Commun. math. Phys. 15, 173—200 (1969)

Fields, Observables and Gauge Transformations II

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Received July 10, 1969

Abstract. We wish to study the construction of charge-carrying fields given the representation of the observable algebra in the sector of states of zero charge. It is shown that the set of those covariant sectors which can be obtained from the vacuum sector by acting with "localized automorphisms" has the structure of a discrete Abelian group $\mathscr{G}$. An algebra of fields $\mathfrak{F}$ can be defined on the Hilbert space of a representation π of the observable algebra $\mathfrak{A}$ which contains each of the above sectors exactly once. The dual group of $\hat{\mathscr{G}}$ acts as a gauge group on $\mathfrak{F}$ in such a way that $\pi(\mathfrak{A})$ is the gauge invariant part of $\mathfrak{F}$. $\mathfrak{F}$ is made up of Bose and Fermi fields and is determined uniquely by the commutation relations between spacelike separated fields.

I. Introduction

In a previous paper [1] we studied how the various inequivalent irreducible representations (superselection sectors) of the "algebra of observables" $\mathfrak{A}$ which occur in an irreducible representation of the "field algebra" $\mathfrak{F}$ are related to each other. The algebra $\mathfrak{A}$ was defined as the "gauge-invariant" part of $\mathfrak{F}$. Several assumptions were made concerning the action of the gauge group $\mathscr{G}$, the representation of the Poincaré group and the local structure of the theory. Under these assumptions we found that the question of whether the gauge group is Abelian or not reflects itself in an interesting difference in the structure of the set of sectors. In the case of an Abelian gauge group all sectors are obtained from a single one by applying "localized automorphisms" to the observable algebra. For a non-Abelian $\mathscr{G}$ one must instead apply localized isomorphisms of $\mathfrak{A}$ onto subalgebras.

Since the physical content of the theory is determined by the algebraic structure of $\mathfrak{A}$, one may regard $\mathfrak{F}$ and $\mathscr{G}$ from the physical point of view as auxiliary constructs. This leads to the question: if we are only given the representation of $\mathfrak{A}$ in the vacuum sector, can we construct all other sectors and define an $\mathfrak{F}$ and a $\mathscr{G}$ in such a way that the structural assumptions of [1] are satisfied?

A complete treatment of this problem should first of all analyse the meaning of the term "all sectors" (when we are only given $\mathfrak{A}$ and not $\mathfrak{F}$). Thus we are considering a family of states over $\mathfrak{A}$ which is larger than the collection of vector states in one irreducible representation, but certainly smaller than the set of all pure states over $\mathfrak{A}$. What criteria single out this family of "physical" states and what is their physical significance? One criterion is certainly that we are only interested in pure states of $\mathfrak{A}$ which behave asymptotically like the vacuum state for observations in far away regions of space[1]. However this intuitive partial answer is neither definitive nor precise.

In the present paper we confine our attention to those sectors which can be generated from the vacuum state by applying localized automorphisms. It should be clear that this procedure will, in general, give only a subset of the family of states described above and, correspondingly, only a subset of the superselection quantum numbers. However it has the virtue that a complete analysis of the possible choices of $\mathfrak{F}$ (within this restricted family of sectors) can be carried through and that furthermore, according to the results of [1], it includes superselection rules of practical importance.

The main results of the analysis here are that this restricted set of sectors always leads in a natural way to an Abelian group of superselection quantum numbers ("charges") and that within this set of sectors there is only the alternative of Bose or Fermi statistics, the statistics being coupled in an intrinsic way to the charge. Given this set of sectors there is still some freedom in the choice of field algebra, but only because the commutation relations of fields carrying different charges are not intrinsically determined. The classification of the possible field algebras is essentially a problem of group extensions which is completely analysed in Section V and Appendix I.

It has to be pointed out that the programme of generating the field algebra from the observable algebra was initiated by Borchers [2]. The difference between his input and ours was discussed in [1]. For the subset of sectors treated in the present paper our main conclusions are identical with those of [2]. However, from our point of view this is a special case and the existence of non-Abelian gauge groups and parastatistics cannot be excluded without additional physical principles.

Our notation here will, with minor modifications, follow that of [1] and so will the assumptions on the vacuum representation of the observable algebra. For the convenience of the reader we shall, however, state the relevant assumptions and definitions here.

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[1] We neither wish to concern ourselves with problems of cosmology nor with the properties of a material medium filling space with a finite density.

If $\mathcal{O}$ is a region in space-time we denote by $\mathfrak{A}(\mathcal{O})$ the algebra generated by all the observables which can be measured within $\mathcal{O}$. We are not interested in the exact way in which $\mathfrak{A}(\mathcal{O})$ depends on the shape of $\mathcal{O}$ and for this reason we shall always suppose in this paper that $\mathcal{O}$ is a closed double cone[2]. The set of closed double cones will be denoted by $\mathcal{K}$. We make the following four groups of assumptions:

1. There is a correspondence

$$\mathcal{O} \to \mathfrak{A}(\mathcal{O}), \qquad \mathcal{O} \in \mathcal{K} \tag{1.1}$$

between closed double cones and von Neumann algebras on a Hilbert space $\mathcal{H}_0$[3]. If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ and $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ whilst if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ commute. The C^*-algebra of quasilocal observables $\mathfrak{A}$ is defined as the uniform closure of $\bigcup_{\mathcal{O} \in \mathcal{K}} \mathfrak{A}(\mathcal{O})$ and is assumed to be weakly dense in the set of all bounded operators on $\mathcal{H}_0$. $\mathfrak{A}(\mathcal{O}')$ denotes the C^*-subalgebra of $\mathfrak{A}$ generated by all $\mathfrak{A}(\mathcal{O}_i)$ with $\mathcal{O}_i \in \mathcal{K}$ spacelike to $\mathcal{O}$.

2. The Poincaré group $\mathcal{L}$ is represented by automorphisms α_L of $\mathfrak{A}$, $L \in \mathcal{L}$. α_L transforms a local subalgebra $\mathfrak{A}(\mathcal{O})$ onto the subalgebra of the transformed region $\mathfrak{A}(L\mathcal{O})$.

3. There is a strongly continuous unitary representation U_0 of $\mathcal{L}$ on $\mathcal{H}_0$ implementing the automorphisms $\alpha_L, L \in \mathcal{L}$:

$$U_0(L)\, A\, U_0(L)^{-1} = \alpha_L(A), \qquad L \in \mathcal{L}. \tag{1.2}$$

The representation U_0 satisfies the following spectrum condition: the energy operator has its spectrum confined to $E \geqq 0$, the eigenvalue zero being nondegenerate and corresponding to the vacuum state ω_0 which is represented by a vector $\Omega \in \mathcal{H}_0$

$$\omega_0(A) = (\Omega, A\Omega), \qquad A \in \mathfrak{A}. \tag{1.3}$$

4. Duality holds for each double cone:

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')', \qquad \mathcal{O} \in \mathcal{K}. \tag{1.4}$$

The last assumption is supported by the analysis of [1; Theorem 4.1] and the known results for the free fields [3 and 1; Appendix].

An automorphism γ of $\mathfrak{A}$ is said to be *localized* in $\mathcal{O} \in \mathcal{K}$ if

$$\gamma(A) = A \quad \text{for} \quad A \in \mathfrak{A}(\mathcal{O}'). \tag{1.5}$$

[2] By a closed double cone we shall understand a closed set with non-void interior which is the intersection of a closed forward light cone with a closed backward light cone. By acting with a suitable Lorentz transformation any closed double cone may be reduced to the standard form

$$\{x \in \boldsymbol{R}^4 \mid |x^0| + |\boldsymbol{x}| \leqq R, R > 0\}.$$

[3] The reason for choosing the local algebras to be weakly closed is that the representations of these local algebras considered here are all unitarily equivalent (cf. [1], Theorem 6.1).

