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Stability of General Relativistic Gaseous Masses and Variational Principles*

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Abstract. The Einstein field equations for a self-gravitating fluid that obeys an equation of state of the form $p = p(w)$, p the pressure and w the energy density may be derived from a variational principle. The perturbations of the metric tensor and the fluid dynamic variables satisfy equations which may be derived from a related variational principle, namely the principle associated with the "second variation problem." It is shown that the variational principle given by Chandrasekhar from which a sufficient criterion may be obtained for deciding when a self gravitating spherical gaseous mass is unstable against spherically symmetric perturbations is that given by the "second variation problem". It is further shown that this criterion is equivalent to requiring that the integral entering into the second variation be negative. The latter form of the criterion may be used in general situations.

1. Introduction

It is the purpose of this paper to apply the variational principle [1] obeyed by self-gravitating fluids which satisfy an equation of state to the discussion of the stability against radial perturbations of a spherically symmetric distribution of such a fluid. We shall show that the variational principle given by Chandrasekhar [2] for determining the stability of a spherically symmetric self gravitating gaseous mass is given by the "second variation problem" associated with the principle referred to above.

Such a result is to be expected for it is well knwon that the equations satisfied by perturbations of solutions of the Euler equations of a variational principle are the Euler equations of another variational principle – the second variation problem. The two variational problems are related as follows: Let $\mathscr{L}(\phi^A; \phi^A_{,\mu})$ be a scalar density formed from some scalar or tensor fields ϕ^A and the derivatives of these fields with respect to the coordinates in space time,

$$\phi^A_{,\mu} = \frac{\partial \phi^A}{\partial x^\mu} \qquad \begin{array}{l} \mu = 1, 2, 3, 4 \\ A = 1, 2, \ldots, N . \end{array}$$

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Then

$$I = \int \mathscr{L}(\phi^A; \phi^A_{,\mu})\, d^4x \tag{1.1}$$

where the integral is carried out over an arbitrary four volume in space time determines a variational principle in the following sense. We assume that the ϕ^A are functions of the x^μ and a parameter e, thus

$$\phi^A = \phi^A(x; e).$$

Then I is also a function of e and we may require that

$$I'(0) = \left.\frac{dI}{de}\right|_{e=0} = 0$$

for arbitrary

$$\phi'^A(0) = \left(\frac{d\phi^A}{de}\right)_{e=0}.$$

We have

$$\begin{aligned} I'(e) &= \int \left[\frac{\partial \mathscr{L}}{\partial \phi^A}\phi'^A + \frac{\partial \mathscr{L}}{\partial \phi^A_{,\mu}}(\phi^A_{,\mu})'\right] d^4x \\ I'(e) &= \int \left[\frac{\partial \mathscr{L}}{\partial \phi^A}\phi'^A + \frac{\partial \mathscr{L}}{\partial \phi^A_{,\mu}}(\phi'^A)_{,\mu}\right] d^4x \end{aligned} \tag{1.2}$$

since

$$\phi'^A = \frac{\partial \phi^A}{\partial e}$$

$$(\phi^A_{,\mu})' = \frac{\partial^2 \phi^A}{\partial x^\mu\, \partial e} = \frac{\partial^2 \phi^A}{\partial e\, \partial x^\mu}.$$

On integrating the above expression for $I'(e)$ by parts we obtain

$$I'(e) = \int F_A(e)\, \phi'^A(e)\, d^4x + \int \left(\frac{\partial \mathscr{L}}{\partial \phi^A_{,\mu}}\phi'^A\right)_{,\mu} d^4x. \tag{1.3}$$

where

$$F_A(e) = \frac{\partial \mathscr{L}}{\partial \phi^A} - \left(\frac{\partial \mathscr{L}}{\partial \phi^A_{,\mu}}\right)_{,\mu}. \tag{1.4}$$

The second integral in Eq. (1.3) may be written as an integral over the hypersurface bounding the four-volume of integration.

The requirement that $I'(0) = 0$ for arbitrary $\phi'^A(0)$, in particular, for those that vanish on the boundary of the region of integration then leads to the Euler equations

$$F_A(0) = \left[\frac{\partial \mathscr{L}}{\partial \phi^A} - \left(\frac{\partial \mathscr{L}}{\partial \phi^A_{,\mu}}\right)_{,\mu}\right]_{e=0} = 0. \tag{1.5}$$

The equations satisfied by the difference between two "almost equal" solutions of these equations, or the equations satisfied by perturbations of solutions to these Euler equations are

$$F'_A(0) = \left(\frac{dF_A}{de}\right)_{e=0} = 0$$

where $F_A(e)$ is given by Eq. (1.4). Thus

$$F'_A(0) = \left[\frac{\partial^2 \mathscr{L}}{\partial^2 \phi^A \partial\phi^B}\phi'^B + \frac{\partial^2 \mathscr{L}}{\partial\phi^A \partial\phi^B_{,\mu}}\phi'^B_{,\mu} - \left(\frac{\partial^2 \mathscr{L}}{\partial\phi^A_{,\mu}\partial\phi^B}\phi'^B + \frac{\partial^2 \mathscr{L}}{\partial\phi^A_{,\mu}\partial\phi^B_{,\nu}}\phi'^B_{,\nu}\right)_{,\mu}\right]_{e=0} = 0\,. \tag{1.6}$$

These are a set of linear equations for the variables $\phi'^A(0)$ whose coefficients depend on the $\phi^A(x;0)$ and their derivatives. The $\phi'^A(0)$ are called the perturbations and the $\phi^A(x;0)$, the unperturbed solutions.

