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Projective Unitary Antiunitary Representations of Locally Compact Groups

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Abstract. We give here a systematic presentation of the theory of projective representations when antiunitary operators are present. In particular the imprimitivity theorem of Mackey is proved in this situation and all the unitary antiunitary representations of the extended Poincaré group are derived.

§ 1. Introduction

In the mathematical formulation of quantum mechanics proposed by von Neumann (cf. [3, 8, 9]) the set of all propositions concerning a quantum mechanical system is an orthocomplemented lattice. The simplest example of such a lattice is the lattice $\mathscr{L}(\mathscr{H})$ of all closed subspaces of a separable Hilbert space $\mathscr{H}$. The observables in such a system turn out to be self adjoint operators in $\mathscr{H}$. In order to construct the standard observables like energy, linear, angular and spin angular momenta etc., in such a system it is necessary to study the effect of coordinate transformations by a group G of symmetries. In this context the representations of G in the group $\mathscr{A}(\mathscr{H})$ of automorphisms of the lattice $\mathscr{L}(\mathscr{H})$ is of great importance.

By the extension of a theorem of Wigner (cf. [8], Theorem 7.27, page 167) it is known that every automorphism τ of $\mathscr{L}(\mathscr{H})$ is induced by a unitary or antiunitary operator U^τ on $\mathscr{H}$. The operator U^τ is determined uniquely only up to a scalar multiple of modulus unity. Suppose $\mathscr{U}(\mathscr{H})$ is the group of all unitary and antiunitary operators on $\mathscr{H}$ and $\mathscr{I}(\mathscr{H})$ is the normal subgroup consisting of scalar multiples of identity. Then Wigner's theorem can be reformulated as follows: the group $\mathscr{A}(\mathscr{H})$ is isomorphic with the quotient group $\mathscr{U}(\mathscr{H})/\mathscr{I}(\mathscr{H})$. We shall denote this quotient by $PUA(\mathscr{H})$ and call it *the projective unitary antiunitary group of* $\mathscr{H}$. $\mathscr{U}(\mathscr{H})$ with the weak (operator) topology (which is equivalent to the strong topology) is a complete and separable metric group in which $\mathscr{I}(\mathscr{H})$ is a compact subgroup. Thus the quotient topology in $PUA(\mathscr{H})$ can be carried over to $\mathscr{A}(\mathscr{H})$ in order to make it a topological group.

In this paper we are interested in studying the continuous representations of G in the group $PUA(\mathscr{H})$ or equivalently $\mathscr{A}(\mathscr{H})$. Even though there is a very detailed account of projective unitary representations in the papers of Mackey [2, 4]; Wigner [10] and others, there does not seem to exist a systematic theory of projective unitary antiunitary or PUA representations. Some examples of such representations in the case of the Poincaré group (or the inhomogeneous full Lorentz group) may be found in the works of Foldy [1] and Streater and Wightman [7]. We shall extend the results of Mackey on transitive imprimitivity systems when PUA representations are considered and derive as an application all the "positive mass" PUA representations of the Poincaré group. This answers some of the questions raised in Foldy [1].

§ 2. The *PUA* Group of a Hilbert Space

In this section we shall study some of the basic properties of unitary and antiunitary operators, describe the projective unitary antiunitary group and finally prove a result concerning its topology and Borel structure.

Definition 2.1. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two complex separable Hilbert spaces with inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ respectively. A map $V:\mathscr{H}_1\to\mathscr{H}_2$ is called *antiunitary* if V is one one, onto and satisfies the following properties:

(1) $V(x+y)=Vx+Vy$ for all $x, y\in\mathscr{H}_1$,

(2) $Vax=\bar{a}\,Vx$ for all $x\in\mathscr{H}_1$ and scalars a,

(3) $(Vx, Vy)_2=(y, x)_1$ for all $x, y\in\mathscr{H}_1$.

If $\mathscr{H}_1=\mathscr{H}_2=\mathscr{H}$, V is called an *antiunitary operator* on $\mathscr{H}$.

From now onwards let $\mathscr{H}$ stand for a fixed complex separable Hilbert space with inner product $(\cdot,\cdot)$. We shall denote by $\mathscr{U}^+(\mathscr{H})$ and $\mathscr{U}^-(\mathscr{H})$ respectively the set of all unitary operators and the set of all antiunitary operators on $\mathscr{H}$. The set $\mathscr{U}(\mathscr{H})=\mathscr{U}^+(\mathscr{H})\cup\mathscr{U}^-(\mathscr{H})$ is a group under the composition operation. This is called *the unitary antiunitary group* or simply *the UA group* of $\mathscr{H}$. If there is no ambiguity we shall drop the $\mathscr{H}$ within brackets and denote these by $\mathscr{U}$, $\mathscr{U}^+$, and $\mathscr{U}^-$. It is obvious that $\mathscr{U}^+$ is a normal subgroup of $\mathscr{U}$ and

$$\mathscr{U}^+\mathscr{U}^-=\mathscr{U}^-\mathscr{U}^+=\mathscr{U}^-;$$

$$\mathscr{U}^-\mathscr{U}^-=\mathscr{U}^+.$$

If $V\in\mathscr{U}^-$, then $V\mathscr{U}^+=\mathscr{U}^+V=\mathscr{U}^-$. In particular, if V is any fixed antiunitary operator then every antiunitary operator W can be written as

UV where U is a unitary operator. Thus the group $\mathscr{U}/\mathscr{U}^+$ has only two elements.

Lemma 2.1. *Let $(X, \mathscr{S})$ be a Borel space and P and Q be two projection valued measures on $\mathscr{S}$. Suppose there exists a $W \in \mathscr{U}^-$ such that*

$$WP(E)W^{-1} = Q(E) \quad \textit{for all} \quad E \in \mathscr{S} .$$

Then there exists a $U \in \mathscr{U}^+$ such that

$$UP(E)U^{-1} = Q(E) \quad \textit{for all } E .$$

Proof. By the Hahn-Hellinger theorem we may, without loss of generality, assume that $\mathscr{H}$ is of the form $\bigoplus_r L_2(\mu_r)$ where μ_r is a sequence of measures on $\mathscr{S}$ and $P(E)$ is just multiplication by χ_E where χ_E is the indicator function of E. Consider the map $V_0 : f \to \bar{f}$ where bar stands for complex conjugation and f is an element of $\mathscr{H}$. V_0 is an antiunitary operator which commutes with all the $P(E)$. Thus the operator $U = WV_0$ is unitary and satisfies the required property. This completes the proof.

In the above lemma we used the fact that complex conjugation is an antiunitary operator. We note that it is an antiunitary operator whose square is identity. The next lemma asserts that every such antiunitary operator is unitarily equivalent to a complex conjugation.

Lemma 2.2. *Let V_1 and V_2 be antiunitary operators such that $V_1^2 = V_2^2 = I$. Then there exists a $U \in \mathscr{U}^+$ such that $UV_1U^{-1} = V_2$.*

Proof. Consider the unitary operator $W = V_2 V_1$. Suppose

$$W = \int_0^{2\pi} e^{i\lambda} P(d\lambda)$$

where P is a projection valued measure in $[0, 2\pi]$. Let

$$U = \int_0^{2\pi} e^{i\lambda/2} P(d\lambda) .$$

We have

$$I = V_2^2 = WV_1WV_1 .$$

Thus $W^{-1} = V_1 W V_1$. This implies

$$\int_0^{2\pi} e^{-i\lambda} P(d\lambda) = V_1 \int_0^{2\pi} e^{i\lambda} P(d\lambda) V_1$$

$$= \int_0^{2\pi} e^{-i\lambda} V_1 P(d\lambda) V_1 .$$

Hence $V_1\, P(E) V_1 = P(E)$ for all Borel sets $E \subset [0, 2\pi]$. Thus

$$\begin{aligned} V_1 U V_1 &= V_1 \int_0^{2\pi} e^{i\lambda/2} P(d\lambda) V_1 \\ &= \int_0^{2\pi} e^{-i\lambda/2} V_1 P(d\lambda) V_1 \\ &= \int_0^{2\pi} e^{-i\lambda/2} P(d\lambda) \\ &= U^{-1}. \end{aligned}$$

Now we have

$$U V_1 U^{-1} = U V_1 (V_1 U V_1) = W V_1 = V_2 .$$

Since U is unitary, this completes the proof of the lemma.

We shall now assign the UA group $\mathscr{U}$ the weak topology. A sequence U_n converges to U in $\mathscr{U}$ if and only if, for every pair $x, y \in \mathscr{H}$, $(U_n x, y) \to (Ux, y)$ as $n \to \infty$. It is easy to prove that this topology is equivalent to the strong topology. Thus $U_n \to U$ in $\mathscr{U}$ if and only if, for every $x \in \mathscr{H}$, $\|U_n x - Ux\| \to 0$ as $n \to \infty$. Introduce the metric

$$d(U, V) = \sum_{i=1}^{\infty} \frac{\|U x_i - V x_i\|}{2^i} \quad \text{for} \quad U, V \in \mathscr{U}$$

where $x_1, x_2, \ldots$ is any fixed orthonormal basis in $\mathscr{H}$. This is a left invariant metric in $\mathscr{U}$ which induces the required topology in $\mathscr{U}$ and makes it a complete and separable metric group.