It follows from this definition and (1.4) that if $\mathcal{O}_1 \in \mathcal{K}$ and $\mathcal{O}_1 \supset \mathcal{O}$ then

$$A \in \mathfrak{A}(\mathcal{O}_1) \quad \text{implies} \quad \gamma(A) \in \mathfrak{A}(\mathcal{O}_1). \tag{1.6}$$

The identity automorphism is denoted by ι and is clearly localized in any $\mathcal{O}$. The automorphisms localized in $\mathcal{O}$ form a subgroup of the automorphism group of $\mathfrak{A}$ which we denote by $\Gamma(\mathcal{O})$. We let

$$\Gamma = \bigcup_{\mathcal{O} \in \mathcal{K}} \Gamma(\mathcal{O}) \tag{1.7}$$

denote the group of all localized automorphisms.

The algebra of observables $\mathfrak{A}$ has been defined by its vacuum representation which we denote by π_0 [4]. We can define further representations π by composing π_0 with a localized automorphism.

$$\pi = \pi_0 \circ \gamma. \tag{1.8}$$

The representations considered in this paper are those of the form (1.8) which satisfy in addition the following covariance condition:

a) There is a strongly continuous unitary representation of $\mathcal{P}$, the covering group of $\mathcal{L}$, implementing the automorphisms α_L

$$U(L)\,\pi(A)\,U(L)^{-1} = \pi(\alpha_L(A)), \qquad A \in \mathfrak{A},\ L \in \mathcal{P}\ [5]. \tag{1.9}$$

The subsets of Γ and $\Gamma(\mathcal{O})$ which give such representations will be denoted by Γ_C and $\Gamma_C(\mathcal{O})$ respectively. Thus the set of sectors considered in this paper will be those corresponding to representations $\pi_0 \circ \gamma$ with $\gamma \in \Gamma_C$.

This set of sectors can also be characterized without using localized automorphisms. They correspond to representations π which satisfy a) above and the following two conditions:

b) Strong local equivalence. There exists an $\mathcal{O}_0 \in \mathcal{K}$ such that

$$\pi \mid \mathfrak{A}(\mathcal{O}_0') \cong \pi_0 \mid \mathfrak{A}(\mathcal{O}_0') \tag{1.10}$$

where the symbol $\cong$ denotes unitary equivalence.

c) Duality. If $\mathcal{O} \in \mathcal{K}$ and $\mathcal{O} \supset \mathcal{O}_0$, then

$$\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}'))'. \tag{1.11}$$

The last condition rules out the occurrence of non-Abelian gauge groups [1; Theorems 4.1 and 5.6] and ensures that the corresponding set of sectors has the structure of an Abelian group. Condition b) is equivalent to saying that the sector contains a state strictly localized in $\mathcal{O}_0$ and cyclic for $\mathfrak{A}(\mathcal{O}_0')$.

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[4] The symbol π_0 is used whenever we wish to stress the fact that the realization of $\mathfrak{A}$ as an operator algebra on $\mathcal{H}_0$ is also a representation of $\mathfrak{A}$.

[5] For simplicity we use the symbol L to denote both an element of $\mathcal{P}$ and the corresponding element of $\mathcal{L}$.

An important feature of the analysis in this paper is that all the structural properties of $\mathfrak{A}$ which are relevant to our task are proved in an elementary fashion in Sections II and III by using localized automorphisms. Typical examples are the Bose-Fermi alternative (Lemma 2.3) and the spectrum condition (Proposition 3.2). It seems that all properties which are usually expressed with the aid of the field algebra can be expressed directly in terms of localized automorphisms and in this sense the reconstruction of the field algebra is perhaps unnecessary.

II. Localized Automorphisms

This section is concerned with some basic facts, which are of an algebraic nature, about localized automorphisms.

Let $\mathscr{A}(\mathcal{O})$ be the set of all unitary elements in $\mathfrak{A}(\mathcal{O})$ and $\mathscr{A}$ the union of all $\mathscr{A}(\mathcal{O})$ for the different double cones $\mathcal{O}$. Then $\mathscr{A}(\mathcal{O})$ (resp. $\mathscr{A}$) is a group under the multiplication in the algebra $\mathfrak{A}$, and generates $\mathfrak{A}(\mathcal{O})$ (resp. a dense subalgebra of $\mathfrak{A}$) by linear combinations. We introduce a group homomorphism σ of $\mathscr{A}$ into Γ, carrying $\mathscr{A}(\mathcal{O})$ into $\Gamma(\mathcal{O})$, by the definition

$$U \in \mathscr{A} \to \sigma_U : \sigma_U(A) = UAU^{-1}, \qquad A \in \mathfrak{A}. \tag{2.1}$$

We denote by $\mathscr{I}(\mathcal{O})$ (resp. $\mathscr{I}$) the image of $\mathscr{A}(\mathcal{O})$ (resp. $\mathscr{A}$) under σ; then $\mathscr{I}$ is a subgroup of the group of inner automorphisms of $\mathfrak{A}$ and a normal subgroup of Γ. In fact, if $\varrho = \sigma_U \in \mathscr{I}$ and $\gamma \in \Gamma$, $\gamma\varrho\gamma^{-1} = \sigma_{\gamma(U)} \in \mathscr{I}$. We can thus define the quotient group $\Gamma/\mathscr{I}$; the canonical homomorphism onto this quotient will be denoted by $\gamma \in \Gamma \to \hat{\gamma} \in \Gamma/\mathscr{I}$.

Acting with $\gamma \in \Gamma$ on the vacuum representation π_0 we obtain a representation $\pi_0 \circ \gamma$:

$$\pi_0 \circ \gamma(A) = \pi_0(\gamma(A)), \qquad A \in \mathfrak{A}.$$

In this way we get a map $\gamma \to \widehat{\pi_0 \circ \gamma}$, where $\hat{\pi}$ denotes the unitary equivalence class of the representation π. Our first lemma says that the kernel of this map is exactly $\mathscr{I}$.

2.1. Lemma. *If $\gamma_1, \gamma_2 \in \Gamma$, $\pi_0 \circ \gamma_1$ is unitarily equivalent to $\pi_0 \circ \gamma_2$ if and only if $\gamma_1\gamma_2^{-1} \in \mathscr{I}$.*

Proof. Clearly the relation $\pi_0 \circ \gamma_1 \cong \pi_0 \circ \gamma_2$ is implied by $\gamma_1\gamma_2^{-1} \in \mathscr{I}$; we have to prove the converse implication.

Let $\mathcal{O}$ be a double cone such that $\gamma_1, \gamma_2 \in \Gamma(\mathcal{O})$, and V a unitary operator on $\mathscr{H}_0$ such that

$$\pi_0(\gamma_2(A)) = V\pi_0(\gamma_1(A))V^{-1}, \qquad A \in \mathfrak{A}.$$

Since $\gamma_1(A) = \gamma_2(A) = A$ if $A \in \mathfrak{A}(\mathcal{O}')$, it follows that $V \in \pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))$ by the duality property, hence $\gamma_1\gamma_2^{-1} \in \mathscr{I}(\mathcal{O})$.

The preceding lemma implies that the map $\gamma \to \widehat{\pi_0 \circ \gamma}$ of Γ into the spectrum $\hat{\mathfrak{A}}$ of $\mathfrak{A}$ sets up a one-to-one correspondence between the quotient group $\Gamma/\mathscr{I}$ and the family of equivalence classes of representations obtained from the vacuum sector by the action of localized automorphisms; this family can accordingly be given a group structure.

As explained in the Introduction, we establish the relation between localized automorphisms and fields only for the subset Γ_C, i.e. for those automorphisms which transform the vacuum representation π_0 into a representation $\pi_0 \circ \gamma$ covariant for the Poincaré group. For those automorphisms we can prove a locality property closely related to the commutativity or anticommutativity of fields at spacelike separations.

