it is easy to see that the results of Ref. [2] holds if we assume that $V(\alpha)$ has a bounded and uniformly continuous derivative $V'(\alpha)$.

The interaction picture annihilation-creation operators corresponding to the cut-off interaction is defined by

$$a_t^{\#}(h) = e^{-itH_{\varepsilon,r}} e^{itH_0} a^{\#}(h) e^{-itH_0} e^{itH_{\varepsilon,r}} \tag{5.1}$$

where h is in $\mathscr{H}$. Let $D_{\frac{1}{2}}$ be the domain of $H_0^{\frac{1}{2}}$, which is also the domain of $(H_{\varepsilon,r} + b)^{\frac{1}{2}}$. Then $a_t^{\#}(h)$ is a closed operator with domain containing $D_{\frac{1}{2}}$ (see Ref. [2]). Since

$$e^{itH_0} a^{\#}(h) e^{-itH_0} = a^*(h_{\pm t}), \tag{5.2}$$

where $+$ goes with a^* and $-$ with a and $h_t(p) = e^{it\omega(p)} h(p)$, we may write (5.1) as

$$a_t^{\#}(h) = e^{-itH_{\varepsilon,r}} a^{\#}(h_{\pm t}) e^{itH_{\varepsilon,r}}. \tag{5.3}$$

The following three theorems are proved in Section 3 of Ref. [2].

Theorem 5. *Let h be in $\mathscr{H}$ and ϕ in $D_{\frac{1}{2}}$. Then $a_t^{\#}(h)\phi$ converge strongly to $a_{\pm}^{\#}(h)\phi$ as t tends to $\pm\infty$. The asymptotic limit operators $a_{\pm}^{\#}(h)$ are closed operators with domain containing $D_{\frac{1}{2}}$, and $a_{\pm}^{\#}(h)$ is a bounded linear map from $\mathscr{H}$ into the Banach space of bounded linear maps from $D_{\frac{1}{2}}$ into $\mathscr{F}$. Moreover $a_{\pm}(\bar{h})$ and $a_{\pm}^{*}(h)$ are adjoints.*

Theorem 6. *Let h and g be in $\mathscr{H}$. Then $a_{\pm}^{\#}(h)$ maps D_0 into the domain of $a_{\pm}^{\#}(g)$, and $a_{\pm}^{\#}(g)\,a_{\pm}^{\#}(h)$ is a bounded linear map from $\mathscr{H}\otimes\mathscr{H}$ into the Banach space of bounded linear maps from D_0 into $\mathscr{F}$. Moreover $a_{\pm}^{\#}(h)$ satisfy the same commutation relations on D_0 as do $a^{\#}(h)$ on H_0. $H_{\varepsilon,r}$ and $a_{\pm}^{\#}(h)$ satisfy the same commutation relations as do H_0 and $a^{\#}(h)$, in the sense that on $D_{\frac{1}{2}}$*

$$e^{itH_{\varepsilon,r}}a_{\pm}(h)e^{-itH_{\varepsilon,r}} = a_{\pm}(h_{-t}),$$

$$e^{itH_{\varepsilon,r}}a_{\pm}^{*}(h)e^{-itH_{\varepsilon,r}} = a_{\pm}^{*}(h_{-t}).$$

Theorem 7. *Let Φ be an eigenvector of $H_{\varepsilon,r}$. Then for any h in $\mathscr{H}$*

$$a_{\pm}(h)\Phi = 0.$$

For the proof of these theorems see Ref. [2].