Now it follows from Eq. (1.2) that

$$\begin{aligned} I''(e) = &\int F_A\,\phi''^A\,d^4x + \int F'_A\,\phi'^A\,d^4x + \int\left(\frac{\partial\mathscr{L}}{\partial\phi^A_{,\mu}}\phi''^A\right)_{,\mu} d^4x \\ &+ \int\left[\frac{\partial^2\mathscr{L}}{\partial\phi^A\,\partial\phi^B_{,\mu}}\phi'^A\,\phi'^B + \frac{\partial^2\mathscr{L}}{\partial\phi^A_{,\mu}\,\partial\phi^B_{,\nu}}\phi'^A_{,\mu}\,\phi'^B\right]_{,\nu} d^4x \end{aligned} \tag{1.7}$$

or

$$\begin{aligned} I''(e) = &\int F_A\,\phi''^A\,d^4x + \int\left(\frac{\partial\mathscr{L}}{\partial\phi^A_{,\mu}}\phi''^A\right)_{,\mu} d^4x \\ &+ \int\left[\frac{\partial^2\mathscr{L}}{\partial\phi^A\partial\phi^B}\phi'^A\phi'^B + \frac{2\partial^2\mathscr{L}}{\partial\phi^A\partial\phi^B_{,\mu}}\phi'^A\phi'^B_{,\mu} + \frac{\partial^2\mathscr{L}}{\partial\phi^A_{,\mu}\partial\phi^B_{,\nu}}\phi'^A_{,\mu}\phi'^B_{,\nu}\right] d^4x\,. \end{aligned} \tag{1.8}$$

From Eq. (1.8) we have

$$I''(0) = \int\left[\frac{\partial^2\mathscr{L}}{\partial\phi^A\partial\phi^B}\phi'^A\phi'^B + \frac{2\partial^2\mathscr{L}}{\partial\phi^A\partial\phi^B_{,\mu}}\phi'^A\phi'^B_{,\mu} + \frac{\partial^2\mathscr{L}}{\partial\phi^A_{,\mu}\partial\phi^B_{,\nu}}\phi'^A_{,\mu}\phi'^A_{,\nu}\right] d^4x \tag{1.9}$$

when the $\phi^A(0)$ are such that $F_A(0) = 0$, that is the $\phi^A(0)$ are unperturbed solutions of the Euler equations associated with I, and the $\phi''^A(0) = 0$, on the boundary of the region of integration. If we now consider the ϕ'^A (not the ϕ^A) functions of x and a parameter f we may define

$$J(f) = I''(0)$$

and examine the Euler equations resulting from the condition

$$\left(\frac{dJ}{df}\right)_{f=0} = 0\,.$$

This is the "second variation problem." We find

$$\delta J = 2\int F'_A\,\delta\phi'^A\,d^4x + 2\int\left(\frac{\partial^2\mathcal{L}}{\partial\phi^A\,\partial\phi^B_{,\mu}}\,\phi'^A\,\delta\phi'^B\right)_{,\mu} d^4x$$
$$+2\int\left(\frac{\partial^2\mathcal{L}}{\partial\phi^A_{,\mu}\,\partial\phi^B_{,\nu}}\,\phi'^A_{,\mu}\,\delta\phi^B\right)_{,\nu} d^4x\,.$$

Hence for variations of the ϕ'^A such that ϕ'^A vanish on the boundary of the region of integration, $I''(0)$ takes on extreme values when the ϕ'^A satisfy the equations

$$F'_A(0)=0\,,$$

the equations satisfied by the perturbations.

Thus the solutions ϕ^A of the Euler equations $F_A(\phi)=0$, when considered as $\phi^A(x;0)$ are such that $I'(0)=0$, for ϕ'^A which vanish on the boundary of the region of integration, and the solutions ϕ'^A of the equations $F'_A(\phi;\phi')=0$, where the ϕ^A satisfy the Euler equations and are coefficients in the linear differential equations, are such that $I''(0)$ takes on extreme values.

2. Comoving Coordinates

The Einstein field equations for a self-gravitating fluid are

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=-kT^{\mu\nu} \tag{2.1}$$

where

$$T^{\mu\nu}=(w+p)\,u^\mu u^\nu-g^{\mu\nu}p\,, \tag{2.2}$$

p is the pressure, w is the energy density, k is the Einstein constant of gravitation and u^μ which is required to satisfy

$$u^\mu u_\mu=1 \tag{2.3}$$

is the four-velocity of the fluid. We shall assume that an equation of state exists, that is

$$p=p(w)\,. \tag{2.4}$$

It has been shown in Ref. [1], that under this assumption, Eqs. (2.1) may be derived as the Euler equations of the variational principle based on the integral

$$I=-\int(R+2kp)\sqrt{g}\,d^4x \tag{2.5}$$

in which R is the scalar curvature of space-time and the pressure p is regarded as a function of the $g_{\mu\nu}$, the metric tensor of space-time.

In deriving this result it was convenient to use comoving coordinates in the family of space-times $g_{\mu\nu}(x; e)$ which arise in discussing such a variational principle. We note that in applying the discussion of the previous section to equations (2.1) to (2.5), we must interpret the ϕ^A as representing the scalars w and p, the four-velocity vector u^μ and the metric tensor $g_{\mu\nu}$. Thus we are considering a one parameter family of space-times with metrics $g_{\mu\nu}(x; e)$ and in each of which there is a congruence of courves determined by the solutions of the ordinary differential equation

$$\frac{dx^{*\mu}}{ds} = u^{*\mu}(x^*; e) \tag{2.6}$$

where the $x^{*\mu}$ are the labels assigned to events in space-time in an arbitrary coordinate system. In this coordinate system the four-velocity vector has components $u^{*\mu}$ and the metric tensor has components $g^*_{\mu\nu}$.