Let $\mathscr{I} \subset \mathscr{U}$ be the subset $\{cI, |c| = 1\}$. Then $\mathscr{I}$ is a compact normal subgroup of $\mathscr{U}$. The quotient group $\mathscr{U}/\mathscr{I}$ is called the *projective unitary antiunitary group* or simply *the PUA group of* $\mathscr{H}$. We shall denote the PUA group by $\tilde{\mathscr{U}}$ and the canonical homomorphism from $\mathscr{U}$ onto $\tilde{\mathscr{U}}$ by $\sim$. Thus $\sim$ map sends a unitary or antiunitary operator U to the coset $\tilde{U} = U\mathscr{I}$ in $\tilde{\mathscr{U}}$. Endowed with the quotient topology, $\tilde{\mathscr{U}}$ becomes a separable metric group. The next lemma implies that $\tilde{\mathscr{U}}$ is actually a complete and separable metric group.

Lemma 2.3. *If G is a complete and separable metric group and $H \subset G$ is a closed subgroup, then the quotient space G/H of left cosets admits a metric which induces the quotient topology and makes G/H a complete and separable metric space.*

Proof[1]. We may assume that the metric d in G is right invariant. Let

$$N_k = \{x : d(g, e) < 2^{-k}\}, \qquad k = 1, 2 \ldots \tag{2.1}$$

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[1] This proof was suggested to the author by A. Tortrat.

be a decreasing sequence of symmetric open neighbourhoods of the identity e.

Let $\dot{g} = \pi g$ be the coset gH for every $g \in G$. π is the canonical map from G onto G/H which takes the point g to the coset containing g. Define

$$d^0(\dot{g}_1, \dot{g}_2) = \inf_{h \in H} d(g_1, g_2 h) .$$

It is easy to verify that d^0 defines a metric in G/H and induces the quotient topology. Now let $\dot{g}_n$ be a sequence of points in G/H such that

$$\lim_{n, m \to \infty} d^0(\dot{g}_n, \dot{g}_m) = 0 .$$

This implies that for every k, there exists an integer n_k such that $\{n_k\}$ is monotonic increasing and

$$\pi^{-1} g_n \subset N_k \pi^{-1} g_m \quad \text{for all} \quad n, m \geqq n_k . \tag{2.2}$$

Choose a sequence $g'_k \in \pi^{-1} g_{n_k}$ as follows: g'_1 is any element in $\pi^{-1} g_{n_1}$. By (2.2) $\pi^{-1} g_n \cap N_1 g'_1 \neq \emptyset$ for all $n \geqq n_1$. Hence choose $g'_2 \in \pi^{-1} g_{n_2} \cap N_1 g'_1$. Then $\pi^{-1} g_n \cap N_2 g'_2 \neq \emptyset$ for all $n \geqq n_2$. Choose $g'_3 \in \pi^{-1} g_{n_3} \cap N_2 g'_2$. Repeat this procedure.

It is clear that

$$g'_l \in N_{l-1} N_{l-2} \ldots N_k g'_k \quad \text{for all} \quad l > k .$$

By (2.1) and the triangle inequality we have

$$N_{l-1} N_{l-2} \ldots N_k \subset N_{k-1} .$$

Hence $g'_l \in N_{k-1} g'_k$ for all $l > k$. Since G is complete under the metric d it follows that $g'_k \to g \in G$ as $k \to \infty$. Thus $\dot{g}'_k \to \dot{g}$ in G/H. Since $\dot{g}'_k = \dot{g}_{n_k}$, it follows that $\dot{g}_n \to \dot{g}$ as $n \to \infty$. Thus every Cauchy sequence converges in G/H. This completes the proof.

Corollary 2.1. *The PUA group $\tilde{\mathcal{U}}$ of a complex separable Hilbert space is a complete and separable metric group. In particular, $\tilde{\mathcal{U}}$ with its Borel structure derived from its topology is a standard Borel space.*

Corollary 2.2. *There exists a one one Borel map η from $\tilde{\mathcal{U}}$ into $\mathcal{U}$ such that $\eta(U^{\sim})^{\sim} = U^{\sim}$.*

Proof. Since $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are complete and separable metric spaces and the map $U \to U^{\sim}$ is continuous, the existence of the map η follows from a theorem of Kuratowski (cf. [5], Theorem 3.9, page 21).

Remark. Since $\sim$ is an open map and $\mathcal{U}^+$ and $\mathcal{U}^-$ are both open closed subsets of $\mathcal{U}$, it follows that their images $\tilde{\mathcal{U}}^+$ and $\tilde{\mathcal{U}}^-$ are disjoint both open closed subsets of $\tilde{\mathcal{U}}$. $\tilde{\mathcal{U}}^+$ is a normal subgroup of $\tilde{\mathcal{U}}$. Further

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}^+ \cup \tilde{\mathcal{U}}^- ; \quad \tilde{\mathcal{U}}^+ \tilde{\mathcal{U}}^- = \tilde{\mathcal{U}}^- \tilde{\mathcal{U}}^+ = \tilde{\mathcal{U}}^- ; \quad \tilde{\mathcal{U}}^- \tilde{\mathcal{U}}^- = \tilde{\mathcal{U}}^+ .$$

§ 3. *PUA* Representations

Throughout this section let G denote a locally compact second countable group and $\tilde{\mathscr{U}}$ be the *PUA* group of a complex separable Hilbert space $\mathscr{H}$. We give G its natural Borel structure and $\tilde{\mathscr{U}}$ the topology described in § 2.

Definition 3.1. A Borel homomorphism from G into $\tilde{\mathscr{U}}$ is called a *projective unitary antiunitary representation* or simply a *PUA representation* of G in $\mathscr{H}$.

Lemma 3.1. *Any PUA representation of G is continuous. For a PUA representation* $g \to U_g^\sim$ *of G, the set* $G^+ = \{g : U_g^\sim \in \tilde{\mathscr{U}}^+\}$ *is a both open closed normal subgroup of G such that* G/G^+ *has at the most two elements. If* $G^- = \{g : U_g^\sim \in \tilde{\mathscr{U}}^-\}$, *then* G^+ *and* G^- *are the cosets which constitute the group* G/G^+.

Proof. By Corollary 2.1 to Lemma 2.3, $\tilde{\mathscr{U}}$ is a separable metric group. Hence by a well known result of Mackey (cf. [6], Theorem 2.2, page 10), the map $g \to U_g^\sim$ is continuous. By the Remark at the end of § 2, $\tilde{\mathscr{U}}^+$ is a both open closed normal subgroup of $\tilde{\mathscr{U}}$. Since G^+ is the inverse image of $\tilde{\mathscr{U}}^+$ through a continuous map, it is both open closed. The rest of the properties are quite straightforward to prove.

Corollary 3.1. *If in a PUA representation* $g \to U_g^\sim$ *of G, there exists a* g_0 *such that* $U_{g_0}^\sim \in \tilde{\mathscr{U}}^-$, *then G cannot be connected.*

Definition 3.2. The decomposition of G into G^+ and G^- occurring in Lemma 3.1 is called the *UA decomposition* of G associated with the *PUA* representation $g \to U_g^\sim$.

We shall now investigate how a *PUA* representation can be lifted to a "multiplier representation" in the *UA* group $\mathscr{U}$. Suppose $g \to U_g^\sim$ is a *PUA* representation of G. Making use of the cross section map η of Corollary 2.2 to Lemma 2.3 we can construct a measurable map $g \to \eta(U_g^\sim)$ from G into $\mathscr{U}$. Since $\eta(U_g^\sim)^\sim = U_g^\sim$, we may, without loss of generality, assume that $U_g = \eta(U_g^\sim)$. Then $g \to U_g$ is a measurable map and for any two elements $g_1, g_2 \in G$, $(U_{g_1} U_{g_2})^\sim = U_{g_1 g_2}^\sim$. Hence there exists a complex number $\sigma(g_1, g_2)$ of modulus unity such that

$$U_{g_1} U_{g_2} = \sigma(g_1, g_2)\, U_{g_1 g_2} \quad \text{for all} \quad g_1, g_2 \in G. \tag{3.1}$$

We may always take $U_e = I$ where e is the identity element of G. Then

$$\sigma(e, g) = \sigma(g, e) = 1 \quad \text{for all} \quad g \in G. \tag{3.2}$$

Suppose that $G = G^+ \cup G^-$ is the associated *UA* decomposition of the *PUA* representation $g \to U_g^\sim$. Computing $U_{g_1} U_{g_2} U_{g_3}$ in two different ways as $U_{g_1}(U_{g_2}U_{g_3})$ and $(U_{g_1}U_{g_2})U_{g_3}$, we obtain

$$\begin{aligned} U_{g_1}(U_{g_2}U_{g_3}) &= \sigma(g_1, g_2g_3)\,\sigma(g_2, g_3)\,U_{g_1g_2g_3} \quad \text{if} \quad g_1 \in G^+, \\ &= \sigma(g_1, g_2g_3)\,\bar{\sigma}(g_2, g_3)\,U_{g_1g_2g_3} \quad \text{if} \quad g_1 \in G^- \end{aligned}$$

and

$$(U_{g_1}U_{g_2})U_{g_3} = \sigma(g_1, g_2)\,\sigma(g_1g_2, g_3)\,U_{g_1g_2g_3} \quad \text{for all} \quad g_1, g_2, g_3 \in G.$$

Thus σ satisfies the following equation:

$$\left.\begin{aligned} \sigma(g_1, g_2)\,\sigma(g_1g_2, g_3) &= \sigma(g_1, g_2g_3)\,\sigma(g_2, g_3) \quad \text{if} \quad g_1 \in G^+, \\ &= \sigma(g_1, g_2g_3)\,\bar{\sigma}(g_2, g_3) \quad \text{if} \quad g_1 \in G^-. \end{aligned}\right\} \tag{3.3}$$

Considering the properties of σ in (3.2) and (3.3), we introduce the following definition.