In Section 4 of Ref. [2] we show that any vector that is annihilated by $a_{\pm}(h)$ for all h in $\mathscr{H}$, is in the domain of $a_{\pm}^{*}(h_n)$ for all $h_1, \ldots, h_n$ in $\mathscr{H}$ and all n. Since the vacuum $\Omega_{\varepsilon,r}$ is an eigenvector of $H_{\varepsilon,r}$ we get by Theorem 5.3 that it is in the domain of $a_{\pm}^{*}(h_1)\ldots a_{\pm}^{*}(h_n)$. Let $\mathscr{F}_{\pm}$ be the smallest closed subspace of $\mathscr{F}$ containing all vectors of the form $a_{\pm}^{*}(h_1)\ldots a_{\pm}^{*}(h_n)\Omega_{\varepsilon,r}$. Due to the commutation relations for $a_{\pm}^{\#}(h)$, we see that $\mathscr{F}_{\pm}$ are Fock spaces with annihilation-creation operators $a_{\pm}^{\#}$ and vacuum $\Omega_{\varepsilon,r}$. By regarding $\mathscr{F}_{\pm}$ as Fock spaces in this way we get a natural identification of the asymptotic Fock spaces $\mathscr{F}_{\pm}$ with the free Fock space $\mathscr{F}$ given by the "wave" operators $W_{\pm}$, where $W_{\pm}$ are defined by

$$W_{\pm}\, a^{*}(h_1)\ldots a^{*}(h_n)\Omega = a_{\pm}^{*}(h_1)\ldots a_{\pm}^{*}(h_n)\Omega_{\varepsilon,r}. \tag{5.4}$$

Due to the commutation relations for $a_{\pm}^{\#}$ we get that $W_{\pm}$ are isometric from $\mathscr{F}$ onto $\mathscr{F}_{\pm}$. We have already seen that the vacuum $\Omega_{\varepsilon,r}$ is unique. It is therefore natural to identify $a_{\pm}^{*}(h_1)\ldots\Omega_{\varepsilon,r}$ with an outgoing (incoming) n-particle state with momentum distribution given by $h_1, \ldots, h_n$, for the cut-off interaction.

The scattering amplitude for the cut-off interaction for m incoming particles with momentum distribution $h_1, \ldots, h_m$; and n outgoing particles with momentum distribution $g_1, \ldots, g_n$ is then given by

$$(a_{+}^{*}(g_1)\ldots a_{+}^{*}(g_n)\Omega_{\varepsilon,r}, a_{-}^{*}(h_1)\ldots a_{-}^{*}(h_m)\Omega_{\varepsilon,r}). \tag{5.5}$$

By (5.4) we get that this is equal to

$$(W_+ a^*(g_1) \dots a^*(g_n)\Omega, W_- a^*(h_1) \dots a^*(h_m)\Omega) \\ = (a^*(g_1) \dots a^*(g_n)\Omega, W_+ W_- a^*(h_1) \dots a^*(h_m)\Omega) . \tag{5.6}$$

Hence we get that the scattering operator for the cut-off interaction is given by

$$S = W_+^* W_- . \tag{5.7}$$

Since $W_\pm$ are isometrics in $\mathscr{F}$ we see that $\|S\| \leqq 1$, and that S is unitary if and only if $\mathscr{F}_+ = \mathscr{F}_-$. Let $E_{\varepsilon,r}$ be the eigenvalue of $\Omega_{\varepsilon,r}$. From the commutation relations for $a_\pm^\#$ and $H_{\varepsilon,r}$ we get from (5.4) that

$$(H_{\varepsilon,r} - E_{\varepsilon,r}) W_\pm = W_\pm H_0 \tag{5.8}$$

and this together with (5.7) gives us that

$$H_0 S = S H_0 . \tag{5.9}$$

We shall now be interested in what happens with S if we keep $V(\alpha)$ and r fixed but let ε tend to zero.

Lemma 6. *Let h be in $\mathscr{H}$. Then $V_{\varepsilon,r}$ leaves the domain of $a^\#(h)$ invariant and*

$$\|[a^\#(h), V_{\varepsilon,r}]\| \leqq C \cdot \sup |\textstyle\int h(p)\, \hat{g}_\varepsilon(p)\, (p)^{-\frac{1}{2}} e^{ixp} dp| ,$$

where C depends only on r and on $|V'|_\infty$. Moreover

$$a_t^\#(h) - a^\#(h) = i \int_0^t e^{-isH_{\varepsilon,r}} [a^\#(h_{\pm s}), V_{\varepsilon,r}] e^{isH_{\varepsilon,r}} ds ,$$

where the integral is taken in the strong sense.