2.2. Lemma. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated double cones, the automorphisms from Γ_C localized in $\mathcal{O}_1$ commute with those localized in $\mathcal{O}_2$.*

Proof. Let $\gamma_1 \in \Gamma_C(\mathcal{O}_1)$, $\gamma_2 \in \Gamma_C(\mathcal{O}_2)$ and $A \in \mathfrak{A}(\mathcal{O})$. Let $\mathcal{O}_3$ and $\mathcal{O}_4$ be translates of the double cones $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively such that

(i) $\mathcal{O} \subset \mathcal{O}_3' \cap \mathcal{O}_4'$,

(ii) there exist spacelike separated double cones $\mathcal{O}_5, \mathcal{O}_6$ with $\mathcal{O}_1 \cup \mathcal{O}_3 \subset \mathcal{O}_5 \subset \mathcal{O}_2'$ and $\mathcal{O}_2 \cup \mathcal{O}_4 \subset \mathcal{O}_6 \subset \mathcal{O}_1'$.

Let γ_1' and γ_2' be translates of γ_1 and γ_2 localized in $\mathcal{O}_3$ and $\mathcal{O}_4$ respectively. Then since $\gamma_1, \gamma_2 \in \Gamma_C$, Lemma 2.1 shows that $\gamma_1' = \sigma_{U_1}\gamma_1$ and $\gamma_2' = \sigma_{U_2}\gamma_2$ with $U_1 \in \mathfrak{A}(\mathcal{O}_5)$ and $U_2 \in \mathfrak{A}(\mathcal{O}_6)$. Now since $\mathcal{O} \subset \mathcal{O}_3' \cap \mathcal{O}_4'$, $\gamma_1'\gamma_2'(A) = \gamma_2'\gamma_1'(A) = A$. Hence $\sigma_{U_1}\gamma_1\sigma_{U_2}\gamma_2(A) = \sigma_{U_2}\gamma_2\sigma_{U_1}\gamma_1(A)$. However $\gamma_1(U_2) = U_2, \gamma_2(U_1) = U_1$ and $U_1U_2 = U_2U_1$, hence

$$\gamma_1\gamma_2(A) = \gamma_2\gamma_1(A).$$

Thus $\gamma_1\gamma_2 = \gamma_2\gamma_1$ as required.

The elements of Γ_C can be further classified into Bose or Fermi automorphisms.

2.3. Lemma. *Let $\gamma_1, \gamma_2 \in \Gamma_C$ be automorphisms leading to the same sector ξ, i.e. $\pi_0 \circ \gamma_1 \cong \pi_0 \circ \gamma_2$, and U the element of $\mathscr{A}$, defined up to a phase, such that $\sigma_U\gamma_1 = \gamma_2$ (Lemma 2.1); then, if γ_1 and γ_2 are localized in spacelike separated double cones, we have*

$$\gamma_1(U) = \pm U \tag{2.2}$$

with the sign depending only upon the sector ξ and not upon the choice of γ_1, γ_2.

Proof. Assume $\gamma_1 \in \Gamma_C(\mathcal{O}_1)$ and $\gamma_2 \in \Gamma_C(\mathcal{O}_2)$ with $\mathcal{O}_1$ and $\mathcal{O}_2$ spacelike separated double cones; by the preceding lemma γ_1 commutes with γ_2, hence with σ_U also. Since $\gamma_1\sigma_U = \sigma_{\gamma_1(U)}\gamma_1$ it follows that $\sigma_{\gamma_1(U)} = \sigma_U$ and therefore that

$$\gamma_1(U) = \varepsilon_{\gamma_1,\gamma_2}U, \quad \varepsilon_{\gamma_1,\gamma_2} \in \boldsymbol{T}^6. \tag{2.2'}$$

[6] $\boldsymbol{T}$ denotes the group of complex numbers of modulus one.

By the Eq. (2.2′) we see that the number $\varepsilon_{\gamma_1,\gamma_2}$ depends only on the pair γ_1, γ_2 and not on the phase of U. Further by definition it is easily seen that

$$\varepsilon_{\gamma_1,\gamma_2} = \varepsilon_{\gamma_2,\gamma_1}^{-1}. \tag{2.3}$$

Let $\gamma_3 \in \Gamma(\mathcal{O}_3)$ be an automorphism leading to the same sector ξ, with $\mathcal{O}_3$ a double cone containing $\mathcal{O}_2$ and spacelike to $\mathcal{O}_1$ [7]; then $\gamma_3 = \sigma_W \gamma_2 = \sigma_{U'} \gamma_1$, with $U' = WU$, and, by the duality property, $W \in \mathcal{A}(\mathcal{O}_3)$; therefore $\gamma_1(W) = W$, and multiplying Eq. (2.2′) by W on the left we obtain

$$\gamma_1(U') = \varepsilon_{\gamma_1,\gamma_2} U' = \varepsilon_{\gamma_1,\gamma_3} U' ;$$

hence $\varepsilon_{\gamma_1,\gamma_2} = \varepsilon_{\gamma_1,\gamma_3}$. Thus the value of $\varepsilon_{\gamma_1,\gamma_2}$ does not change when we vary the second argument within the specified limitations. Eq. (2.3) tells us the same is true of variations in the first argument.

Applying this process repeatedly we find that

$$\varepsilon_{\gamma_1,\gamma_2} = \varepsilon_{\gamma_1',\gamma_2'} = \varepsilon_\xi \tag{2.4}$$

for any pair $\gamma_1', \gamma_2' \in \Gamma_C$ localized in spacelike separated double cones and leading to the same sector ξ. Therefore by Eq. (2.3) we get $\varepsilon_\xi^2 = 1$; hence $\varepsilon_\xi = \pm 1$, and the Lemma is proved.

It can be easily seen that

$$\varepsilon_{\xi_1} \varepsilon_{\xi_2} = \varepsilon_{\xi_1 \xi_2},$$

i.e. ε is a homomorphism of $\Gamma_C/\mathcal{I}$ into the two element group $\{\pm 1\}$. The kernel of ε is the subgroup of $\Gamma_C/\mathcal{I}$ of *Bose sectors* and elements mapping onto -1 form the coset of *Fermi sectors*. In a similar way, we define in Γ_C the subgroup of *Bose automorphisms* and the coset of *Fermi automorphisms*, whose elements map onto Bose and Fermi sectors respectively.

A localized automorphism $\gamma \in \Gamma_C$ may correspond to a "multiplicative" superselection rule in the sense that there is a power of γ which leaves the vacuum sector invariant, i.e.

$$\gamma^\nu \in \mathcal{I};$$

in this case, if γ is a Fermi automorphism ν is even.

The next Lemma is of a purely technical nature but will be essential when reconstructing fields corresponding to multiplicative superselection rules.

2.4. Lemma. *If $\varrho \in \Gamma_C$ is such that $\pi_0 \circ \varrho^\nu \cong \pi_0$ for an integer ν, then there is a γ in the $\mathcal{I}$-coset $\mathcal{I}\varrho$ such that*

$$\gamma^\nu = \iota .$$

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[7] Notice that, for $\gamma \in \Gamma_C$, there are sufficiently many spacelike separated elements in the coset $\mathcal{I}\gamma$, in particular the space translates $\alpha_x \gamma \alpha_x^{-1}$, for suitable translations $\boldsymbol{x}$.

Proof. Let $U \in \mathscr{A}$ be such that

$$\varrho^{\nu} = \sigma_U .$$

Since $\varrho\varrho^{\nu} = \varrho^{\nu}\varrho$, we have $\varrho(U) = \lambda U$ with $\lambda \in \boldsymbol{T}$. We first show that $\lambda = 1$.

Let $W \in \mathscr{A}$ be such that $\sigma_W \varrho$ and ϱ are localized in spacelike separated regions (see footnote 7), then $\sigma_W \varrho(U) = U$, and therefore

$$\lambda W U W^{-1} = U .$$

However by Lemma 2.3 $\varrho(W) = \pm W$, hence $\varrho^{\nu}(W) = (\pm 1)^{\nu} W$ i.e.

$$(\pm 1)^{\nu} W U W^{-1} = U ;$$

therefore $\lambda = (\pm 1)^{\nu}$; since the $(-)$ sign can occur only if ν is even, $\lambda = 1$. Now let $\mathscr{O}$ be a double cone such that $\varrho \in \Gamma(\mathscr{O})$, whence $U \in \mathfrak{A}(\mathscr{O})$. The automorphism ϱ is normal[8] on the von Neumann algebra $\mathfrak{A}(\mathscr{O})$ and $\varrho(U) = U$; hence ϱ acts as the identity on the von Neumann algebra $\mathfrak{M} \subset \mathfrak{A}(\mathscr{O})$ generated by U. If the unitary element $V \in \mathfrak{M}$ is a νth root of U^{-1}, it follows that $\varrho(V) = V$ and ϱ commutes with σ_V. The desired automorphism γ can now be written as $\gamma = \sigma_V \varrho$.