Definition 3.3. Suppose G^+ is a both open closed normal subgroup of G such that the group G/G^+ has at the most two cosets G^+ and G^- (where G^- can be empty). A Borel function σ defined on $G \times G$ and taking values on the unit circle of the complex plane is called a *multiplier* with respect to G^+ if it satisfies Eqs. (3.2) and (3.3). A Borel map $g \to U_g$ from G into $\mathcal{U}$ is called a *multiplier representation* if there exists a multipler σ such that Eq. (3.1) is satisfied. In this case it is also called a *σ representation.*

Thus our previous discussion can be summed up in the form of a theorem.

Theorem 3.1. *Let G be a locally compact second countable group and $g \to U_g^\sim$ be a PUA representation of G. Then there exists a multiplier representation $g \to V_g$ of G such that $V_g^\sim = U_g^\sim$ for all $g \in G$. Conversely every multiplier representation $g \to V_g$ of G determines a PUA representation $g \to V_g^\sim$ of G.*

Definition 3.4. Let $g \to U_g^\sim$ be a *PUA* representation of G. Any multiplier representation $g \to V_g$ of G such that $V_g^\sim = U_g^\sim$ for all $g \in G$ is called a *version* of the given *PUA* representation.

Remark. Suppose now that $g \to V_g$ and $g \to W_g$ are two versions of a *PUA* representation with multipliers σ and σ'. Since $V_g^\sim = W_g^\sim$ for all $g \in G$, there exists a Borel function $a(g)$ on G such that $|a(g)| = 1$ and $W_g = a(g)\,V_g$ for all $g \in G$. Hence σ and σ' satisfy the following identity:

$$\begin{aligned} \sigma'(g_1, g_2) &= \frac{a(g_1)\,a(g_2)}{a(g_1g_2)}\,\sigma(g_1, g_2) \quad \text{if} \quad g_1 \in G^+, \\ &= \frac{a(g_1)\,\overline{a(g_2)}}{a(g_1g_2)}\,\sigma(g_1, g_2) \quad \text{if} \quad g_1 \in G^- \end{aligned} \tag{3.4}$$

where $G = G^+ \cup G^-$ is the associated UA decomposition.

Definition 3.5. A multiplier representation $g \to U_g$ of G is said to be *irreducible* if there exists no non trivial proper closed subspace invariant under all the $U_g, g \in G$, A PUA representation is said to be *irreducible* if it has an irreducible version.

Two multiplier representations $g \to U_g^{(i)}$, $i = 1, 2$ in Hilbert spaces $\mathscr{H}_i$, $i = 1, 2$ respectively are said to be equivalent if there exists a unitary or antiunitary operator U from $\mathscr{H}_1$ onto $\mathscr{H}_2$ such that $U U_g^{(1)} U^{-1} = U_g^{(2)}$ for all $g \in G$.

Two PUA representations are said to be equivalent if they have equivalent versions.

Remark. Two equivalent PUA representations of G have the same associated UA decomposition for the group G. If the equivalence is effected by a unitary operator, then they have versions with the same multiplier. If it is effected by an antiunitary operator, then they have versions whose multipliers σ_1 and σ_2 satisfy the equation $\sigma_2 = \bar{\sigma}_1$.

Because of the Remark after Definition 3.4 and the comments made above we shall introduce the following definition.

Definition 3.6. Two multipliers σ and σ' are said to be *trivially equivalent* if they are defined with respect to the same normal subgroup G^+ and there exists a Borel function $a(g)$ on G such that $|a(g)| = 1$ and Eq. (3.4) is satisfied. σ and σ' are said to be *equivalent* if either σ and σ' are trivially equivalent or $\bar{\sigma}$ and σ' are trivially equivalent.

§ 4. Imprimitivity Systems

Let G be a locally compact second countable group and $H \subset G$ be a closed subgroup. We shall denote by $X = G/H$ the homogeneous space of left cosets in which G acts transitively through left translation. Let x denote an arbitrary point of X and x_0 the point corresponding to the coset H. We shall denote by π the map from G onto X which takes the element g to the coset gH. The space X with the quotient topology becomes a locally compact second countable space. Let $\mathscr{B}_X$ be the Borel σ-field of X.

Following Mackey we shall define the notion of an "imprimitivity system".

Definition 4.1. Let $\mathscr{H}$ be a complex separable Hilbert space. By an *imprimitivity system* for G on X, we mean a pair consisting of a multiplier representation $g \to U_g$ of the group G and a projection valued measure $E \to P(E)$, $E \in \mathscr{B}_X$ in $\mathscr{H}$ such that

$$U_g P(E) U_g^{-1} = P(gE) \quad \text{for all} \quad g \in G, \ E \in \mathscr{B}_X .$$

We shall denote this system by $\{\mathscr{H}, U_g, P(E)\}$.

Remark. It is clear that the above definition is meaningful if X is replaced by any Borel space where G acts as a group of Borel automorphisms. We shall, however, confine our attention to the transitive case since other "ergodic G actions" have not been completely understood even in the unitary case. In this connection the reader is referred to Mackey [4].

Definition 4.2. An imprimitivity system $\{\mathscr{H}, U_g, P(E)\}$ for the group G on the homogeneous space X is said to be *irreducible* if there is no proper non zero subspace which is invariant under all the operators U_g and $P(E)$, $g \in G$, $E \in \mathscr{B}_X$.

Two imprimitivity systems $\{\mathscr{H}^i, U_g^{(i)}, P^{(i)}(E)\}$, $i = 1, 2$ for the group G on the homogeneous space X are said to be *equivalent* if there exists a unitary or antiunitary map V from $\mathscr{H}^1$ onto $\mathscr{H}^2$ such that

$$\begin{aligned} (VU_g^{(1)}V^{-1})^{\sim} &= U_g^{(2)\sim} \qquad \text{for all} \quad g \in G, \\ VP^{(1)}(E)V^{-1} &= P^{(2)}(E) \qquad \text{for all} \quad E \in \mathscr{B}_X. \end{aligned}$$

We shall now study the problem of classifying the imprimitivity systems of G up to equivalence. More or less we follow Mackey and use his results in many places. However, there are some differences and these will become clear as we proceed.

By a *quasi invariant measure* on $(X, \mathscr{B}_X)$ we mean a non zero σ-finite measure μ on $\mathscr{B}_X$ such that $\mu(E) = 0$ if and only if $\mu(gE) = 0$ for all $g \in G$, $E \in \mathscr{B}_X$. It is well known that quasi invariant measures exist on X and any two quasi invariant measures are equivalent in the sense of measure theory. We shall choose and fix a quasi invariant measure μ throughout our discussion in this section.

For every finite or countable cardinal n, $L_2(\mu, n)$ will stand for the Hilbert space which is the direct sum of n copies of the Hilbert space $L_2(\mu)$ of complex square integrable functions. Let $\mathfrak{C}^n$ denote the n dimensional complex Hilbert space if $n < \infty$ and the Hilbert space of square summable sequences of complex numbers, i.e.,

$$\left\{ \boldsymbol{a} : \boldsymbol{a} = (a_1, a_2, \ldots) : \sum_{i=1}^{\infty} |a_i|^2 < \infty \right\} \quad \text{if} \quad n = \infty .$$

We shall denote an arbitrary element of $\mathfrak{C}^n$ by $\boldsymbol{a}$ and the inner product in $\mathfrak{C}^n$ by $\langle .,. \rangle$. The inner product in $L_2(\mu, n)$ will be denoted by $(.,.)$. Any element of $L_2(\mu, n)$ can be written as $\boldsymbol{f}(x) = (f_1(x), f_2(x), \ldots)$ where $\int \sum_i |f_i(x)|^2 d\mu(x) < \infty$. The space $L_2(\mu, n)$ admits a *canonical projection valued measure* $P^0(E)$, $E \in \mathscr{B}_X$ defined by

$$P^0(E)\boldsymbol{f} = (\chi_E f_1, \chi_E f_2, \ldots)$$

for all $f = (f_1, f_2, \ldots) \in L_2(\mu, n)$ where χ_E is the indicator function of the set E.

We shall now prove a series of lemmas before proceeding to the statement of the main results.

Lemma 4.1. *Let $\{\mathscr{H}, U_g, P(E)\}$ be an imprimitivity system for G on X. Then there exists an equivalent imprimitivity system $\{L_2(\mu, n), V_g, P^0(E)\}$ where μ is a quasi invariant measure on X, n is a finite or countable cardinal and $P^0(E)$ is the canonical projection valued measure.*

Proof. Definition 4.1 implies that the projection valued measures $P(E)$ and $P(gE)$, $E \in \mathscr{B}_X$ are equivalent through the unitary or antiunitary operator U_g. Hence by Lemma 2.1, they are unitarily equivalent. Now an application of the Hahn-Hellinger theorem and the argument of Mackey [2] yield the proof of the lemma.