We may write the solutions of Eqs. (2.6) as

$$x^{*\mu} = x^{*\mu}(\xi^i, s; e) \qquad (i = 1, 2, 3) \tag{2.7}$$

where

$$x^{*\mu}_0 = x^{*\mu}(\xi^i, 0; e)$$

are required to be the parametric equations of a hypersurface $\Sigma(e)$. The four variables ξ^i, s which we shall denote as x^μ, form a comoving coordinate system in each of the space-times. Eqs. (2.7) which may be written more generally as

$$x^{*\mu} = x^{*\mu}(x; e) \tag{2.8}$$

with

$$\begin{aligned} x^i &= \xi^i, \\ x^4 &= x^4(\xi^i, s), \end{aligned} \tag{2.9}$$

may be regarded as the transformation between the x^* coordinate system and a general comoving one which uses the x^μ as labels for events. Eq. (2.6) is then to be understood as

$$\frac{\partial x^{*\mu}}{\partial s} = u^{*\mu}(x^*(x; e); e) \tag{2.10}$$

where in the partial differentiation the x^i are kept constant for when these variables are fixed a particular world-line is selected.

In the general comoving coordinate system we have the components of the four velocity vector given by

$$u^\mu(x) = u^{*\sigma}(x^*)\frac{\partial x^\mu}{\partial x^{*\sigma}} = \frac{\partial x^\mu}{\partial s}$$

as follows from Eqs. (2.10). Hence

$$u^{\mu} = \frac{\delta_4^{\mu}}{\sqrt{g_{44}}} \tag{2.11}$$

where $g_{\mu\nu}$ are the components of the metric tensor in the comoving coordinate system. Eqs. (2.11) are a consequence of Eqs. (2.9) and (2.3).

We shall be using a comoving coordinate system in each space-time of the one parameter family of space-times with which we shall be concerned. In a particular one of these the metric tensor in the comoving coordinate system will be written as $g_{\mu\nu}(x; e)$. The tensor

$$g'_{\mu\nu}(x; e) = \frac{\partial g_{\mu\nu}}{\partial e}, \tag{2.12}$$

with x^{μ} kept constant will measure the change in the metric tensor evaluated in the comoving coordinate system at an event labelled by the coordinates x^{μ}, produced by a change in the parameter e. Similar statements will apply to other tensor fields which depend on e. In particular we shall have

$$u'^{\mu} = -\frac{1}{2} u^{\mu} \frac{g'_{44}}{g_{44}} = -\frac{1}{2} u^{\mu} g'_{\sigma\tau} u^{\sigma} u^{\tau}. \tag{2.13}$$

That is, the transformations given by Eqs. (2.8) for various values of e, produce comoving coordinates in each of the space-times associated with that value of e.

We shall use the notation

$$\dot{V}^{*\mu}(x^*; e) = \frac{\partial V^{*\mu}}{\partial e} \tag{2.14}$$

with $x^{*\mu}$ kept constant, where the $V^{*\mu}$ are the components of a vector field in a general coordinate system using the labels $x^{*\mu}$. It is of interest to determine the relation between V'^{μ} and $\dot{V}^{\mu}$. To do this we define

$$\xi^{*\mu} = \frac{\partial x^*}{\partial e} \tag{2.15}$$

where $x^{*\mu}$ is given as a function of x and e by equations (2.8) and x^{μ} is kept constant under the differentiation. Since

$$\frac{\partial x^{*\mu}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{*\varrho}} = \delta_{\varrho}^{\mu}$$

must hold for all values of e, it follows from the differentiation of this equation with respect to e keeping x^{μ} fixed that

$$\frac{\partial}{\partial e}\left(\frac{\partial x^{\nu}}{\partial x^{*\varrho}}\right) = -\frac{\partial x^{\nu}}{\partial x^{*\mu}} \frac{\partial \xi^{*\mu}}{\partial x^{*\varrho}} \tag{2.16}$$

and we have used the fact that

$$\frac{\partial}{\partial e}\left(\frac{\partial x^{*\nu}}{\partial x^{\varrho}}\right) = \frac{\partial}{\partial x^{\varrho}} \xi^{*\nu} . \tag{2.17}$$

From the transformation law of vectors we have

$$V^{\mu}(x; e) = V^{*\nu}(x^{*}(x; e); e) \frac{\partial x^{\mu}}{\partial x^{*\nu}} .$$

On differentiating this equation with respect to e, keeping x^{μ} fixed we find

$$V'^{\mu} = \left(\frac{\partial V^{*\nu}}{\partial x^{*\varrho}} \xi^{*\varrho} + \dot{V}^{*\nu}\right) \frac{\partial x^{\mu}}{\partial x^{*\nu}} + V^{*\mu} \frac{\partial}{\partial e}\left(\frac{\partial x^{\mu}}{\partial x^{*\nu}}\right).$$

In virtue of Eq. (2.16) we may write this as

$$V'^{\mu} = (\dot{V}^{*\nu} + \mathscr{L}_{\xi^{*}} V^{*\nu}) \frac{\partial x^{\mu}}{\partial x^{*\nu}} \tag{2.18}$$

where

$$\mathscr{L}_{\xi^{*}} V^{*\nu} = V^{*\nu}{}_{;\varrho} \xi^{*\varrho} - \xi^{*\nu}{}_{;\varrho} V^{*\varrho} \tag{2.19}$$

and is of course the Lie derivative of the vector $V^{*\nu}$ with respect to $\xi^{*\mu}$. It may be shown by similar arguments that for any tensor the operation of taking the prime derivative of the tensor components differs from the transform of taking the dot derivative by the appropriate Lie derivative of the tensor.

In particular for a scalar we have

$$f'(x; e) = \dot{f}^{*} + \frac{\partial f^{*}}{\partial x^{\varrho *}} \xi^{*\varrho} \tag{2.20}$$

where

$$f^{*}(x^{\mu}; e) = f(x(x^{*}; e); e) .$$

3. The Equations $T^{\mu\nu}{}_{;\nu} = 0$

These equations are consequences of the Bianchi identities and the Einstein field equations, Eqs. (2.1). When Eqs. (2.2) hold they may be written as

$$w_{,\nu} u^{\nu} + (w + p) u^{\mu}{}_{;\mu} = 0 \tag{3.1}$$

and

$$(w + p) u^{\lambda}{}_{;\nu} u^{\nu} = p_{,\nu} h^{\nu\lambda} \tag{3.2}$$

where

$$h^{\nu\lambda} = g^{\nu\lambda} - u^{\nu} u^{\lambda} . \tag{3.3}$$

When Eq. (2.4) holds, that is, when an equation of state exists, we may define a thermodynamic variable $\sigma(w)$ by the equations

$$\frac{d\sigma}{\sigma}=\frac{dw}{w+p}. \tag{3.4}$$

Eq. (3.1) then becomes

$$(\sigma u^{\mu});\mu=0.$$

In the comoving coordinate system we then have

$$\sqrt{-g}\,\sigma=\sqrt{g_{44}}\,f(x^i) \qquad i=1,2,3 \tag{3.5}$$

where f is a function of x^1, x^2, x^3 but not of x^4. Eq. (3.5) is the integrated form of the equation of conservation of σ.