The reader will notice that Lemmas 2.2, 2.3, and 2.4 can be rephrased for automorphisms in Γ – but not necessarily in Γ_C – which belong to an $\mathscr{I}$-coset containing "sufficiently many" spacelike separated automorphisms.

III. The Abelian Group of Superselection Quantum Numbers

We defined Γ_C in the Introduction as that subset of Γ consisting of all localized automorphisms γ for which $\pi_0 \circ \gamma$ is covariant with respect to the Poincaré group. If $\mathfrak{A}$ describes a physically sensible theory, we might expect that each locally normal pure state ω, which approaches the vacuum state ω_0 rapidly at large spacelike distances, gives rise to a sector π_ω where the covariance and spectrum conditions are automatically satisfied. This would in particular imply that $\Gamma_C = \Gamma$. Our present framework may be too wide to realize this situation but we can at least prove that Γ_C is a group and disregard any of the more singular objects $\gamma \in \Gamma$, $\gamma \notin \Gamma_C$ if they should exist at all.

If π is an irreducible representation of $\mathfrak{A}$, there is at most one representation U of the covering group of the Poincaré group such that (π, U) is covariant, since $\mathscr{P}$ has no non-trivial one-dimensional representations. Let U_0 and U be *the* continuous unitary representations of $\mathscr{P}$ corresponding in this way to the vacuum representation π_0 and to $\pi = \pi_0 \circ \gamma$, $\gamma \in \Gamma_C$, respectively. Hence

$$\pi_0 \circ \gamma(\alpha_L^{-1}(A)) = U(L)^{-1} \pi_0 \circ \gamma(A)\, U(L), \qquad A \in \mathfrak{A},\ L \in \mathscr{P}. \tag{3.1}$$

[8] Every automorphism of a von Neumann algebra is normal ([10]; Chapter 1, § 4).

Given $L \in \mathscr{P}$ and $\gamma \in \Gamma_C$, let $\mathcal{O}$ be a double cone such that $\gamma \in \Gamma(\mathcal{O}) \cap \Gamma(L^{-1}\mathcal{O})$, then we get from (3.1)

$$U_0(L)^{-1}\pi_0(A)\,U_0(L) = U(L)^{-1}\pi_0(A)\,U(L), \qquad A \in \mathfrak{A}(\mathcal{O}'). \tag{3.2}$$

Hence by duality in the vacuum sector

$$U_0(L)\,U(L)^{-1} = \pi_0(U_L(\gamma)) \qquad U_L(\gamma) \in \mathscr{A}(\mathcal{O}). \tag{3.3}$$

Further since $\pi_0(\alpha_L^{-1}(A)) = U_0(L)^{-1}\pi_0(A)\,U_0(L)$, $A \in \mathfrak{A}$, $L \in \mathscr{P}$, we get, comparing with (3.1) and using (3.3),

$$\gamma_L \equiv \alpha_L \gamma \alpha_L^{-1} = \sigma_{U_L(\gamma)}\gamma\,. \tag{3.4}$$

The meaning of this relation can be understood in the following way: γ_L and γ are automorphisms leading to the same sector, hence by Lemma 2.1, $\gamma_L\gamma^{-1}$ is an inner automorphism which determines a unitary element of $\mathscr{A}$ up to a phase. The definition of $U_L(\gamma)$ by (3.3) amounts to a special choice of this phase which is canonical in view of the following multiplication law

$$U_{LL'}(\gamma) = \alpha_L(U_{L'}(\gamma))\,U_L(\gamma)\,. \tag{3.5}$$

Suppose now that $\gamma' \in \Gamma_C$ and U' is the continuous unitary representation of $\mathscr{P}$ corresponding to $\pi' = \pi_0 \circ \gamma'$, then

$$\begin{aligned}\pi_0 \circ \gamma\gamma'(\alpha_L^{-1}(A)) &= \pi_0 \circ \gamma(\alpha_L^{-1}\gamma'_L(A))\\ &= U(L)^{-1}\pi_0 \circ \gamma(\gamma'_L(A))\,U(L)\,.\end{aligned}$$

Hence by (3.4) for γ'

$$\pi_0 \circ \gamma\gamma'(\alpha_L^{-1}(A)) = U(L)^{-1}\pi_0 \circ \gamma(\sigma_{U_L(\gamma')}\gamma'(A))\,U(L)$$

so that

$$U''(L) \equiv \pi_0(\gamma(U_L(\gamma')))^{-1}\,U(L) \tag{3.6}$$

implements α_L in the representation $\pi_0 \circ \gamma\gamma'$. A routine computation using (3.5) and (3.6) shows that $L \to U''(L)$ is a representation of $\mathscr{P}$. Now U_0 and U are strongly continuous representations of $\mathscr{P}$. Since the unitary operators form a topological group under the strong topology $L \to U_L(\gamma')$ is strongly continuous (see (3.3)). Further $L \to \pi_0(\gamma(U_L(\gamma')))$ is also strongly continuous since localized automorphisms are locally normal. Hence the representation U'' defined in (3.6) is strongly continuous and $\gamma\gamma' \in \Gamma_C$.

In a similar way we can show that $\gamma \in \Gamma_C$ implies $\gamma^{-1} \in \Gamma_C$. In fact

$$\tilde{U}(L) \equiv \pi_0(\gamma^{-1}(U_L(\gamma)))\,U_0(L) \tag{3.7}$$

implements α_L in the representation $\pi_0 \circ \gamma^{-1}$ and $L \to \tilde{U}(L)$ is a strongly continuous unitary representation of $\mathscr{P}$.

3.1. Theorem. *Γ_C is a subgroup of Γ and $\Gamma_C/\mathscr{I}$ is an Abelian group.*

Proof. We have just shown that Γ_C is a subgroup of Γ so it only remains to show that the quotient group $\Gamma_C/\mathscr{I}$ is Abelian. If $\gamma, \varrho \in \Gamma_C$ and $L \in \mathscr{P}$, then $\hat{\gamma}\hat{\varrho} = \hat{\gamma}\hat{\varrho}_L = \widehat{\gamma\varrho_L}$ and $\hat{\varrho}\hat{\gamma} = \hat{\varrho}_L\hat{\gamma} = \widehat{\varrho_L\gamma}$. But by Lemma 2.2 we may choose L such that $\gamma\varrho_L = \varrho_L\gamma$. Hence $\hat{\gamma}\hat{\varrho} = \hat{\varrho}\hat{\gamma}$ and $\Gamma_C/\mathscr{I}$ is an Abelian group.

If we replace $\mathscr{P}$ by the group of translations in space-time, Theorem 3.1 still holds but the representations of the translation group in the different sectors are now defined only to within a continuous one-dimensional unitary representation of the translation group[9].