For the proof of this lemma see Section 3, Ref. [2]. In Ref. [2] we assume that $V(\alpha)$ is the Fourier transform of a bounded measure with a bounded first order moment. The technique for the proof when $V(\alpha)$ has a bounded uniformly continuous derivative $V'(\alpha)$ is to be found in the proof of Lemma 1, Ref. [3].

Let $\mathscr{H}_0$ be the set of functions in $\mathscr{H}$ which has compact support and is zero in a neighborhood of zero. Let h be in $\mathscr{H}_0$. From Lemma 6 we then see that $\|[a^\#(h_{\pm s}), V_{\varepsilon,r}]\|$ converge to zero faster than any inverse power in $|s|$, and the convergence is uniform in ε. From Lemma 6 we also get that

$$\|a_+^\#(h) - a_t^\#(h)\| \leqq \int_t^\infty \|[a^\#(h_{\pm s}), V_{\varepsilon,r}]\| ds .$$

Hence for any $\delta > 0$ we may choose t so large that $\|a_+^\#(h) - a_t^\#(h)\| < \delta$, and this choice of t may be done independently of ε.

Lemma 7. *Assume also that $V(\alpha)$ tends to zero at infinity. Then for h in $\mathscr{H}_0$, $a^{\#}_{\pm}(h) - a^{\#}(h)$ is norm bounded uniformly in ε and converge strongly to zero as ε tends to zero.*

Proof. The uniform norm boundedness follows from Lemma 6. To prove the strong convergence write

$$a^{\#}_{+}(h) - a^{\#}(h) = (a^{\#}_{+}(h) - a^{\#}_{t}(h)) + (a^{\#}_{t}(h) - a^{\#}(h)).$$

We have already seen that the first term can be made smaller than δ and the choice of t does not depend on ε. By Lemma 3 the last term tends strongly to zero on D_0. Hence we get that $a^{\#}_{+}(h) - a^{\#}(h)$ converge strongly to zero on D_0. The uniform boundedness then gives us strong convergence on all of $\mathscr{F}$. This proves the lemma.

Theorem 8. *Let $V(\alpha)$ be a differentiable real function which tends to zero at infinity, and assume that $V'(\alpha)$ is bounded and uniformly continuous. If $r^3 \|V\|_\infty \leqq C$, where C is the constant of Theorem 4, then the scattering operator S converges weakly to 1 as ε tends to zero.*

Proof. From Theorem 4 we get that $\Omega_{\varepsilon,r}$ converge strongly to Ω as ε tends to zero. Let h be in $\mathscr{H}_0$. From Lemma 7 we then see that $a^*(h)\Omega_{\varepsilon,r}$ converge strongly to $a^*(h)\Omega$, since

$$a^*_{\pm}(h)\,\Omega_{\varepsilon,r} - a^*(h)\,\Omega = (a^*_{\pm}(h) - a(h))\,\Omega_{\varepsilon,r} + a^*(h)\,(\Omega_{\varepsilon,r} - \Omega).$$

In the same way we see that for $h_1, \ldots, h_n$ in $\mathscr{H}_0\, a^*_{\pm}(h_1) \ldots a^*_{\pm}(h_n)\Omega_{\varepsilon,r} = W_{\pm}\, a^*(h_1) \ldots a^*(h_n)\Omega$ converge strongly to $a^*(h_1) \ldots a^*(h_n)\Omega$. Hence $W_{\pm}$ converge strongly to 1 on a dense subset. Since $W_{\pm}$ are isometries we conclude that they converge strongly to 1 on $\mathscr{F}$. It follows then from (5.7) that S converges weakly to 1. This proves the theorem.

Remark. From the definition of the scattering operator we see that if $V_1(\alpha) - V_2(\alpha) =$ constant then the cut-off interaction corresponding to $V_1(\alpha)$ and $V_2(\alpha)$ has the same scattering operator. This fact of course extends the validity of Theorem 8 somewhat.