It has been shown in [1], that Eqs. (3.2) may also be integrated in the comoving coordinate system to give

$$g_{44}=e^{2\phi}, \tag{3.6}$$

$$g_{4i}=e^{2\phi}\,C_i(x^j) \qquad i,j=1,2,3 \tag{3.7}$$

where

$$e^{\phi}=\frac{\sigma}{w+p}. \tag{3.8}$$

Thus in this coordinate system we have

$$u^{\mu}=e^{-\phi}\,\delta^{\mu}_{4} \tag{3.9}$$

and

$$u_{\mu}=e^{\phi}\,C_{\mu}(x^i) \tag{3.10}$$

with

$$C_{\mu}=g_{\mu 4}/g_{44}.$$

In particular we have $C_4=1$ and C_i functions of the variables x^i alone.

The three functions $C_i(x)$ are simply related to the vorticity and hence to the rotation as may be seen by evaluating the vector

$$v^{\mu}=\frac{1}{\sqrt{-g}}\,\varepsilon^{\mu\nu\sigma\tau}\,u_{\nu}u_{\sigma,\tau}$$

in the comoving coordinate system.

When

$$C_{i,j}-C_{j,i}=0$$

that is, when

$$C_i=\psi_{,i}$$

and the flow is irrotational, we may find a transformation of the form

$$\bar{x}^4 = x^4\,,$$
$$\bar{x}^i = \bar{x}^i(x^j)\,,$$

which preserves the comoving character of the coordinate system and in which the metric tensor is such $\bar{g}_{4i} = 0$.

The subsequent discussion will be carried out in a comoving coordinate system for in this system we shall be able to make immediate use of the integrals obtained above. We may reformulate the discussion to apply to an arbitrary coordinate system by using the discussion of the preceding section. We observe that Eqs. (3.4), (3.8), and (3.6) enable us to express the pressure p as a function of g_{44} in the comoving coordinate system.

4. The Variational Principle

We now turn to a discussion of the integral defined by Eq. (2.5), namely

$$I = -\int (R + 2kp)\sqrt{-g}\, d^4x\,. \tag{4.1}$$

We shall evaluate $I(e)$ and its derivatives with respect to e by using the comoving coordinate systems discussed above. We shall also use the fact that in such a coordinate system Eqs. (3.4), (3.6), and (3.8) obtain. Hence

$$p' = -(w+p)\,\phi' = -(w+p)\frac{1}{2}\frac{g'_{44}}{g_{44}} = -(w+p)\frac{1}{2}\, g'_{\sigma\tau} u^\sigma u^\tau\,. \tag{4.2}$$

Since

$$(\sqrt{-g})' = \tfrac{1}{2}\sqrt{-g}\, g^{\sigma\tau} g'_{\sigma\tau}\,, \tag{4.3}$$

$$R' = -R^{\sigma\tau} g'_{\sigma\tau} + g^{\sigma\tau} R'_{\sigma\tau}\,, \tag{4.4}$$

and

$$R'_{\sigma\tau} = (\Gamma'^{\nu}_{\sigma\nu}\,\delta^\lambda_\tau - \Gamma'^{\lambda}_{\sigma\tau})_{;\lambda} \tag{4.5}$$

where

$$\Gamma'^{\lambda}_{\sigma\tau} = \tfrac{1}{2} g^{\lambda\sigma}(g'_{\sigma\varrho;\tau} + g'_{\tau\varrho;\sigma} - g'_{\sigma\tau;\varrho})\,, \tag{4.6}$$

it follows that

$$I'(e) = +\int (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + k T^{\mu\nu})\sqrt{-g}\, g'_{\mu\nu}\, d^4x$$
$$- \int \sqrt{-g}\, \lambda^\sigma{}_{;\sigma}\, d^4x$$

where

$$\lambda^\lambda = g^{\sigma\tau}(\Gamma'^{\nu}_{\sigma\nu}\,\delta^\lambda_\tau - \Gamma'^{\lambda}_{\sigma\tau})$$

or

$$\lambda^\lambda = (g^{\lambda\varrho} g^{\sigma\tau} - g^{\lambda\tau} g^{\sigma\varrho})\, g'_{\sigma\tau;\varrho} \tag{4.7}$$

and $T^{\mu\nu}$ is given by Eq. (2.2).

Hence Eqs. (2.1) are the Euler equations determined by the requirement that $I'(0)=0$ for arbitrary $g'_{\mu\nu}$ which vanish on the boundary of the region of integration. Further we have

$$I''(0)= + \int (G'^{\mu\nu} + k\,T'^{\mu\nu})\sqrt{-g}\,g'_{\mu\nu}\,d^4x - \int [\sqrt{-g}(\lambda^\sigma)_{;\sigma}]'\,d^4x \quad (4.8)$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu} R$$

and the prime denotes the derivative with respect to e in the comoving coordinate system. The $g_{\mu\nu}(x,0)$ which enter into the integral are required to satisfy Eqs. (2.1). The first integral in the right hand side of Eq. (4.8) is quadratic in $g'_{\mu\nu}$ and its derivatives.