Let $\gamma, \gamma' \in \Gamma_C$ and suppose that the energy-momentum spectrum in the representation defined by γ is $S(\gamma)$. We show that

$$S(\gamma\gamma') \supset S(\gamma) + S(\gamma'). \tag{3.8}$$

Let $\mathscr{N}$ and $\mathscr{N}'$ be open sets intersecting $S(\gamma)$ and $S(\gamma')$ respectively and let U, U' and U'' be the representations of the covering group of the Poincaré group corresponding to $\pi_0 \circ \gamma$, $\pi_0 \circ \gamma'$ and $\pi_0 \circ \gamma\gamma'$ respectively. Now by (3.3) $U(x)A\Omega = U_x(\gamma)^{-1}\alpha_x(A)\Omega$ so that we may pick $A, A' \in \mathfrak{A}$ such that the functions $x \to U_x(\gamma)^{-1}\alpha_x(A)$ and $x \to U_x(\gamma')^{-1}\alpha_x(A')$ have Fourier transforms with supports in $\mathscr{N}$ and $\mathscr{N}'$ respectively. Now

$$U''(x)\Phi_y \equiv U''(x)\,\gamma(U_y(\gamma')^{-1}\alpha_y(A'))A\Omega = \gamma(U_x(\gamma'))^{-1}\,U(x)\Phi_y$$

by (3.6). Hence

$$U''(x)\Phi_y = \gamma\big(U_x(\gamma')^{-1}\alpha_x(U_y(\gamma'))^{-1}\alpha_{x+y}(A')\big)\,U_x(\gamma)^{-1}\alpha_x(A)\Omega$$

and by (3.5)

$$U''(x)\Phi_y = \gamma(U_{x+y}(\gamma')^{-1}\alpha_{x+y}(A'))\,U_x(\gamma)^{-1}\alpha_x(A)\Omega.$$

However the strongly continuous function $x \to \gamma(U_{x+y}(\gamma')^{-1}\alpha_{x+y}(A'))$ has a Fourier transform with support in $\mathscr{N}'$ by choice of A'. Hence the vector Φ_y considered in the representation $\pi_0 \circ \gamma\gamma'$ has energy-momentum spectrum in $\mathscr{N} + \mathscr{N}'$ and it suffices to show that we may choose y such that $\Phi_y \neq 0$. Now $\|\Phi_y\|^2 = (\Omega, A^*\gamma(\alpha_y(A'^*A'))A\Omega)$ and since γ is strictly localized we may apply the cluster decomposition property to deduce that $\|\Phi_y\|^2 \to \|A\Omega\|^2\,\|A'\Omega\|^2$ as $|y| \to \infty$. In particular for suitable $\boldsymbol{y}$, $\Phi_{\boldsymbol{y}} \neq 0$ and we have proved (3.8). Putting $\gamma' = \gamma^{-1}$ in (3.8) we get $S(\iota) \supset S(\gamma) + S(\gamma^{-1})$ and hence

[9] In the case of the translation group, it is also natural to consider automorphisms leading to sectors carrying covariant projective representations. These can again be shown to form a subgroup Γ_a of Γ. If $\mathscr{H}_0$ is a separable Hilbert space, Γ_a may be identified with the set of those $\gamma \in \Gamma$ such that $\pi_0 \circ \gamma$ carries a Borel measurable covariant representation up to a factor and we have a natural homomorphism of Γ_a into the second cohomology group of the translation group with coefficients in $\boldsymbol{T}$.

3.2. Proposition. *The energy-momentum spectrum for the representation* $\pi_0 \circ \gamma$, $\gamma \in \Gamma_C$, *lies in the closed forward light cone.*

We terminate this section with two relations which will be used later to define Lorentz transformations of fields. Eqs. (3.3) and (3.6) give

$$U_L(\gamma\gamma') = U_L(\gamma)\,\gamma(U_L(\gamma')) \tag{3.9}$$

and Eqs. (3.3) and (3.9) give

$$U'(L)\,U(L)^{-1} = \pi'(U_L(\gamma'^{-1}\gamma)). \tag{3.10}$$

IV. Field Group and Field Algebra for Independent Charges

As proved above, the family of equivalence classes of Poincaré covariant representations obtained by acting on the vacuum representation with localized automorphisms, has the structure of an Abelian group being in one-to-one correspondence with $\Gamma_C/\mathscr{I}$. These representations automatically satisfy the spectrum condition, i.e. positivity of the energy and reality of the mass. By [1; Proposition 6.4], this family coincides with the collection of all "sectors" which are equivalent to the vacuum sector when restricted to the subalgebra of some outer region and satisfy a duality property. The corresponding subset of superselection quantum numbers accordingly forms an Abelian group.

Our task in this and the next section will be to construct fields carrying those charges. The construction in [1; Theorem 6.2], suggests that the automorphisms in Γ_C should be implemented by unitary elements of the field algebra in a representation π of $\mathfrak{A}$ which contains each sector in $\Gamma_C/\mathscr{I}$ exactly once, the unitaries implementing inner automorphisms being observables. Since $\varrho \in \mathscr{I}$ determines a $U \in \mathscr{A}$ such that $\sigma_U = \varrho$ only up to a phase, $\varrho \to \pi(U)$ is a representation of $\mathscr{I}$ up to a factor in $\boldsymbol{T}$. Therefore we cannot choose the unitaries which implement the automorphisms in Γ_C in such a way that they form a true representation of the group Γ_C[10]. We shall however see that we can obtain a representation of Γ_C up to a factor in $\boldsymbol{T}$.

We divide the problem into two stages. First, find a group $\mathscr{F}$ (whose elements will be denoted by ψ, ψ' etc.) with the following properties

(i) $\mathscr{A}$ is a normal subgroup of $\mathscr{F}$;

(ii) the homomorphism σ of $\mathscr{A}$ onto $\mathscr{I}$ extends to a homomorphism $\psi \to \sigma_\psi$ of $\mathscr{F}$ onto Γ_C, satisfying

(iii) $\sigma_\psi(U) = \psi U \psi^{-1}$ if $\psi \in \mathscr{F}$, $U \in \mathscr{A}$.

(iv) Kernel $\sigma = \boldsymbol{T} =$ Kernel $(\sigma \mid \mathscr{A})$.

[10] Note that such a choice would also contradict Lemma 2.2 in the case of Fermi fields.

Note that because of condition (iii) the group $\mathscr{F}$ determines the homomorphism σ from $\mathscr{F}$ onto Γ_C; a group $\mathscr{F}$ satisfying conditions (i) to (iv) will be called a *field group*.

The second stage is to obtain the field operators by considering a covariant representation (π, V) of $\{\mathfrak{A}, \sigma\}$. Here π is a representation of the observable algebra and V is a true unitary representation of $\mathscr{F}$ which implements the automorphisms σ_ψ in the representation π and moreover coincides with π when restricted to $\mathscr{A}$.

The field group $\mathscr{F}$ will contain in its abstract group structure all the formal properties we look for. For instance, if $\psi_1, \psi_2 \in \mathscr{F}$ are spacelike separated and "carry the same charge", i.e. $\sigma_{\psi_1}, \sigma_{\psi_2}$ are localized in spacelike separated double cones and lead from π_0 to the same sector, then properties (i) to (iv) imply $\psi_1\psi_2 = \pm\psi_2\psi_1$ [11]. For in that case $\sigma_{\psi_2} = \sigma_U\sigma_{\psi_1}$, with $U \in \mathscr{A}$ and $\sigma_{\psi_1}(U) = \pm U$ by Lemma 2.3; by (iv) $\psi_2 = \lambda U\psi_1, \lambda \in \boldsymbol{T}$ and then $\psi_1\psi_2 = \pm\psi_2\psi_1$ by (iii). We say that ψ is a Bose or Fermi field according as σ_ψ is a Bose or Fermi automorphism.

However conditions (i) to (iv) by no means define a unique solution; in particular the commutation relations between different spacelike separated fields may or may not be normal.

In this section we construct $\mathscr{F}$ explicitly for a group $\Gamma_C/\mathscr{I}$ with a family of independent generators which is at most countable. We postpone the existence theorem for a field group corresponding to any $\Gamma_C/\mathscr{I}$ to the next section.

Let $\xi_1, \xi_2, \ldots$ be a countable family of independent generators of $\Gamma_C/\mathscr{I}$. If the cyclic subgroup generated by ξ_i is finite, let v_i be the smallest positive integer such that

$$\xi_i^{v_i} = \iota \tag{4.1}$$

where $\iota = \hat{\imath}$ denotes the identity of $\Gamma_C/\mathscr{I}$. If this cyclic subgroup is infinite, we put $v_i = 0$. Let $\boldsymbol{Z}_{v_i}$ denote the additive group of integers modulo v_i. The independence of the generators $\xi_1, \xi_2, \ldots$ is expressed by the fact that

$$(n_1, n_2, \ldots) \in \prod_i \boldsymbol{Z}_{v_i} \to \xi_1^{n_1}\xi_2^{n_2} \ldots \in \Gamma_C/\mathscr{I}$$

is a group *isomorphism* of the restricted[12] product $\prod_i \boldsymbol{Z}_{v_i}$ onto $\Gamma_C/\mathscr{I}$.