In the space $L_2(\mu, n)$, the complex conjugation which maps f to $\bar{f}$ is a canonical antiunitary operator. By the discussion in § 2, it follows that every antiunitary operator is the product of a unitary operator and this conjugation. Making use of this fact and following the arguments of Mackey [2] one can prove the following lemma.

Lemma 4.2. *Let $\{L_2(\mu, n), V_g, P^0(E)\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$ be the UA decomposition of G associated with the PUA representation $g \to V_g$. Then there exist functions $C(g, x)$ and $D(g, x)$ defined respectively on $G^+ \times X$ and $G^- \times X$ and taking values in the space of unitary operators in $\mathfrak{C}^n$ such that*

$$\left.\begin{aligned} (V_g f)(x) &= \left[\frac{d\mu}{d\mu^g}(g^{-1}x)\right]^{\frac{1}{2}} C(g, g^{-1}x) f(g^{-1}x) \quad \text{if} \quad g \in G^+, \\ &= \left[\frac{d\mu}{d\mu^g}(g^{-1}x)\right]^{\frac{1}{2}} D(g, g^{-1}x) \bar{f}(g^{-1}x) \quad \text{if} \quad g \in G^- \end{aligned}\right\} \tag{4.1}$$

where μ^g is the quasi invariant measure defined by the equation $\mu^g(E) = \mu(gE)$, $E \in \mathscr{B}_X$.

If G^- is empty then we have only a unitary imprimitivity system and this case has been studied in great detail by Mackey [2]. Hence we shall concentrate our attention on the case when $G^- \neq 0$. In such a situation two different types of imprimitivity systems arise according as the action of the subgroup G^+ on X is transitive or not.

a) The Case when G^+ Action in X is Transitive and $G^- \neq 0$

Suppose $\{L_2(\mu, n), V_g, P^0(E)\}$ is an imprimitivity such that the subgroup $G^+ = \{g : V_g \text{ is unitary}\}$ acts transitively on X. Since $G^+ G^- = G^-$, this implies that the stability subgroup H of the point x_0 has non empty intersection with G^-. Let

$$H^+ = H \cap G^+, \qquad H^- = H \cap G^-.$$

Then by Lemma 3.1, H^+ is a both open closed normal subgroup of H such that H/H^+ consists of two cosets H^+ and H^-. Choose and fix a point $h_0 \in H^-$. Then $G^- = G^+ h_0$, $H^- = H^+ h_0$. The element h_0 induces an automorphism of the group G^+ through the map $g \to h_0 g h_0^{-1}$. We shall write $h_0[g]$ for $h_0 g h_0^{-1}$. H^+ is invariant under this automorphism. Further $h_0^2 \in H^+$. The map π from G onto X is continuous and in particular measurable. Hence by a result of Kuratowski (cf. [5], Theorem 3.9, page 21) we can construct a one one measurable map γ from X into G such that $\pi\gamma$ is the identity map. Thus $\gamma(x)x_0 = x$ for all x. γ is called a *Borel cross section map* for π. We shall choose and fix such a cross section map throughout our discussion.

Lemma 4.3. *Let* $\{L_2(\mu, n), V_g, P^0(E)\}$ *be an imprimitivity system and let* σ *be the multiplier of the representation* $g \to V_g$. *Then there exists an equivalent imprimitivity system with the same* $L_2(\mu, n)$ *and the canonical projection valued measure* P^0 *such that the function* $C(g, x)$ *of Lemma 4.2 is given by the formula*

$$C(g, x) = \frac{\sigma(g, \gamma(x))}{\sigma(\gamma(gx), \gamma(gx)^{-1} g\gamma(x))} L_{\gamma(gx)^{-1} g\gamma(x)} \quad \text{if} \quad g \in G^+ \tag{4.2}$$

where $h \to L_h$ *is a* σ *unitary representation of the subgroup* H^+.

Proof. For the subgroup G^+ and its transitive action in the space X, the pair consisting of the σ representation $g \to V_g$ and the projection valued measure $E \to P^0(E)$ is a transitive imprimitivity system in the sense of Mackey. Every V_g, $g \in G^+$ is unitary. The first equation in (4.1) and the Eq. (4.2) simply express the fact that the σ representation $g \to V_g$ of G^+ is induced by the σ-representation $h \to L_h$ of the subgroup H^+. Hence Lemma 4.3 is just a restatement of Mackey's result [2]. This completes the proof.

From now onwards we shall tacitly assume that $C(g, x)$ is defined by (4.2). We shall choose the orthonormal basis $v_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ where 1 is written at the ith position in the vector for $i = 1, 2, \ldots$. In this basis we shall construct the matrices of the operators $C(g, x)$ and $D(g, x)$ in the Hilbert space $\mathfrak{C}^n$. Let us denote these matrices by the same symbols. Thus $C(g, x)$ and $D(g, x)$ will be $n \times n$ unitary matrices. For any matrix A, let $\bar{A}$ be the matrix obtained by taking the complex conjugate of every entry.

By writing a.e.x. we shall refer with respect to the quasi invariant measure μ on X. "Almost everywhere" statements in a group will always be relative to Haar measure. With these conventions we have the following lemma.

Lemma 4.4. *Let* $\{L_2(\mu, n), V_g, P^0(E)\}$ *be an imprimitivity system and let* σ *be the multiplier of* $g \to V_g$. *Suppose* V_g *is defined by* (4.1) *and* $C(g, x)$ *is defined according to* (4.2). *Then the matrix valued functions* $C(g, x)$ *and* $D(g, x)$ *satisfy the following equations:*

$$C(g_1, g_2 x) C(g_2, x) = \sigma(g_1, g_2)\, C(g_1 g_2, x) \qquad \textit{for all} \quad x \in X,\ g_1, g_2 \in G^+, \tag{4.3}$$

$$D(g_1, g_2 x)\, \overline{D(g_2, x)} = \sigma(g_1, g_2)\, C(g_1 g_2, x) \qquad \textit{a.e.x. for every} \quad g_1, g_2 \in G^-, \tag{4.4}$$

$$C(g_1, g_2 x)\, D(g_2, x) = \sigma(g_1, g_2)\, D(g_1 g_2, x) \qquad \textit{a.e.x. for every} \quad g_1 \in G^+,\ g_2 \in G^-, \tag{4.5}$$

$$D(g_1, g_2 x)\, \overline{C(g_2, x)} = \sigma(g_1, g_2)\, D(g_1 g_2, x) \qquad \textit{a.e.x. for every} \quad g_1 \in G^-,\ g_2 \in G^+. \tag{4.6}$$

Proof. The first equation follows by direct verification. The last three equations can be deduced easily from (4.1) and the equation $V_{g_1} V_{g_2} = \sigma(g_1, g_2) V_{g_1 g_2}$.

Lemma 4.5. *Under the same conditions as in Lemma 4.4, for every* $g \in G^+$, *the function* $D(g h_0, x)$ *is given by the equation*

$$D(g h_0, x) = \bar{\sigma}(g, h_0) C(g, h_0 x)\, D(x) \quad \textit{a.e.x.} \tag{4.7}$$

where

$$D(x) = a(x)\, L_{\gamma(h_0 x)^{-1} h_0[\gamma(x)]} \Gamma, \quad \textit{a.e.x.} \tag{4.8}$$

Γ *is a constant unitary matrix with the property*

$$L_{h_0[h]} = \alpha(h) \Gamma \bar{L}_h \Gamma^{-1} \quad \textit{for all} \quad h \in H^+, \tag{4.9}$$

$$\Gamma \bar{\Gamma} = c L_{h_0^2}. \tag{4.10}$$

and $a(x)$ *and* $\alpha(h)$ *are scalar functions of modulus unity.*

Proof. Put $g_2 = h_0$, $g_1 = g \in G^+$ in (4.5). Then we get (4.7) with $D(x) = D(h_0, x)$. For any matrix valued function $A(x)$ defined on X, let $A^0(g) = A(\pi(g))$. If A is measurable on X, then A^0 is measurable on G. Putting $g_1 = h_0$, $g_2 = g \in G^+$ in (4.6) we obtain

$$D(g x) = \sigma(h_0, g)\, D(h_0[g] h_0, x)\, \overline{C(g, x)}^{-1} \quad \text{a.e.x. for every} \quad g \in G^+.$$

By "lifting" both sides of this equation to the group G and using Fubini's theorem we obtain

$$D^0(g_1 g_2) = \sigma(h_0, g_1)\, \bar{\sigma}(h_0[g_1], h_0)\, C^0(h_0[g_1], h_0 g_2)\, D^0(g_2)\, \overline{C^0(g_1, g_2)}^{-1} \qquad \text{a.e.} \quad g_1 \in G^+,\ g_2 \in G.$$

Let $g_2 \in G^+$ be a fixed point such that the above equation holds a.e. $g_1 \in G^+$. Let $D^0(g_2) = \Gamma_1$ at that point. Then

$$D^0(g) = a_1(g)\, C^0(h_0[g g_2^{-1}], h_0 g_2)\, \Gamma_1 \overline{C^0(g g_2^{-1}, g_2)}^{-1} \quad \text{a.e.} \quad g \in G^+ \tag{4.11}$$

where $a_1(g)$ is scalar. By (4.3) we conclude the existence of another constant matrix Γ_2 and a scalar function $a_2(g)$ such that

$$D^0(g) = a_2(g)\, C^0(h_0[g], h_0)\, \Gamma_2 \overline{C^0(g, e)}^{-1} \quad \text{a.e.} \quad g \in G^+ .$$

Since $D^0(gh) = D^0(g)$ for every $h \in H^+$, we obtain from (4.2),

$$L_{h_0[h]} \Gamma_2 \bar{L}_h^{-1} = \alpha(h) \Gamma_2 , \qquad h \in H^+$$

for some scalar $\alpha(h)$. Putting $g_1 = g_2 = h_0$ in (4.4), and using (4.2), we have

$$\Gamma_2 \bar{\Gamma}_2 = c L_{h_0^2}$$

where c is a scalar of modulus unity. Writing $\Gamma_2 = \Gamma$ and putting $g = \gamma(x)$ in (4.11) we obtain the required result.