The extreme values of $I''(0)$ are attained for the $g'_{\mu\nu}$ which satisfy the linear equations

$$G'^{\mu\nu} + k\,T'^{\mu\nu} = 0\,, \quad (4.9)$$

the perturbed Einstein equations. The explicit form of $G'^{\mu\nu}$ may be calculated from the results given above. The calculation of $T'^{\mu\nu}$ in the comoving coordinate system involves the evaluation of u'^{μ}, v' and w' as is evident from Eqs. (2.2). We have already discussed the evaluation of the first two quantities. We shall make an additional assumption which will aid in the evaluation of the fluid quantity w'. Namely we shall assume that the perturbations in the fluid motion are adiabatic. That is, if the entropy of a given element of the unperturbed motion is S then for this element in the perturbed motion the entropy is still S or in other words $S'=0$.

For a general fluid motion the pressure p may be expressed as a function of the energy density w and the entropy S. Hence in general we have

$$p' = \left(\frac{\partial p}{\partial w}\right) w' + \left(\frac{\partial p}{\partial S}\right) S'$$

or

$$p' = \alpha^2 w' + \left(\frac{\partial p}{\partial S}\right)_w S'$$

where

$$\alpha^2 = \left(\frac{\partial p}{\partial w}\right)_S$$

is the ratio of the velocity of sound to the special theory of relativity velocity of light. Our assumption then implies that

$$p' = \alpha^2 w'\,. \quad (4.10)$$

It should be noted that the assumption that an equation of state exists for the unperturbed flow means that we are assuming that

$$p(0) = p(w, S; 0) = p(w)\,.$$

That, is

$$\left(\frac{\partial p}{\partial S}\right)_{e=0} = 0\,.$$

Hence the derivatives of w with respect to the coordinates for $e=0$ can be related to those of p. Thus

$$(p(0))_{,\mu} = \alpha^2 (w(0))_{,\mu}\,. \tag{4.11}$$

5. The Spherically Symmetric Case

In a spherically symmetric space-time we may write the line element as

$$ds^2 = e^{2\phi}\,dt^2 - e^{2\psi}\,dr^2 - e^{2\mu}\,d\Omega^2 \tag{5.1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\,dx^2\,, \tag{5.2}$$

ϕ, ψ and μ are functions of r, t and a parameter e and these are comoving coordinates for each value of the parameter e. It then follows [3] that the non-vanishing components of $G^{\mu}{}_{\nu}$ are

$$\begin{aligned}
-\left(R^4_4 - \frac{R}{2}\right) &= e^{2\phi}(\mu_t^2 + 2\mu_t\psi_t) - e^{-2\psi}(2\mu_{rr} + 3\mu_r^2 - 2\mu_r\psi_r) + e^{-2\mu}\\
-\left(R^1_1 - \frac{R}{2}\right) &= e^{-2\phi}(2\mu_{tt} + 3\mu_t^2 - 2\mu_t\phi_t) - e^{-2\psi}(\mu_r^2 + 2\mu_r\phi_r) + e^{-2\mu}\\
-\left(R^2_2 - \frac{R}{2}\right) &= -\left(R^3_3 - \frac{R}{2}\right)\\
&= e^{-2\phi}[\psi_{tt} + \mu_{tt} + \mu_t^2 + \psi_t^2 - \psi_t\phi_t + \mu_t(\psi_t - \phi_t)]\\
&\quad - e^{-2\phi}[\phi_{rr} + \mu_{rr} + \mu_r^2 + \phi_r^2 - \phi_r\psi_r + \mu_r(\phi_r - \psi_r)]\\
R^4_1 &= 2e^{-2\phi}[\mu_{rt} - \mu_t\phi_r - \mu_r\psi_t + \mu_t\mu_r]\\
R^1_4 &= -2^{-2\psi}[\mu_{rt} - \mu_t\phi_r - \mu_r\psi_t + \mu_t\mu_r]
\end{aligned} \tag{5.3}$$

where the subscripts r and t denote the derivatives with respect to these variables.

The Einstein equations become

$$\begin{aligned}
-F_\phi &\equiv (R^4_4 - \tfrac{1}{2}R) + kw = 0\,,\\
-F_\psi &\equiv (R^1_1 - \tfrac{1}{2}R) - kp = 0\,,\\
-F_\mu &\equiv (R^2_2 - \tfrac{1}{2}R) - kp = 0\,,
\end{aligned} \tag{5.4}$$

and

$$R_4^1 = 0. \tag{5.5}$$

The four Eqs. (5.4) and (5.5), are not all independent in view of the Bianchi identities. It may be shown that the solution of these equations is determined by the solution of Eq. (5.5) and $F_\psi = 0$ for a range of values of t and of $F_\phi = 0$ for $t = 0$.

The unperturbed solution we shall consider will be assumed to be static, that is ϕ, ψ and μ will be assumed to be functions of r alone. In that case it is no restriction to take

$$\mu = \log r.$$

Eq. (5.5) is identically satisfied and Eqs. (5.4) reduce to

$$\begin{aligned} \frac{1}{r^2} - e^{-2\psi}\left(\frac{1}{r^2} - \frac{2}{r}\psi_r\right) &= k w, \\ \frac{1}{r^2} - e^{-2\psi}\left(\frac{1}{r^2} + \frac{2}{r}\phi_r\right) &= -k p, \\ e^{-2\psi}\left[\phi_{rr} + \phi_r^2 - \phi_r \psi_r + \frac{1}{r}(\phi_r - \psi_r)\right] &= k p. \end{aligned} \tag{5.6}$$

It is a consequence of these equations that

$$2e^{-2\psi}(\phi_r + \psi_r) = k(w+p)r. \tag{5.7}$$

It is a further consequence of Eqs. (5.6) that

$$(w+p)\,\phi_r = -p_r. \tag{5.8}$$

The last equation also follows from the equation of state assumption.