A *section* s is defined by picking a representative element $s(\xi) \in \Gamma_C$ in each class $\xi \in \Gamma_C/\mathscr{I}$; we now choose a section s which is moreover a group homomorphism.

Let $\widehat{\gamma_i'} = \xi_i$, and $\mathscr{O}_i$ be a double cone with $\gamma_i' \in \Gamma_C(\mathscr{O}_i)$. Since an automorphism from Γ_C and its translates under the Poincaré group lead to

[11] Notice that $\mathscr{F} \supset \mathscr{A} \supset \boldsymbol{T}$ so $-1 \in \mathscr{F}$.

[12] This means that only sequences with a finite number of non-zero terms are considered.

the same sector, we may suppose that $\mathcal{O}_i$ is spacelike to $\mathcal{O}_j$, $i \neq j$. From Eq. (4.1) we see that $\gamma_i'^{\nu_i} \in \mathcal{I}$; hence from Lemma 2.4 we can choose γ_i localized in the same region as γ_i', satisfying

$$\gamma_i^{\nu_i} = \iota; \quad \widehat{\gamma_i} = \xi_i .$$

By Lemma 2.2, $\gamma_i \gamma_j = \gamma_j \gamma_i$ for all $i, j = 1, 2, \ldots$; we see then that the section $\xi \to s(\xi)$ defined by

$$\xi = \xi_1^{n_1} \xi_2^{n_2} \ldots \to s(\xi) = \gamma_1^{n_1} \gamma_2^{n_2} \ldots, \qquad (n_1, n_2, \ldots) \in \prod_i \boldsymbol{Z}_{\nu_i}$$

is a homomorphism.

We now introduce the semidirect product of $\mathcal{A}$ and $\Gamma_C/\mathcal{I}$ with respect to the action defined by s:

$$\mathcal{F}_0 \equiv \mathcal{A} \times_s \Gamma_C/\mathcal{I} . \tag{4.2}$$

In other words, if $\psi = (U, \xi) \in \mathcal{F}_0$ with $U \in \mathcal{A}$ and $\xi \in \Gamma_C/\mathcal{I}$, the group operation is given by

$$\psi\psi' = (U s(\xi)(U'), \xi\xi') . \tag{4.3}$$

If $\psi \in \mathcal{F}_0$, we shall write $\psi = (U_\psi, \xi_\psi)$ whenever we need to emphasize the dependence of U and ξ on ψ.

Define a representation σ of $\mathcal{F}_0$ by automorphisms of $\mathfrak{A}$ setting

$$\sigma_\psi = \sigma_{U_\psi} s(\xi_\psi) . \tag{4.4}$$

It is immediately seen that the pair $\mathcal{F}_0, \sigma$ defined in Eqs. (4.2) to (4.4) satisfies the conditions (i) to (iv) above.

For any field group $\mathcal{F}$ we define the subgroups corresponding to local regions by

$$\mathcal{F}(\mathcal{O}) = \{\psi \in \mathcal{F} \mid \sigma_\psi \in \Gamma(\mathcal{O})\} . \tag{4.5}$$

4.1. Theorem. *Let $\mathcal{F}$ be any field group and $\psi_1 \in \mathcal{F}(\mathcal{O}_1)$, $\psi_2 \in \mathcal{F}(\mathcal{O}_2)$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ spacelike separated double cones. Then $\varepsilon \equiv \psi_1 \psi_2 \psi_1^{-1} \psi_2^{-1}$ is a phase factor depending only upon ξ_{ψ_1} and ξ_{ψ_2}, $\varepsilon = \varepsilon(\xi_{\psi_1}, \xi_{\psi_2})$. The function $\varepsilon(\xi, \xi')$ satisfies the relations*

$$\varepsilon(\xi, \xi') = \varepsilon(\xi', \xi)^{-1}; \qquad \varepsilon(\xi, \xi) = \varepsilon_\xi$$

where $\varepsilon_\xi = \pm 1$ according as ξ is a Bose or Fermi sector, see Eq. (2.4).

In particular if $\mathcal{F} = \mathcal{F}_0$ then $\varepsilon(\xi, \xi') = \pm 1$ for all $\xi, \xi' \in \Gamma_C/\mathcal{I}$ and is given by

$$\varepsilon(\xi, \xi') = \prod_i \varepsilon_{\xi_i}^{n_i n_i'} \tag{4.6}$$

if $\xi = \xi_1^{n_1} \xi_2^{n_2} \ldots, \xi' = \xi_1^{n_1'} \xi_2^{n_2'} \ldots$.

Proof. $\xi_{\psi_1\psi_2\psi_1^{-1}\psi_2^{-1}} = \iota$ so $\varepsilon \in \mathscr{A}$. Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, $\sigma_{\psi_1}\sigma_{\psi_2} = \sigma_{\psi_2}\sigma_{\psi_1}$ and $\sigma_\varepsilon = \iota$. Hence $\varepsilon \in \boldsymbol{T}$. An argument similar to the one in the proof of Lemma 2.3 shows that the phase factor ε depends only upon ξ_{ψ_1} and ξ_{ψ_2}. By definition ε is antisymmetric, and the remarks following conditions (i) to (iv) establish that $\varepsilon(\xi, \xi) = \varepsilon_\xi = \pm 1$.

Now let $\mathscr{F} = \mathscr{F}_0$; the fields $(1, \xi_i) \in \mathscr{F}_0(\mathcal{O}_i)$ commute with one another and $\mathcal{O}_i$ is spacelike to $\mathcal{O}_j$, $i \neq j$; hence $\varepsilon(\xi_i, \xi_j) = +1$ for $i \neq j$. Since the commutator ε between the fields ψ and ψ' with spacelike separated supports depends only upon their charges ξ, ξ', we may prove (4.6) by specializing ψ and ψ' to be monomials in fields carrying "elementary charges" ξ_i, all support regions being mutually spacelike.

If $\mathscr{F}$ is any field group, the associativity of the product also implies the cocycle identity $\delta\varepsilon = 1$[13]; we will call ε the *commutator cocycle* of $\mathscr{F}$.

We see by Theorem 4.1 that the particular solution $\mathscr{F}_0$ of our problem does not correspond to normal commutation relations. We will obtain the normal solution $\mathscr{F}$ by a Klein transformation.

Let $\mathscr{C}$ denote the group of all functions from $\Gamma_C/\mathscr{I}$ to $\boldsymbol{T}$ under pointwise multiplication: if $f_1, f_2 \in \mathscr{C}, (f_1 f_2)(\xi) = f_1(\xi) f_2(\xi)$, $\xi \in \Gamma_C/\mathscr{I}$.

We define an action of $\mathscr{F}_0$ on $\mathscr{C}$ by setting

$$f \to {}^\psi f : {}^\psi f(\xi') = f(\xi'\xi_\psi), \qquad f \in \mathscr{C},\ \psi \in \mathscr{F},\ \xi' \in \Gamma_C/\mathscr{I}.$$

Let $\mathscr{C} \times \mathscr{F}_0$ be the corresponding semidirect product, the group operation being

$$(f_1, \psi_1)(f_2, \psi_2) = (f_1 {}^{\psi_1} f_2, \psi_1 \psi_2).$$

Let $\tilde{\mathscr{F}}$ be the quotient group modulo the equivalence relation $(f, \lambda\psi) \sim (\lambda f, \psi)$, $\lambda \in \boldsymbol{T}$, i.e. $\tilde{\mathscr{F}} = \mathscr{C} \times \mathscr{F}_0 / N$ where $N = \{(\lambda, \lambda^{-1}) \mid \lambda \in \boldsymbol{T}\}$. We extend σ to $\tilde{\mathscr{F}}$ by setting $\sigma_{(f, \psi)} = \sigma_\psi$.