Lemma 4.6. *In the Hilbert space* $\mathfrak{C}^n$, *define the operators* M_h, $h \in H$ *as follows:*

$$M_h \boldsymbol{a} = L_h \boldsymbol{a} \quad \textit{if} \quad h \in H^+ , \ \boldsymbol{a} \in \mathfrak{C}^n$$
$$M_{h_0} \boldsymbol{a} = \overline{\sigma(h, h_0)} L_h \Gamma \bar{\boldsymbol{a}} \quad \textit{if} \quad h h_0 \in H^- , \ \boldsymbol{a} \in \mathfrak{C}^n$$

where L_h, $h \in H^+$ *and* Γ *are as in Lemma* 4.5. *Then the map* $h \to M_h$ *is a multiplier representation of the group* H *whose multiplier* $\tilde{\sigma}$ *is given by*

$$\begin{aligned} \tilde{\sigma}(h_1, h_2) &= \sigma(h_1, h_2) \quad && \textit{if} \quad h_1 \in H^+ , \ h_2 \in H \\ &= \chi(h_2) && \textit{if} \quad h_1 \in H^- , \ h_2 \in H \end{aligned}$$

where χ *is a character on* H.

Proof. That $h \to M_h$ defines a multiplier representation of H is an immediate consequence of Eqs. (4.9) and (4.10) of Lemma 4.5. Since $M_h = L_h$ for $h \in H^+$, it is clear that $\tilde{\sigma} = \sigma$ on $H^+ \times H^+$. If $h_1 \in H^+$ and $h_2 = h h_0$, $h \in H^+$, then for any $\boldsymbol{a} \in \mathfrak{C}^n$,

$$\begin{aligned} M_{h_1} M_{h h_0} \boldsymbol{a} &= L_{h_1} \bar{\sigma}(h, h_0) L_h \Gamma \bar{\boldsymbol{a}} \\ &= \bar{\sigma}(h, h_0)\, \sigma(h_1, h) L_{h_1 h} \Gamma \bar{\boldsymbol{a}} \\ &= \bar{\sigma}(h, h_0)\, \sigma(h_1, h)\, \sigma(h_1 h, h_0)\, M_{h_1 h h_0} \boldsymbol{a} \\ &= \sigma(h_1, h h_0) M_{h_1 h h_0} \boldsymbol{a} . \end{aligned}$$

Thus $\tilde{\sigma}(h_1, h h_0) = \sigma(h_1, h h_0)$. Since $\tilde{\sigma}$ and σ are multipliers with respect to the same normal subgroup $H^+ \subset H$, it follows that $\tilde{\sigma}\bar{\sigma}$ is also a multiplier for H with respect to H^+. Let $\phi = \tilde{\sigma}\bar{\sigma}$ on $H \times H$. We have $\phi(h_1, h_2) = 1$ if $h_1, h_2 \in H^+$. A straightforward computation now shows that $\phi(h_1 h_0, h_2)$

$= \phi(h_0, h_2)$ for all $h_1 \in H^+$ and $\phi(h_0, hh') = \phi(h_0, h)\, \phi(h_0, h')$ for all $h, h' \in H$. This completes the proof of the lemma.

Now we are ready to state and prove the first main theorem of this section.

Theorem 4.1. *Let G be a locally compact second countable group, $H \subset G$ a closed subgroup and $X = G/H$ the homogeneous space of left cosets. Let $\{\mathscr{H}, U_g, P(E)\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$ be the UA decomposition of G with respect to PUA representation $g \to U_g$, $g \in G$. Suppose that G^+ acts transitively on X and σ is the multiplier of the representation $g \to U_g$. Let γ be a one one Borel map from X into G^+ such that $\pi\gamma$ is the identity map of X onto itself. Then there exists an equivalent imprimitivity system $\{L_2(\mu, n), V_g, P^0(E)\}$ where*

$$(V_g f)(x) = \frac{\sigma(g, \gamma(g^{-1}x))}{\sigma(\gamma(x), \gamma(x)^{-1} g \gamma(g^{-1}x))} \left\{ \left(\frac{d\mu}{d\mu^g} \right) (g^{-1}x) \right\}^{\frac{1}{2}} \cdot M_{\gamma(x)^{-1} g \gamma(g^{-1}x)} f(g^{-1}x), \tag{4.12}$$

μ is a quasi invariant measure, n is a finite or countable cardinal, $h \to M_h$ is a σ representation of H and P^0 is the canonical projection valued measure on $\mathscr{B}_X$. This imprimitivity system is irreducible if and only if the σ representation $h \to M_h$ of H is irreducible.

Proof. Lemmas 4.1–4.3 and 4.5–4.6 imply the existence of an equivalent imprimitivity system $\{L_2(\mu, n), V_g, P^0(E)\}$ where V_g is given by the formula

$$V_g f(x) = \left\{ \frac{d\mu}{d\mu^g} (g^{-1}x) \right\}^{\frac{1}{2}} \frac{\sigma(g, \gamma(g^{-1}x))}{\sigma(\gamma(x), \gamma(x)^{-1} g \gamma(g^{-1}x))} \cdot \alpha(g, g^{-1}x)\, M_{\gamma(x)^{-1} g \gamma(g^{-1}x)} f(g^{-1}x)$$

where $\alpha(g, x)$ is a scalar function of modulus unity, $h \to M_h$ is a $\tilde{\sigma}$ representation of H and $\tilde{\sigma}$ is a multiplier of the form given by Lemma 4.6. Further $\alpha(g, x) = 1$ if $g \in G^+$. Writing the equation $V_{g_2} V_{g_1} = \sigma(g_2, g_1) V_{g_2 g_1}$ for $g_1, g_2 \in G^-$ and putting $y = g_2^{-1} x$, we obtain

$$\alpha(g_2, y) = \alpha(g_1, g_1^{-1} y)\, \chi(\gamma(y)^{-1} g_1 \gamma(g_1^{-1} y)) \quad \text{a.e.y.}$$

Thus $\alpha(g, y)$ is independent of $g \in G^-$. Writing $\alpha(y)$ for $\alpha(g, y)$ we have

$$\alpha(y)\, \overline{\alpha(g_1^{-1} y)} = \chi(\gamma(y)^{-1} g_1 \gamma(g_1^{-1} y)) \quad \text{a.e.y.}$$

Writing $\alpha^0(g) = \alpha(\pi(g))$, we obtain

$$\alpha^0(g)\, \overline{\alpha^0(g_1^{-1} g)} = \chi(\gamma(\pi(g))^{-1} g_1 \gamma(\pi(g_1^{-1} g)) \quad \text{a.e.g.}$$

Putting $g_1^{-1}g = g'$, and noting that χ is a homomorphism on H, we obtain

$$\alpha^0(g)\,\overline{\alpha^0(g')} = \chi([\gamma\pi(g)]^{-1}g)\,\overline{\chi([\gamma\pi(g')]^{-1}g')}\,.$$

Hence

$$\alpha^0(g) = c\chi([\gamma\pi(g)]^{-1}g) \quad \text{a.e.g.}$$

where c is a constant. Since $\alpha^0(gh) = \alpha^0(g)$ for every $h \in H$, it follows that $\chi(h) = 1$ for all $h \in H$ and α is a constant. Without loss of generality we may take this to be unity. Now $h \to M_h$ is indeed a σ representation of H. This completes the proof of the first part. The irreducibility part is proved exactly as in the unitary case.

Remark. 1. If $H^- = H \cap G^- \neq \emptyset$, then M_h is antiunitary for all $h \in H^-$. Further V_g is antiunitary for $g \in G^-$.

2. Changing the quasi invariant measure μ yields only an equivalent representation.

3. One may call the representation $g \to V_g$ defined by (4.12) as the σ representation *induced by* $h \to M_h$.

b) The Case when G^+ Action on X is not Transitive and $G^- \neq \emptyset$

In this case the stability subgroup H of the point x_0 (corresponding to the coset H) is contained in G^+. Hence $H^- = \emptyset$. Choose and fix a point $g_0 \in G^-$. The homogeneous space X splits into two orbits X^+ and X^- under the action of G^+ where $X^+ = G^+x_0$, $X^- = G^+(g_0x_0) = G^-x_0$. Let the restrictions of the quasi invariant measure μ to X^+ and X^- be μ^+ and μ^- respectively. The Hilbert space $L_2(\mu, n)$ splits into a direct sum $L_2(\mu^+, n) \oplus L_2(\mu^-, n)$. The point g_0 maps X^+ onto X^- in a one one manner. For any set $E \subset X$, let $E^+ = E \cap X^+$, $E^- = E \cap X^-$.