V. Removal of the Cut-Off, and the Scattering Operators for Scalar Fields

In the last section we proved that the scattering operator $S_{\varepsilon,r}(V)$ existed for the cut-off interaction $\int\limits_{|x|<r} V(\phi_\varepsilon(x))dx$, under the assumption that $V(\alpha)$ has a bounded and uniformly continuous derivative $V'(\alpha)$. Since $\|S_{\varepsilon,r}(V)\| \leqq 1$ for all ε, r and V, we may use the fact that the unit ball in $\mathscr{F}$ is weakly compact to construct a scattering operator $S_{\varepsilon,r}(F)$ for a larger class of cut-off interactions $\int\limits_{|x|\leqq r} F(\phi_\varepsilon(x))dx$.

Let $F(\alpha)$ be any continuous function of α, and let $V_n(\alpha)$ be a sequence of functions with bounded and uniformly continuous derivatives such that $V_n(\alpha)$ converge pointwise to $F(\alpha)$. We know that such a sequence always exists. Since $\|S_{\varepsilon,r}(V_n)\| \leqq 1$ there is a subsequence $V_m(\alpha)$ such that $S_{\varepsilon,r}(V_m)$ converges weakly to an operator $S_{\varepsilon,r}(F)$. $S_{\varepsilon,r}(F)$ may not be unique since it may depend on the sequence $V_n(\alpha)$ and on the subsequence $V_m(\alpha)$ we choose. But from (5.9) we will always have that $S_{\varepsilon,r}(F)$ commutes with H_0, and since the unit ball is weakly closed we have also that $\|S_{\varepsilon,r}(F)\| \leqq 1$.

Having constructed in this way for each ε and r and $S_{\varepsilon,r}(F)$, we may again use weak compactness to select sequences ε_n tending to zero and r_n tending to infinity such that $S_{\varepsilon_n,r_n}(F)$ converge weakly to a limit $S(F)$. $S(F)$ will commute with H_0 and $\|S(F)\| \leqq 1$. It is natural to interpret $S(F)$ constructed in this way as the relativistic scattering operator corresponding to the interaction density $F(\phi(x))$. It is probable from what we have seen in the previous sections that $S(F)$ is the identity of $F(\alpha)$ is bounded and tend to zero at infinity even though we have not been able to prove this.

A more general class of scattering operator are obtained by using renormalized interaction densities. A renormalized interaction density is given by a family of continuous functions $F^{\varepsilon}(\alpha)$. As the scattering operator for the renormalized interaction we take a weak limit point of $S_{\varepsilon,r}(F^{\varepsilon})$ as ε tends to zero and r to infinity. As an example consider the ϕ^4 theory. Here

$$F^{\varepsilon}(\alpha) = a_0(\varepsilon)\alpha^4 + a_1(\varepsilon)\alpha^3 + a_2(\varepsilon)\alpha^2 + a_3(\varepsilon)\alpha^1$$

where $a_i(\varepsilon)$ will be functions depending on ε. A weak limit point of $S_{\varepsilon,r}(F^{\varepsilon})$ as ε tends to zero and r to infinity is then a scattering operator for the ϕ^4 theory.

We may also introduce the set $\mathscr{S}$ of all local relativistic scattering operators in the following way. Let $\mathscr{S}_{\varepsilon,r}$ be the weak closure of the set $\{S_{\tilde{\varepsilon},\tilde{r}}(V);\ \tilde{\varepsilon} \leqq \varepsilon,\ \tilde{r} \geqq r$ and V real with a bounded and uniformly continuous derivative$\}$. We then define

$$\mathscr{S} = \bigcap\nolimits_{\varepsilon,r} \mathscr{S}_{\varepsilon,r}\,, \tag{6.1}$$

$\mathscr{S}$ is then a closed non-empty subset of the unit ball of operators in $\mathscr{F}$. Due to (5.9) all the elements in $\mathscr{S}$ commutes with H_0. An element S in $\mathscr{S}$ is a local relativistic scattering operator corresponding to a generalized renormalization scheme, in the sense that S would be a weak limit point of a sequence $S_{\varepsilon_n,r_n}(V^{\varepsilon_n,r_n})$, where the functions $V^{\varepsilon,r}(\alpha)$ may depend in an arbitrary manner on ε and r. It is natural to identify the set $\mathscr{S}$ with the set of all local relativistic scattering operators for scalar fields.

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