We next introduce special elements f_i, h_i, h_ξ of $\mathscr{C}$ by the relations

$$f_i(\xi') = \begin{vmatrix} +1 \text{ } \textit{if } \gamma_i \textit{ is a Bose automorphism;} \\ (-1)^{n'_i} \text{ } \textit{if } \gamma_i \textit{ is a Fermi automorphism and } \xi' = \xi_1^{n'_1} \xi_2^{n'_2} \ldots; \end{vmatrix}$$

$$h_i = f_1 f_2 \ldots f_{i-1}; \ h_\xi = h_1^{n_1} h_2^{n_2} \ldots \text{ if } \xi \in \Gamma_C/\mathscr{I},\ \xi = \xi_1^{n_1} \xi_2^{n_2} \ldots .$$

The Klein transformation is the map of $\mathscr{F}_0$ into $\tilde{\mathscr{F}}$ defined by

$$\psi \in \mathscr{F}_0 \to (h_{\xi_\psi}, \psi). \tag{4.7}$$

The range of the map (4.7) is a subgroup $\mathscr{F}$ of $\tilde{\mathscr{F}}$ which, together with the restriction to it of the homomorphism σ of $\tilde{\mathscr{F}}$ onto Γ_C, yields another solution of (i) to (iv) above. As the map (4.7) is one-to-one we will identify $\mathscr{F}$ and $\mathscr{F}_0$ as sets, and also denote the elements of $\mathscr{F}$ by

[13] By writing $\psi_1\psi_2\psi_3 = \lambda\psi_3\psi_2\psi_1$ for mutually spacelike separated fields and calculating λ from ε in two different ways, we deduce an identity for ε, which by definition (A.1.1) reads $\delta\varepsilon = 1$ (cf. also Theorem 5.3).

ψ, ψ_1, etc. ..., remembering that the product operations in $\mathscr{F}$ and $\mathscr{F}_0$ may differ by a sign.

Combining our construction with Eq. (4.6) we have

4.2. Theorem. *Let $\psi_1 \in \mathscr{F}(\mathcal{O}_1)$, $\psi_2 \in \mathscr{F}(\mathcal{O}_2)$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ spacelike separated double cones; then*

$$\psi_1 \psi_2 = \pm \psi_2 \psi_1$$

with the $(-)$ sign holding if and only if both ψ_1 and ψ_2 are Fermi fields. In other words, we have normal commutation relations in $\mathscr{F}$.

Before discussing representations, we introduce the gauge group $\mathscr{G}$ and define the action of $\mathscr{G}$ and of the group $\mathscr{P}$ on the field group $\mathscr{F}$.

In the present situation, $\Gamma_C/\mathscr{I}$ is a countable, discrete Abelian group; let $\mathscr{G}$ be its dual group. $\mathscr{G}$ is the separable, compact Abelian group consisting of all elements $g \in \mathscr{C}$ such that $g(\xi_1 \xi_2) = g(\xi_1) g(\xi_2)$ (g is a character) with the topology of pointwise convergence. Then $\Gamma_C/\mathscr{I}$ is canonically identified with the dual $\hat{\mathscr{G}}$ of the compact group $\mathscr{G}$ [4; § 31, Theorem 32] by $\chi_\xi(g) = g(\xi)$.

Being a subgroup of $\mathscr{C} \subset \tilde{\mathscr{F}}$, $\mathscr{G}$ induces inner automorphisms of $\tilde{\mathscr{F}}$. Computing the action of $g \in \mathscr{G}$ on $(f, \psi) \in \tilde{\mathscr{F}}$ explicitly, we get

$$(f, \psi) \rightarrow (f, {}_g\psi) \quad \text{where} \quad {}_g\psi = g(\xi_\psi)\psi = \chi_{\xi_\psi}(g)\psi\,. \tag{4.8}$$

Both the sets $\mathscr{F}_0$ and $\mathscr{F}$ are invariant under this action, and in this way we obtain faithful representations of $\mathscr{G}$ by automorphisms of $\mathscr{F}_0$ or $\mathscr{F}$, since the characters χ_ξ, $\xi \in \Gamma_C/\mathscr{I}$, separate $\mathscr{G}$.

Now let $L \in \mathscr{P}$ and $\psi \in \mathscr{F}$; we define

$${}_L\psi = \left(\alpha_L(U_\psi)\, U_L(s(\xi_\psi)), \xi_\psi\right), \tag{4.9}$$

where $U_L(\gamma)$, $\gamma \in \Gamma_C$, was introduced in Eq. (3.3). Assigning to $L \in \mathscr{P}$ the map $\psi \rightarrow {}_L\psi$, relations (3.5) and (3.9) show that we get a representation of $\mathscr{P}$ by automorphisms of the group $\mathscr{F}$ which satisfy

$${}_L\{\mathscr{F}(\mathcal{O})\} = \mathscr{F}(L\mathcal{O}), \qquad L \in \mathscr{P}.$$

The two actions $g \in \mathscr{G} \rightarrow {}_g\psi$, $L \in \mathscr{P} \rightarrow {}_L\psi$, $\psi \in \mathscr{F}$, clearly commute with each other, and we have a representation of the product group $\mathscr{G} \times \mathscr{P}$ by automorphisms of $\mathscr{F}$. The action of $(g, L) \in \mathscr{G} \times \mathscr{P}$ on $\mathscr{A} \subset \mathscr{F}$ reduces to that of α_L.

We will now construct field operators acting on a Hilbert space by displaying a suitable covariant representation (π, V) of $\{\mathfrak{A}, \sigma\}$.

We first define the representation π

$$\pi = \bigoplus_{\xi \in \Gamma_C/\mathscr{I}} \pi_\xi, \quad \text{with} \quad \pi_\xi = \pi_0 \circ s(\xi)\,. \tag{4.10}$$

It acts on the Hilbert space

$$\mathscr{H} = \bigoplus_{\xi \in \Gamma_C/\mathscr{I}} \mathscr{H}_\xi, \mathscr{H}_\xi \equiv \mathscr{H}_0 . \tag{4.10'}$$

Since the representations $\pi_\xi \equiv \pi_0 \circ s(\xi)$ are irreducible and pairwise inequivalent, π is multiplicity free and the commutant $\pi(\mathfrak{A})'$ coincides with the set of all bounded operators which reduce to a multiple of the identity on each $\mathscr{H}_\xi$. Therefore the representation of the group $\mathscr{C}$ defined by

$$f \in \mathscr{C} \to \mathscr{U}_f : (\mathscr{U}_f \Psi)(\xi) = f(\xi)\, \Psi(\xi), \qquad \Psi \in \mathscr{H} \tag{4.11}$$

maps $\mathscr{C}$ onto the unitary group of $\pi(\mathfrak{A})'$.

We introduce in $\mathscr{H}$ a unitary representation of $\Gamma_C/\mathscr{I}$, which is the "canonically conjugate" representation to $\mathscr{U} \mid \mathscr{G}$. Namely we define $\xi \in \Gamma_C/\mathscr{I} \to \mathscr{V}_\xi$:

$$(\mathscr{V}_\xi \Psi)(\xi') = \Psi(\xi' \xi), \qquad \Psi \in \mathscr{H},\ \xi' \in \Gamma_C/\mathscr{I}, \tag{4.12}$$

so that $\mathscr{V}_\xi$ implements $s(\xi)$, i.e.

$$\mathscr{V}_\xi \pi(A) \mathscr{V}_\xi^{-1} = \pi(s(\xi)(A)), \qquad A \in \mathfrak{A},\ \xi \in \Gamma_C/\mathscr{I}. \tag{4.12'}$$

The representation of the covering group of the Poincaré group is defined by

$$L \to U(L) = \bigoplus_{\xi \in \Gamma_C/\mathscr{I}} U_\xi(L) \tag{4.13}$$

where U_ξ is *the* continuous unitary representation of $\mathscr{P}$ on $\mathscr{H}_\xi \equiv \mathscr{H}_0$ such that (π_ξ, U_ξ) is covariant. Then (π, U) is a covariant representation of $\{\mathfrak{A}, \alpha\}$.