We can take $\mu^- = \mu^+ g_0^{-1}$ without loss of generality. We shall now adopt the matrix notation for an operator in $\mathscr{H}_1 \oplus \mathscr{H}_2$, i.e., an operator A in $\mathscr{H}_1 \oplus \mathscr{H}_2$ will be written as $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{ij} is an operator from $\mathscr{H}_j$ into $\mathscr{H}_i$, $i, j = 1, 2$. If P^0, P^+ and P^- are the canonical projection valued measures in $L_2(\mu, n)$, $L_2(\mu^+, n)$ and $L_2(\mu^-, n)$ respectively, we have

$$P^0(E) = \begin{pmatrix} P^+(E^+) & 0 \\ 0 & P^-(E^-) \end{pmatrix}, \quad E \in \mathscr{B}_X\,.$$

Let P_0 be the projection valued measure defined with respect to the Hilbert space $L_2(\mu^+, n) \oplus L_2(\mu^+, n)$ and given by the equation

$$P_0(E) = \begin{pmatrix} P^+(E^+) & 0 \\ 0 & P^+(g_0E^-) \end{pmatrix}, \quad E \in \mathscr{B}_X\,. \tag{4.13}$$

With these notations we have the following theorem.

Theorem 4.2. *Let G be a locally compact second countable group, $H \subset G$ a closed subgroup and $X = G/H$ the homogeneous space of left cosets. Let $\{\mathscr{H}, U_g, P(E)\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$ be the UA decomposition for the PUA representation $g \to U_g^{\sim}$ of G. Suppose that $H \subset G^+$, $G^- \neq \emptyset$ and $g_0 \in G^-$ is any fixed point. Let σ be the multiplier of $g \to U_g$. Then there exists an equivalent imprimitivity system $\{L_2(\mu^+, n) \oplus L_2(\mu^+, n), V_g, P_0(E)\}$ where V_g is given by the equations*

$$V_g = \begin{pmatrix} V_g^+ & 0 \\ 0 & \overline{\alpha(g)}\, s^+ V^+_{g_0[g]} s^+ \end{pmatrix} \quad \text{if} \quad g \in G^+$$

$$V_{g_0} = \begin{pmatrix} 0 & s^+ \\ \overline{\sigma(g_0, g_0)}\, s^+ V^+_{g_0^2} & 0 \end{pmatrix},$$

$$g_0[g] = g_0 g g_0^{-1}, \qquad g \in G^+,$$

$$\alpha(g) = \sigma(g_0, g)\, \overline{\sigma(g_0, g_0^{-1})}\, \sigma(g_0 g, g_0^{-1}), \qquad g \in G^+,$$

$g \to V_g^+$ is the unitary σ representation in $L_2(\mu^+, n)$ for the group G^+ induced by a unitary σ representation $h \to L_h$ of the subgroup H, s^+ denotes the complex conjugation in $L_2(\mu^+, n)$ and P_0 is the projection valued measure defined by (4.13). *This imprimitivity system is irreducible if and only if the σ representation $h \to L_h$ of H is irreducible.*

Proof. Using Lemmas 4.1–4.2 we can replace $\{\mathscr{H}, U_g, P(E)\}$ by $\{L_2(\mu, n), W_g, P^0(E)\}$ where the σ representation $g \to W_g$ is given by the right hand side of (4.1). By the comments made before the statement of the theorem, the Hilbert space $L_2(\mu, n)$ splits into a direct sum of two Hilbert spaces $L^+ = L_2(\mu^+, n)$ and $L^- = L_2(\mu^-, n)$. Since G^+ leaves X^+ and X^- invariant it follows from Mackey's theory [2] that W_g has the form

$$W_g = \begin{pmatrix} W_g^+ & 0 \\ 0 & W_g^- \end{pmatrix} \quad \text{if} \quad g \in G^+ \tag{4.14}$$

where $g \to W_g^{\pm}$ in $L^{\pm}$ are two σ representations of G^+, induced from H. Since g_0 maps X^+ onto X^- and vice versa and W_{g_0} is antiunitary, it follows from (4.1) that W_{g_0} has the form

$$W_{g_0} = \begin{pmatrix} 0 & Bs^- \\ As^+ & 0 \end{pmatrix} \tag{4.15}$$

where s^+ and s^- are complex conjugations in L^+ and L^- respectively and A and B are unitary maps from L^+ onto L^- and L^- onto L^+ respectively.

Since $W_g, g \in G^+$ and W_{g_0} generate a σ representation for some multiplier σ, it follows that

$$\left.\begin{aligned} W_{g_0}^2 &= \sigma(g_0, g_0)\, W_{g_0^2}, \\ W_{g_0} W_g W_{g_0}^{-1} &= \alpha(g)\, W_{g_0[g]} \quad \text{for all} \quad g \in G^+ \end{aligned}\right\} . \tag{4.16}$$

Substituting (4.14) and (4.15) in (4.16) we have

$$B s^- A s^+ = \sigma(g_0, g_0)\, W^+_{g_0^2},$$

$$(B s^-)\, W_g^- (B s^-)^{-1} = \alpha(g)\, W^+_{g_0[g]} .$$

Thus the σ-representation $g \to W_g^-$ is "antiunitarily" equivalent to the σ representation $g \to W^+_{g_0[g]}$. Consider the unitary map from $L^+ \oplus L^-$ onto $L^+ \oplus L^+$ which sends an element (f, f') to $(f, s^+ B s^- f')$. Through this map the σ representation $g \to W_g$ of G becomes equivalent to the σ representation $g \to V_g$ in $L^+ \oplus L^+$ given by

$$V_g = \begin{pmatrix} W_g^+ & 0 \\ 0 & \overline{\alpha(g)}\, s^+ W^+_{g_0[g]} s^+ \end{pmatrix} \quad \text{if} \quad g \in G^+, \tag{4.17}$$

$$V_{g_0} = \begin{pmatrix} 0 & s^+ \\ \overline{\sigma(g_0, g_0)}\, s^+ W^+_{g_0^2} & 0 \end{pmatrix} . \tag{4.18}$$

It is easy to verify that (4.17) and (4.18) determine a σ representation. Under this equivalence the canonical projection valued measure P^0 in $L^+ \oplus L^-$ is taken into the projection valued measure P_0 in $L^+ \oplus L^+$ defined by (4.13). Writing V instead of W in (4.17) and (4.18) we complete the proof of the first part.

In order to prove the second part we first observe that if the σ representation $h \to L_h$ of H is a direct sum of two σ representations, then the imprimitivity system described in the statement of the theorem decomposes into a direct sum of two systems. Thus, if L is reducible then the corresponding imprimitivity system is reducible. To prove the converse, let us assume that L is irreducible. Let

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

be an operator in $L^+ \oplus L^+$ which commutes with all $V_g, g \in G$ and $P_0(E)$, $E \in B_X$. The commutativity of K and $P_0(E)$ for all $E \in \mathscr{B}_X$ implies that $K_{12} = K_{21} = 0$. K_{11} and K_{22} commute with all $P^+(E)$, $E \subset X^+$. Further K_{11} and K_{22} commute with all $W_g, g \in G^+$. Hence from Mackey's imprimitivity theorem it follows that $K_{11} = aI$, $K_{22} = bI$ where a and b are scalars. The matrix $K = \begin{pmatrix} aI & 0 \\ 0 & bI \end{pmatrix}$ commutes with W_{g_0}. This implies that

$b=\bar{a}$. Thus the only projection operator which commutes with all W_g, $g\in G$ and $P_0(E)$, $E\in\mathscr{B}_X$ is either 0 or I. This completes the proof of the theorem.

§ 5. The *PUA* Representations of the Extended Poincaré Group

Following Mackey [2] and using the results of § 4 one can construct the irreducible *PUA* representations of any locally compact second countable group which is a "regular semidirect product" of a connected abelian group and a group of its automorphisms. We shall do this in the special case when the group under consideration is the semidirect product of the four dimensional real Euclidean group $\mathfrak{R}^4$ and the group of its automorphisms generated by $SL(2,\mathfrak{C})$, the space reflection and the time reflection. We shall call this *the extended Poincaré group*.

We shall now describe the extended Poincaré group in slightly greater detail. In order to do this we need the notion of a semidirect product.

Definition 5.1. Let N and H be locally compact second countable groups. Suppose H acts on N as a group of automorphisms such that the map $(n,h)\to h(n)$ from $N\times H$ into N is continuous. Then the product space $N\times H$ can be made into a locally compact second countable group by assigning the product topology and defining the multiplication operation as

$$(n_1,h_1)\,o(n_2,h_2)=(n_1h_1(n_2),h_1h_2),\qquad n_1,n_2\in N,\ h_1,h_2\in H\,.$$

The group obtained in this manner is called the *semidirect product* of N and H and is denoted by $N\odot H$.

Remark. Note that N and H can be considered as subgroups of $N\odot H$ in a natural manner. In such a case N is a closed normal subgroup and H is a closed subgroup.