The equations satisfied by the perturbations, ϕ', ψ' and μ' are obtained by differentiating Eqs. (5.4) and (5.5) with respect to e and setting $e = 0$. We then obtain from Eq. (5.5) and the last of (5.4) the equation

$$\mu'_{rt} - \mu'_t \phi_r - \frac{1}{r}\psi'_t + \frac{1}{r}\mu'_t = 0 \tag{5.9}$$

where now ϕ_r is determined by Eqs. (5.6). The solution of Eq. (5.9) is given by

$$\psi' - \psi'_0 = e^{\phi}(e^{-\phi} r \mu')_r \tag{5.10}$$

where

$$\psi'_0 = \psi'(r, 0)$$

and we have chosen our comoving coordinates so that

$$\mu'_0 = \mu'(r, 0) = 0. \tag{5.11}$$

This can always be achieved by a coordinate from transformation involving r alone.

The function ϕ' may be evaluated by using the integral of the field equations given by Eq. (3.5) which holds for all values of e. That equation may be written as

$$\sigma e^{\psi+2\mu} = \sigma_0 e^{\psi_0+2\mu_0} \tag{5.12}$$

where now the subscript zero on $f(r, t, e)$ is defined by

$$f_0 = f(r, 0; e).$$

On differentiating Eq. (5.12) with respect to e, setting $e=0$ and using Eqs. (3.4), (4.10), and (5.11) we obtain

$$\alpha^{-2}\phi' - \alpha_0^{-2}\phi_0' = (3\mu' + r\mu_r' - \phi_r r\mu') \tag{5.13}$$

or

$$\alpha^{-2}\phi' - \alpha_0^{-2}\phi_0' = r^{-2} e^{\phi}(r^3\mu' e^{-\phi})_r \tag{5.14}$$

where α is the velocity of sound in the unperturbed fluid, ϕ is given as above and ϕ_0' is $\phi'(r, 0)$ for $e=0$.

Thus ϕ' and ψ' are determined in terms of μ'. This function may be determined by solving the equation

$$F_\psi' = 0.$$

The quantities ϕ' and ψ' enter into this equation but may be eliminated by means of Eqs. (5.10) and (5.14). We shall discuss this equation in the next section

When the field equations, Eqs. (5.4) and (5.5) are applied to a problem in which there exists a hypersurface in space-time across which the stress-energy tensor is discontinuous, the equations must be supplemented by conditions satisfied by the metric tensor, its derivatives and the stress energy tensor on this hypersurface. Thus for the problem we wish to consider, namely that of a gas occupying a limited region of space-time and bounded by a vacuum there exists the hypersurface Σ defined by

$$r = r_b$$

where r_b is the constant comoving coordinate of the boundary element of the material.

It is well known (cf. [4]) that the conditions referred to above become in this case

$$p(r_b, t) = 0$$

and that ϕ, ψ, and μ are continuous across the hypersurface Σ. In addition all first derivatives of these quantities except ψ_r must be continuous across Σ. These conditions must hold for the perturbed as well as for the unperturbed equations. Hence we must have

$$p'(r_b, t) = -[(w+p)\,\phi']_{r=r_b} = 0. \tag{5.15}$$

In veiw of Eq. (5.14) this condition becomes a boundary condition on the function μ'.

Another condition is the requirement that for the perturbed and the unperturbed solutions the function

$$R = e^{\mu} = r = 0$$

at the origin. This function is the analogue of the Eulerian coordinate of an element of the fluid which has the Lagrange coordinate r. Hence we must have

$$R' = e^{\mu}\,\mu' = r\,\mu' = 0 \tag{5.16}$$

at the origin.

Eqs. (5.15) and (5.16) provide boundary conditions for the second order partial differential equation $F_{\psi'} = 0$.

We close this section with a discussion of the implication of the Bianchi identities.

If we define

$$-K^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \tfrac{1}{2}\,\delta^{\mu}{}_{\nu}\,R + k\,T^{\mu}{}_{\nu},$$

these identities are

$$K^{\mu}_{\nu;\mu} = \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,K^{\mu}_{\nu})_{,\mu} - K^{\mu}_{\varrho}\,\Gamma^{\varrho}_{\mu\nu} = 0.$$

They hold for all values of e. If the above equations are differentiated with respect to e and then evaluated for $e=0$, and if it is assumed that $K^{\mu}{}_{\nu}(x;0)=0$, it follows that

$$\frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,K'^{\mu}{}_{\nu})_{,\mu} - K'^{\mu}{}_{\varrho}\,\Gamma^{\varrho}_{\mu\nu} = 0 \tag{5.17}$$

where now $g_{\mu\nu} = g_{\mu\nu}(x;0)$ is the unperturbed metric, and $\Gamma^{\varrho}_{\mu\nu}$ is determined from this metric and this metric satisfies the field equations.

We now evaluate Eqs. (5.17) for the case considered above, when the unperturbed metric is spherically symmetric and static and the perturbed metric depends on time but is still spherically symmetric. In that case Eqs. (5.17) reduce to two equations corresponding to $\nu=4$ and $\nu=1$. These are

$$r^2\,e^{\psi}(F'_{\phi})_t - (e^{\psi}\,r^2\,R'^{1}_{4})_r = 0 \tag{5.18}$$

and

$$-r^2 (R'^4_1)_t + (r^2 e^{\phi} F'_{\psi})_r e^{-\phi} - 2r F'_{\mu} = 0 \tag{5.19}$$

respectively.