The space $\mathfrak{R}^4$ can be identified with the space of all 2×2 Hermitian matrices through the map

$$\tau:\boldsymbol{x}=(x_0,x_1,x_2,x_3)\to\begin{pmatrix}x_0+x_1 & x_2+ix_3\\ x_2-ix_3 & x_0-x_1\end{pmatrix}.$$

Any $h\in SL(2,\mathfrak{C})$ induces an automorphism in the translation group $\mathfrak{R}^4$ as follows:

$$h:\boldsymbol{x}\to\tau^{-1}(h\tau(\boldsymbol{x})h^*)\,. \tag{5.1}$$

This is the well known covering map from $SL(2,\mathfrak{C})$ onto the proper Lorentz group. We shall denote by S and T, the space and time reflections defined in $\mathfrak{R}^4$ as follows:

$$\left.\begin{array}{l}S:(x_0,x_1,x_2,x_3)\to(x_0,-x_1,-x_2,-x_3),\\ T:(x_0,x_1,x_2,x_3)\to(-x_0,x_1,x_2,x_3).\end{array}\right\} \tag{5.2}$$

Let F be the finite group (consisting of four elements) generated by S and T. $SL(2,\mathfrak{C})$ acts on $\mathfrak{R}^4$ as a group of automorphisms according to (5.1). F acts on $SL(2,\mathfrak{C})$ and $\mathfrak{R}^4 \odot SL(2,\mathfrak{C})$ as follows:

$$\left.\begin{array}{r}S: h \to h^{*-1}\\ T: h \to h^{*-1}\\ ST: h \to h\end{array}\right\} \text{ if } \quad h \in SL(2,\mathfrak{C}),$$

$$\left.\begin{array}{r}S: (x,h) \to (Sx, h^{*-1}),\\ T: (x,h) \to (Tx, h^{*-1}),\\ ST: (x,h) \to (STx, h).\end{array}\right\} \text{ if } \quad (x,h) \in \mathfrak{R}^4 \odot SL(2,\mathfrak{C})$$

Thus one can form the semidirect products $[\mathfrak{R}^4 \odot SL(2,\mathfrak{C})] \odot F$ and $SL(2,\mathfrak{C}) \odot F$. The group $SL(2,\mathfrak{C}) \odot F$ acts on $\mathfrak{R}^4$ as a group of automorphisms (by successive application). Hence one can also form $\mathfrak{R}^4 \odot [SL(2,\mathfrak{C}) \odot F]$. Then we have the following lemma.

Lemma 5.1. $[\mathfrak{R}^4 \odot SL(2,\mathfrak{C})] \odot F = \mathfrak{R}^4 \odot [SL(2,\mathfrak{C}) \odot F]$.

Proof. The proof is straightforward and left to the reader.

Definition 5.2. The group $\mathscr{P} = \mathfrak{R}^4 \odot [SL(2,\mathfrak{C}) \odot F]$ is called the *extended Poincaré group.*

We shall now proceed to the description of all the multipliers of the extended Poincaré group. The connected component of the identity of $\mathscr{P}$ is $\mathscr{P}_0 = \mathfrak{R}^4 \odot SL(2,\mathfrak{C})$. It is a well known result of Bargmann and Whitehead (cf. [6] Theorem 5.5, page 34, and Corollary 1, page 51) that every multiplier of $\mathscr{P}_0$ is trivial. Since by Lemma 5.1 $\mathscr{P} = \mathscr{P}_0 \odot F$, any element $g \in \mathscr{P}$ can be uniquely written as $g^0 a$ where $g^0 \in \mathscr{P}_0$ and $a \in F$. An analysis similar to that of Mackey in the unitary situation (cf. Theorem 5.6, page 41 in [6]) and the fact that $\mathscr{P}_0$ has no nontrivial characters yields the following lemma.

Lemma 5.2. *Let ω be any multiplier for F. Then the function σ defined by*

$$\sigma_\omega(g_1^0 a_1, g_2^0 a_2) = \omega(a_1, a_2) \quad \textit{for all} \quad g_1^0, g_2^0 \in \mathscr{P}_0, \qquad a_1, a_2 \in F$$

is a multiplier for $\mathscr{P}$. Conversely every multiplier σ for the group is equivalent to a σ_ω for some multiplier ω of the group Γ.

Thus the problem of describing all the multipliers of $\mathscr{P}$ reduces to describing those of the finite group F. We recall from Definition 3.3 that every multiplier of F is defined with respect to a subgroup F^+ which has either two or four elements. Upto equivalence the multipliers of F are described by the following lemma.

Lemma 5.3. *Let Φ be a finite abelian group consisting of four elements e, a, b, c such that e is the identity, $a^2 = b^2 = c^2 = e$, $ab = c$, $bc = a$ and $ca = b$.*

If $\Phi^+ = \Phi$, *then every multiplier with respect to* Φ^+ *is equivalent to the one which is identically equal to unity or the multiplier* ω_0 *given by the following table:*

Table (i)

	e	a	b	c
e	1	1	1	1
a	1	1	i	$-i$
b	1	$-i$	1	i
c	1	i	$-i$	1

If $\Phi^+ = \{e, a\}$, *then every multiplier with respect to* Φ^+ *is equivalent to a multiplier* $\omega^a_{\alpha\beta}$ *given by the following table:*

Table (ii)

	e	a	b	c
e	1	1	1	1
a	1	1	1	1
b	1	$\alpha\beta$	α	β
c	1	$\alpha\beta$	α	β

$\alpha = \pm 1, \ \beta = \pm 1$.

Proof. Since the proof is a straightforward algebra we leave it to the reader.

Remark. The extended Poincaré group $\mathscr{P}$ has four distinct normal subgroups $\mathscr{P}^+$ such that $\mathscr{P}/\mathscr{P}^+$ has at the most two elements. They are given by

(1) $\mathscr{P}^+ = \mathscr{P}$

(2) $\mathscr{P}^+ = [\mathfrak{R}^4 \odot SL(2, \mathfrak{C})] \odot F_a$, $a = S, T$ or ST

where F_a denotes the subgroup of F which contains the identity and the element a.

By Lemmas 5.2–5.3, there are two multipliers upto equivalence in the first case. One of them is identically equal to unity and other σ_0 is given by

$$\sigma_0(g_1^0 a_1, g_2^0 a_2) = \omega_0(a_1, a_2) \quad \text{for all} \quad g_1^0, g_2^0 \in \mathscr{P}_0, \qquad a_1, a_2 \in F$$

where ω_0 is determined by Table (i) of Lemma 5.3.

In the second case there are four multipliers $\sigma^a_{\alpha\beta}, \alpha = \pm 1, \beta = \pm 1$, given by

$$\sigma^a_{\alpha\beta}(g_1^0 a_1, g_2^0 a_2) = \omega^a_{\alpha\beta}(a_1, a_2) \quad \text{for all} \quad g_1^0, g_2^0 \in \mathscr{P}_0, \qquad a_1, a_2 \in F$$

where $\omega^a_{\alpha\beta}$ is determined by Table (ii) of Lemma 5.3. Since a can be any one of S, T and ST we obtain twelve distinct multipliers in this case.

Thus, on the whole, there are fourteen inequivalent multipliers for the group $\mathscr{P}$.

We shall now proceed to describe the irreducible multiplier representations of $\mathscr{P}$ by following the method of Mackey [2]. Let $G = SL(2, \mathfrak{C}) \odot F$. Then $\mathscr{P} = \mathfrak{R}^4 \odot G$. The G action in $\mathfrak{R}^4$ introduces a natural action in the character group R^4 of $\mathfrak{R}^4$. Under this action R^4 splits into various orbits. Let O be a typical orbit and H be the stability subgroup of some fixed point in O. The orbit O can then be considered as the homogeneous space G/H of left cosets. We shall denote an arbitrary point of O by $\boldsymbol{p} = (p_0, p_1, p_2, p_3)$ and define the value of the character $\boldsymbol{p}$ at $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ as $\exp - i(p_0 x_0 - p_1 x_1 - p_2 x_2 - p_3 x_3)$. Any point of $\mathscr{P}$ can be denoted by $(\boldsymbol{x}, g)$ where $\boldsymbol{x} \in \mathfrak{R}^4$ and $g \in G$. Let σ be a multiplier of $\mathscr{P}$ satisfying $\sigma((\boldsymbol{x}_1, g_1), (\boldsymbol{x}_2, g_2)) = \sigma(g_1, g_2)$ for all $(\boldsymbol{x}_1, g_1)$, $(\boldsymbol{x}_2, g_2)$. Then σ defines a multiplier for G. Suppose $\{\mathscr{H}, V_g, P(E)\}$ is an imprimitivity system for the group G on O where the representation $g \to V_g$ has the multiplier σ. Define the operator $U_{\boldsymbol{x},g}$ by

$$U_{(\boldsymbol{x},g)} = \{\int (\exp - i[p_0 x_0 - p_1 x_1 - p_2 x_2 - p_3 x_3])\, P(d\boldsymbol{p})\}\, V_g\,. \quad (5.3)$$

Then $(\boldsymbol{x}, g) \to U_{(\boldsymbol{x},g)}$ is a σ representation of $\mathscr{P}$. This is irreducible if and only if the imprimitivity system $\{\mathscr{H}, V_g, P(E)\}$ is irreducible. Every multiplier representation of $\mathscr{P}$ is determined in this way upto equivalence.

Following the procedure outlined above we shall now classify the "positive mass" representations of $\mathscr{P}$. Define for every $m > 0$ the set $O_m \subset R^4$ by

$$O_m = \{\boldsymbol{p} : p_0 \neq 0, p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2\}\,.$$

O_m is an orbit of the group G in R^4. Indeed, it is the orbit generated by the point $(m, 0, 0, 0)$. The stability subgroup of the point $(m, 0, 0, 0)$ is generated by SU_2 and the space reflection S. In fact, it is the direct product of SU_2 and the space reflection S. In fact, it is the direct product of SU_2 and F_S where F_S is the group consisting of the identity and S only.

The irreducible multiplier representations corresponding to the orbit O_m are now determined by the irreducible imprimitivity systems for G on O_m. According to Theorems 4.1 and 4.2 such systems are in turn determined by the irreducible multiplier representations of $SU_2 \times F_S$.

We shall now describe the invariant measure in O_m and a cross section map γ from O_m into G. To this end, define

$$O_m^+ = \{\boldsymbol{p} : \boldsymbol{p} \in O_m, p_0 > 0\}\,,$$
$$O_m^- = \{\boldsymbol{p} : \boldsymbol{p} \in O_m, p_0 < 0\}\,.$$

There exists a G invariant measure μ on O_m given by

$$\begin{aligned} d\mu(\boldsymbol{p}) &= \frac{dp_1\, dp_2\, dp_3}{p_0} \quad \text{if} \quad \boldsymbol{p} \in O_m^+\,, \\ &= \frac{dp_1\, dp_2\, dp_3}{-p_0} \quad \text{if} \quad \boldsymbol{p} \in O_m^-\,. \end{aligned}$$

For any $\boldsymbol{p} \in O_m$, define

$$\tau'(\boldsymbol{p}) = \begin{pmatrix} p_0 + p_1 & p_2 + i p_3 \\ p_2 - i p_3 & p_0 - p_1 \end{pmatrix}.$$

If $\boldsymbol{p} \in O_m^+$, $\tau'(\boldsymbol{p})$ is a positive definite Hermitian matrix. If $\boldsymbol{p} \in O_m^-$, $\tau'(\boldsymbol{p})$ is negative definite. Since every positive definite matrix has a unique positive definite square root, we can define

$$\gamma^+(\boldsymbol{p}) = m^{-\frac{1}{2}} [\tau'(\boldsymbol{p})]^{\frac{1}{2}} \quad \text{if} \quad \boldsymbol{p} \in O_m^+ .$$

Then γ^+ is a mapping from O_m^+ into $SL(2, \mathfrak{C})$. If $\boldsymbol{p} \in O_m^-$, then $T\boldsymbol{p} \in O_m^+$. Hence we can define

$$\gamma^-(\boldsymbol{p}) = T\gamma^+(T\boldsymbol{p}) \quad \text{if} \quad \boldsymbol{p} \in O_m^-$$

where the product of T and $\gamma^+(T\boldsymbol{p})$ is taken in the group G. Now we write

$$\begin{aligned} \gamma(\boldsymbol{p}) &= \gamma^+(\boldsymbol{p}) \quad \text{if} \quad \boldsymbol{p} \in O_m^+ , \\ &= \gamma^-(\boldsymbol{p}) \quad \text{if} \quad \boldsymbol{p} \in O_m^- . \end{aligned}$$

It is now easy to verify that γ is a cross section map, i.e., γ is a one one Borel map from O_m into G such that $\gamma(\boldsymbol{p})(m, 0, 0, 0) = (p_0, p_1, p_2, p_3)$.

Representations arrived at by formula (5.3) on the basis of the orbit O_m are called representations with mass m. We shall now classify these according to the multipliers described in the Remark after Lemma 5.3 and the irreducible σ representations L of $H = SU_2 \times F_S$.

Case 1. $\mathscr{P}^+ = \mathscr{P}$, $\sigma \equiv 1$.

In this case L is an irreducible unitary representation of H. In particular $L_S^2 = I$ and

$$L_u L_S = L_S L_u = L_{uS} \quad \text{for all} \quad u \in SU_2 .$$

Thus $L_S = \pm I$ and $u \to L_u$ is a spin j representation of SU_2 where j is either an integer or half integer. For any spin j, the representations with $L_S = +I$ and $L_S = -I$ are inequivalent.

Case 2. $\mathscr{P}^+ = \mathscr{P}$, $\sigma = \sigma_0$.

Since $\sigma_0(S, S) = 1$, and L_S is unitary we have $L_S^2 = I$; $L_u L_S = L_S L_u$. Thus $L_S = \pm I$ and the required representation of $\mathscr{P}$ is given by (5.3) where the σ representation $g \to V_g$ is induced by the σ representation L.

Case 3. $\mathscr{P}^+ = [\mathfrak{R}^4 \odot SL(2, \mathfrak{C})] \odot F_S$, $\sigma = \sigma_{\alpha\beta}^S$.

In this case the stability subgroup H is included in $\mathscr{P}^+$. In (5.3), the σ representation $g \to V_g$ is determined according to Theorem 4.2. L is determined by a spin j representation for SU_2 and $L_S = \pm I$ if $\alpha = +1$ and $L_S = \pm iI$ if $\alpha = -1$.

Case 4a. $\mathscr{P}^+ = [\mathfrak{R}^4 \odot SL(2, \mathfrak{C})] \odot F_T$, $\sigma = \sigma_{\alpha\beta}^T$, $\alpha = +1$.

In this case $H \cap \mathscr{P}^- \neq \emptyset$. In (5.3), the σ representation $g \to V_g$ is induced by a σ representation L of H where L_S is antiunitary. Since

$\sigma(S,S)=1$, $L_S^2=I$. By Lemma 2.2, L_S is a conjugation. Since $L_u L_S = L_S L_u$ for all $u \in SU_2$, $u \to L_u$ has a real character. Hence the spin must be an integer. L_S can now be chosen to be the complex conjugation in the basis in which the matrix entries of $u \to L_u$ are real.

Case 4b. $\mathscr{P}^+ = [\mathfrak{R}^4 \odot SL(2,\mathfrak{C})] \odot F_T$, $\sigma = \sigma_{\alpha\beta}^T$, $\alpha = -1$.

In (5.3), the σ representation $g \to V_g$ is induced by a σ representation L of H where L_S is antiunitary. Further

$$L_S^2 = -I, \qquad L_S L_u = L_u L_S, \qquad u \in SU_2$$

Suppose $u \to L_u$ is a spin j representation for SU_2. If j is an integer there is a vector $\boldsymbol{v}$ such that $L_u \boldsymbol{v} = \boldsymbol{v}$ for all u in a one parameter subgroup K of SU_2.

If $\boldsymbol{v}'$ is another vector such that $L_u \boldsymbol{v}' = \boldsymbol{v}'$ for all $u \in K$, then $\boldsymbol{v}'$ is a constant multiple of $\boldsymbol{v}$. Hence $L_S \boldsymbol{v} = c\boldsymbol{v}$ where c is a scalar of modulus unity. We have

$$-\boldsymbol{v} = L_S^2 \boldsymbol{v} = L_S(c\boldsymbol{v}) = \bar{c} L_S \boldsymbol{v} = \boldsymbol{v}\,.$$

Hence $\boldsymbol{v} = 0$. This implies that j cannot be an integer. It is now easy to show that in a weight basis, L_S is given by

$$L_S = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix} s, \qquad P = \begin{pmatrix} 0 & 0 & \dots & & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & & 0 \end{pmatrix}$$

where P is a $j+\frac{1}{2} \times j+\frac{1}{2}$ matrix and s is complex conjugation in the chosen basis.

Cases 5a and 5b. $\mathscr{P}^+ = [\mathfrak{R}^4 \odot SL(2,\mathfrak{C})] \odot F_{ST}$, $\sigma = \sigma_{\alpha\beta}^{ST}$, $\alpha = \pm 1$.

The representations are determined exactly as in cases 4a and 4b. The only difference is in replacing T by ST.

References

1. Foldy, L.: Synthesis of covariant particle equations. Phys. Rev. **102**, 568—581 (1956).
2. Mackey, G. W.: Unitary representations of group extensions I. Acta Math. **99**, 265—311 (1958).
3. — Mathematical foundations of quantum mechanics. New York: Benjamin 1963.
4. — Group representations. Mimeographed notes of lectures delivered at the Oxford Mathematics Institute 1966—67.
5. Parthasarathy, K. R.: Probability measures on metric spaces. New York: Academic Press 1967.
6. — Multipliers on locally compact groups, Vol. 93, Lecture Notes in Mathematics. Berlin-Heidelberg-New York: Springer 1969.

7. Streater, R. F., Wightman, A. S.: PCT, spin, statistics, and all that. New York: Benjamin 1964.
8. Varadarajan, V. S.: Geometry of quantum theory, Vol. I. Princeton: Van Nostrand 1968.
9. Von Neumann, J.: Mathematical foundations of quantum mechanics. Princeton: University Press 1955 (English translation).
10. Wigner, E. P.: On the unitary representations of the inhomogeneous Lorentz group. Ann. Math. **40**, 149—204 (1939).

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