Hence when $R'^1_4 = (R'^1_4)_t = 0$ as is the case when Eq. (5.10) holds, Eq. (5.18) becomes

$$F'_{\phi}(r, t) = F'_{\phi}(r, 0) \tag{5.20}$$

and Eq. (5.19) becomes

$$2r F'_{\mu} = (r^2 e^{\phi} F'_{\psi})_r e^{-\phi}. \tag{5.21}$$

The first of these equations implies that the equation $F'_{\phi} = 0$ is only a restriction on the functions ϕ'_0 and ψ'_0. It may be verified that on substituting Eqs. (5.10) into the expression for F'_{ϕ} one obtains

$$-F'_{\phi} = \frac{2}{r^2} (r e^{-2\psi} \psi'_0)_r + \frac{k}{\alpha^2} (w + p) \phi'_0 = 0. \tag{5.22}$$

6. The Equation $F'_{\psi} = 0$

The equation $F'_{\psi} = 0$ is derived by differentiating the second of Eqs. (5.4) into which Eqs. (5.3) have been substituted setting $e = 0$, and making use of the values of the unperturbed solution. One then obtains

$$\begin{aligned} F'_{\psi} = 2\Bigg[& e^{-2\phi} \mu'_{tt} - \mu'_r e^{-2\psi} \left(\frac{1}{r} + \phi_r \right) - \frac{\mu'}{r^2} - \frac{1}{r} e^{-2\psi} \phi'_r \\ & + e^{-2\psi} \psi' \left(\frac{1}{r^2} + \frac{2}{r} \phi_r \right) \Bigg] - k(w+p)\phi' = 0 \end{aligned} \tag{6.1}$$

when Eqs. (5.10) and (5.14) are used to express ψ' and ϕ' in terms of μ', one finds that

$$\begin{aligned} \frac{r e^{2\psi}(w+p)}{2} F'_{\psi} = {} & e^{2\psi - 2\phi}(w+p) \xi_{tt} + \frac{4}{r} p_r \xi - \frac{1}{w+p} p_r^2 \xi \\ & - e^{-\psi - 2\phi} \left[e^{3\phi + \psi} \frac{(w+p)\alpha^2}{r^2} (e^{-\phi} r^2 \xi)_r \right]_r \\ & + k e^{2\psi}(w+p) p \xi + \psi'_0 \left(\frac{1}{r^2} + \frac{2}{r} \phi_r \right) - e^{-\psi - \phi} (e^{\phi + \psi} \phi'_0)_r \end{aligned} \tag{6.2}$$

where

$$\xi = r \mu'. \tag{6.3}$$

The equation $F'_{\psi} = 0$ where F'_{ψ} is given by equation (6.2), has a boundary conditions Eqs. (5.15) and (5.16). It is the equation given in [3] for the case of the radial perturbations of a self gravitating fluid when the equation of state was such that the fluid was isentropic. In that case $\sigma = \varrho$, the rest mass density of the fluid. When $\phi_0' = \psi_0' = 0$, the equation is the same as the equation given by Chandrasekhar [2] as may be seen by writing

$$(w+p)\,\alpha^2 = \gamma p\,,$$

and thus defining γ. This definition of γ is that given by Chandrasekhar as may be verified by writing

$$w = N(1 + u(p, N))$$

where u is the internal energy. If one then computes $(\partial p/\partial w)_\beta$ and remembers that

$$T\,dS = du + p\,d\left(\frac{1}{N}\right)$$

one verifies that the definition of γ given above is that used by Chandrasekhar.

7. The Evaluation of $I''(0)$

In this section we shall use the results obtained above to express $I''(0)$ in terms of μ', ψ_0', and ϕ_0'. We begin by observing that when I is defined by Eq. (4.1) and when the perturbed and unperturbed metrics are of the form given by Eq. (5.1), then it is sufficient for the purpose of calculating $I'(e)$ and $I''(e)$ to evaluate $I(e)$ in the coordinate system in which Eq. (5.1) holds.

Thus we have

$$\frac{1}{8\pi} I(e) = -\iint \{e^{\phi+\psi} + e^{\phi-\psi+2\mu}(\mu_r^2 + 2\mu_r\phi_r) - e^{-\phi+\psi+2\mu}(\mu_t^2 + 2\mu_t\psi_t) \\ + kpe^{\phi+\psi+2\mu} - (e^{-\phi}(e^{2\mu+\psi})_t)_t + (e^{-\psi}(e^{2\mu+\phi})_r)_r\}\,dr\,dt\,. \tag{7.1}$$

Hence

$$\frac{1}{8\pi} I'(e) = -\iint \{e^{\phi+\psi+2\mu}(F_\phi\phi' + F_\psi\psi' + 2F_\mu\mu')\,dr\,dt - S(e) \tag{7.2}$$

where F_ϕ, F_ψ, and F_μ are defined by Eqs. (5.4) and (5.3) and

$$S(e) = \iint (A_t - B_r)\,dr\,dt \tag{7.3}$$

with

$$A = e^{-\phi+\psi+2\mu}(-\phi'(2\mu_t + \psi_t) + \psi'\psi_t + 2\mu'\mu_t + 2\mu_t' + \psi_t')\,, \tag{7.4}$$

$$B = e^{\phi-\psi+2\mu}(-\psi'(2\mu_r + \phi_r) + \phi'\phi_r + 2\mu'\mu_r + 2\mu_r' + \phi_r')\,. \tag{7.5}$$

The integration in Eqs. (7.1) to (7.3) may be taken to be the region bounded by the inequalities

$$0 \leqq r \leqq \infty \qquad 0 \leqq t \leqq t_1 . \tag{7.6}$$

Across the boundary

$$r = r_b . \tag{7.7}$$

There is a discontinuity in the stress energy tensor. The pressure p must be continuous at $p = r_b$ but the energy density w need not be. The requirement that

$$\frac{1}{8\pi} I'(0) = 0$$

for arbitrary ϕ', ψ', and μ' which vanish together with their derivatives on the boundary of the region given by the inequalities (7.6) and such that ψ', ϕ', μ' and ϕ'_r and μ'_r may take on arbitrary values on the interior boundary given by Eq. (7.6) leads to the field Eqs. (5.4) and the boundary conditions discussed in Section 5 (cf. [4]).

We also have

$$\frac{1}{8\pi} I''(0) = - \iint (F'_\phi \phi' + F'_\psi \psi' + 2F'_\mu \mu')\, e^{\phi+\psi} r^2\, dr\, dt - S'(0)$$

where ϕ, ψ and μ are evaluated for $e = 0$ and these functions satisfy the unperturbed equations. In view of Eqs. (5.10), (5.7), and (6.3) we have

$$e^{\phi+\psi} r^2 F'_\psi \psi' = e^{\phi+\psi} r^2 \psi'_0 F'_\psi + (e^{\phi+\psi} r^2 \xi F'_\psi)_r$$
$$- e^{\phi+\psi} r^2 \left(\frac{\mu'}{r} e^{-\phi} (e^{\phi} r^2 F'_\psi)_r + \frac{\xi}{2} r e^{2\psi} k(w+p) F'_\psi \right).$$

On using Eq. (5.21) we obtain

$$e^{\phi+\psi} r^2 (F'_\psi \psi' + 2F'_\mu \mu') = e^{\phi+\psi} r^2 \left(\psi'_0 F'_\psi - \frac{\xi}{2} r e^{2\psi} k(w+p) F'_\psi \right)$$
$$+ (e^{\phi+\psi} r^2 \xi F'_\psi)_r .$$

This equation holds for all values of r, however for $r \geqq r_b$ $w = p = 0$ and for $r \leqq r_b$ we may use Eq. (6.2). Hence we have

$$\frac{1}{8\pi} I''(0) = J + J_1 + \Sigma \tag{7.8}$$

where

$$J = k \int_0^{t_1} \int_0^{r_b} \Big[(w+p) r^2 e^{3\psi - \phi} \xi \xi_{tt} + e^{3\phi+\psi} \frac{(w+p)\alpha^2}{r^2} (r e^{-\phi} \xi)_r^2 + r^2 e^{\phi+\psi} \xi^2 \left(k e^{2\psi} (w+p) p + \frac{4}{r} p_r - \frac{1}{w+p} p_r{}^2 \right) \Big] dr\, dt \tag{7.9}$$

$$J_1 = - \int_0^{t_1} \int_0^{\infty} r^2 e^{\phi+\psi} [F'_\phi \phi' + F_\psi \psi'_0] \, dr\, dt + k \int_0^{t_1} \int_0^{r_b} \Big[r^2 e^{\phi+\psi} \xi \psi'_0 \left(\frac{1}{r^2} + \frac{2}{r} \phi_r \right) - r^2 \xi (e^{\phi+\psi} \phi'_0)_r \Big] dr\, dt \tag{7.10}$$

and

$$\Sigma = - \int_0^{t_1} \int_0^{\infty} (e^{\phi+\psi} r^2 \xi F'_\psi)_r \, dr\, dt - S'(0) - \int_0^{t_1} \int_0^{r_b} (k \xi e^{2\phi+\psi} (w+p) \alpha^2 (e^{-\phi} r^2 \xi)_r)_r \, dr\, dt \tag{7.11}$$

with F'_ψ given in terms of μ', ψ' and ϕ' by Eq. (6.1).

If the functions ϕ', ψ' and ξ are to be such that $I'(0)=0$, that is if they and their derivatives are to vanish on the exterior boundaries, and if the boundary conditions on ξ are to hold at $r=r_b$ and $r=0$ we must have

$$\frac{1}{8\pi} I''(0) = J \tag{7.12}$$

where J is given by Eq. (7.9).

The Euler equations of the variational principle

$$\frac{1}{8\pi} \delta I''(0) = \delta J = 0 \tag{7.13}$$

is the equation

$$\frac{r e^{2\psi}(w+p)}{2} F'_\psi = 0 \tag{7.14}$$

where the explicit form of this equation is given by equation (6.2). This equation is equivalent to $F'_\psi = 0$ and the variational principle defined by Eqs. (7.13) and (7.9) was of course to be expected in view of the general discussion given in the introduction.

8. The Stability Criterion

The variational principle defined by Eqs. (7.13) and (7.9) may be related to that given by Chandrasekhar in [2] by observing that if one writes

$$\xi = \sin(\sigma t + a)\,\zeta(r) \tag{8.1}$$

we have

$$\frac{1}{8\pi} I''(0) = J = \int_0^{t_1} \sin^2(\sigma t + a)\,\mathscr{I}\,dt \tag{8.2}$$

where

$$\mathscr{I} = \int_0^{r_b} -\sigma^2 e^{3\psi-\phi} r^2 (w+p)\xi^2\,dr + \mathscr{I}_1 \tag{8.3}$$

where

$$\begin{aligned}\mathscr{I}_1 = \int_0^{r_b} \Bigg[& r^2 e^{\phi+\psi} \xi^2 \left(k e^{2\psi}(w+p)p + \frac{4p_r}{r} - \frac{1}{(w+p)} p_r{}^2 \right) \\ & + e^{+3\phi+\psi}(w+p)\frac{\alpha^2}{r^2}(r e^{-\phi}\xi)_r^2 \Bigg] dr\,.\end{aligned} \tag{8.4}$$

The variational problem

$$\delta \mathscr{I} = 0 \tag{8.5}$$

has as its Euler equation, Eq. (7.14) with ξ given by Eq. (8.3). The functions $\zeta(r)$ satisfying this Euler equation, that is the extremal $\zeta(r) = \zeta_e(r)$ are such that

$$\mathscr{I}[\zeta_e] = 0\,.$$

Chandrasekhar has pointed out (cf. [2]) that the variational problem given by Eq. (8.4) expresses a minimum principle for the determination of the lowest value of σ^2 and that a *sufficient condition for the dynamical instability of a mass is that $\mathscr{I}_1 = 0$ for some "trial function" ξ which satisfies the required boundary conditions.*

However, if such a trial function exists we shall have $\mathscr{I} \leqq 0$ and in view of Eq. (8.2), for this trial function

$$\frac{1}{8\pi} I''(0) \leqq 0\,. \tag{8.6}$$

Thus the sufficient condition for instability used by Chandrasekhar is equivalent to the condition that there exists a trial function such that the inequality (8.6) holds. The latter criterion may be applied to discussion of the stability of general solutions of the Einstein field equations. We need not restrict ourselves to a static unperturbed solution and consider perturbations of such solutions which depend on the then defined time coordinate in a exponential manner.

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