Now we can define a unitary representation of $\tilde{\mathscr{F}}$ on $\mathscr{H}$ assigning to $(f, \psi) = (f, U_\psi, \xi_\psi) \in \tilde{\mathscr{F}}$ the operator

$$\mathscr{U}_f \pi(U_\psi) \mathscr{V}_{\xi_\psi} . \tag{4.14}$$

From the definition (4.4) of σ, the fact that π is a representation and relation (4.12'), we see that as f runs through $\mathscr{C}$ and ψ is kept fixed, the operator (4.14) runs through the set of all unitary operators which implement the automorphism σ_ψ in the multiplicity free representation π.

We are interested in particular in the restriction of the representation (4.14) to the two subgroups of $\tilde{\mathscr{F}}$, consisting of the normal field group $\mathscr{F}$ and the gauge group $\mathscr{G}$; we will denote those restrictions by $\psi \in \mathscr{F} \to V(\psi)$ and $g \in \mathscr{G} \to U(g)$ respectively. Since (4.14) is a group representation we have (cf. Eq. (4.8)).

$$U(g)\, V(\psi)\, U(g)^{-1} = V({}_g\psi) = \chi_{\xi_\psi}(g)\, V(\psi), \qquad g \in \mathscr{G},\ \psi \in \mathscr{F}.$$

4.3. Remark. Let $\xi \to v(\xi)$ be the right regular representation of $\Gamma_C/\mathscr{I}$ acting on the Hilbert space $l^2(\Gamma_C/\mathscr{I})$ of square summable sequences on $\Gamma_C/\mathscr{I}$, and let $g \in \mathscr{G} \to u(g)$ be the representation of the dual group canonically conjugate to it, i.e. acting on an element of $l^2(\Gamma_C/\mathscr{I})$ by multiplying the ξ-term by $g(\xi)$. We have then

$$\mathscr{H} = \mathscr{H}_0 \otimes l^2(\Gamma_C/\mathscr{I}),$$
$$\mathscr{V}_\xi = I \otimes v(\xi),$$
$$U(g) = I \otimes u(g).$$

4.4. Remark. Let ψ be an element of $\mathscr{F}$; the operator $V(\psi)$ on $\mathscr{H}$ has the meaning of a destruction operator for the "charge" $\widehat{\sigma_\psi} = \xi_\psi \in \Gamma_C/\mathscr{I}$; indeed we see from Eqs. (4.10) and (4.10′) that

$$V(\psi)\mathscr{H}_\xi = \mathscr{H}_{\xi'}, \quad \text{where} \quad \xi' = \xi_\psi^{-1}\xi. \tag{4.15}$$

Next we verify the Lorentz covariance of the operators $V(\psi)$. Recalling the definitions of the transformed field (4.9) of $U(L)$ (4.13) and of $V(\psi)$ (4.14), we see that properties (4.15) and (3.9) imply

$$U(L)\,V(\psi)\,U(L)^{-1} = V({}_L\psi), \qquad L \in \mathscr{P},\ \psi \in \mathscr{F}.$$

Let $\mathfrak{F}(\mathscr{O})$ denote the von Neumann algebra generated by $\{V(\psi)\mid\psi \in \mathscr{F}(\mathscr{O})\}$, with $\mathscr{O}$ a double cone in Minkowski space, and $\mathfrak{F}$ the C^*-algebra generated by all $\mathfrak{F}(\mathscr{O})$'s. Clearly $\mathfrak{F}(\mathscr{O})$ has the property

$$U(L)\,\mathfrak{F}(\mathscr{O})\,U(L)^{-1} = \mathfrak{F}(L\mathscr{O}), \qquad L \in \mathscr{P}; \tag{4.16}$$

and, if $\mathscr{O}_1, \mathscr{O}_2$ are spacelike separated double cones,

$$\mathfrak{F}(\mathscr{O}_1) \subset \pi(\mathfrak{A}(\mathscr{O}_2))'. \tag{4.17}$$

4.5. Proposition. $\pi(\mathfrak{A}(\mathscr{O})) = \mathfrak{F}(\mathscr{O}) \cap U(\mathscr{G})'$; $\pi(\mathfrak{A}) = \mathfrak{F} \cap U(\mathscr{G})'$.

Proof. Consider the following sets: first, the linear span of all $V(\psi)$, $\psi \in \mathscr{F}(\mathscr{O})$; secondly, the linear span of all $\pi(U)$, $U \in \mathscr{A}(\mathscr{O})$, $\mathscr{O}$ being a double cone. They are both *-algebras because of the group property, the first includes the second and they are weakly dense in $\mathfrak{F}(\mathscr{O})$ and $\pi(\mathfrak{A}(\mathscr{O}))$ respectively. Define the linear map m on $\mathscr{B}(\mathscr{H})$ by $B \in \mathscr{B}(\mathscr{H}) \to m(B) = \int_{\mathscr{G}} U(g)\,B\,U(g)^{-1}\,d\mu(g)$, when μ denotes normalized Haar measure on $\mathscr{G}$. Then m is a normal, positive projection map of $\mathscr{B}(\mathscr{H})$ onto $U(\mathscr{G})'$ of norm 1 (see [1], Lemma 3.1). Clearly $m(V(\psi)) = 0$ unless $\xi_\psi = \iota$, i.e. $V(\psi) \in \pi(\mathfrak{A})$. Hence m maps the first of the two *-algebras considered above onto the second, and, being normal, it maps $\mathfrak{F}(\mathscr{O})$ onto $\pi(\mathfrak{A}(\mathscr{O}))$.

This proves the first statement in the lemma; the second is then a consequence of the norm continuity of m.

V. The Construction and Classification of Field Groups

Our task in this section is twofold; we shall show that we can construct field groups $\mathscr{F}$ without making restrictive assumptions on the nature of the discrete Abelian group $\Gamma_C/\mathscr{I}$ of sectors, and we shall also classify all possible field groups. For convenience we denote $\Gamma_C/\mathscr{I}$ by $\hat{\mathscr{G}}$ in recognition of the fact that it may be interpreted as the dual group of an Abelian gauge group $\mathscr{G}$. If we can construct a group satisfying (i) to (iv) of the last section, we get a commutative diagram, where 1 denotes the group of one element.

$$\begin{array}{ccccc} \boldsymbol{T} & \rightarrow & \mathscr{A} & \xrightarrow{\sigma} & \mathscr{I} \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{T} & \rightarrow & \mathscr{F} & \xrightarrow{\sigma} & \Gamma_C \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & \hat{\mathscr{G}} & \rightarrow & \hat{\mathscr{G}} \end{array} \tag{5.1}$$

Each row or column is *a short exact sequence* of groups so that the third group in any row or column is the quotient of the second group modulo the first. In other words the second group of any row or column is an *extension* of the first by the third. In particular we may regard $\mathscr{F}$ as an extension of $\mathscr{A}$ by $\hat{\mathscr{G}}$ and this makes it possible to reduce the problem of constructing $\mathscr{F}$ to a standard cohomological problem. This problem is solved explicitly in this section and the reader is referred to Appendix I for the basic definitions of the cohomology theory of groups and the connexion with group extensions.

Before constructing a field group we investigate the extension Γ_C of $\mathscr{I}$ by $\hat{\mathscr{G}}$. Let $j:\hat{\mathscr{G}}\rightarrow\Gamma_C$ be a section for the canonical map $\Gamma_C\rightarrow\hat{\mathscr{G}}$ with $j(\iota)=\iota$. Then

$$j(\xi_1)j(\xi_2)=a(\xi_1,\xi_2)j(\xi_1\xi_2) \quad \text{for} \quad \xi_1,\xi_2\in\hat{\mathscr{G}} \tag{5.2}$$

where $a(\xi_1,\xi_2)\in\mathscr{I}$ satisfies the identity

$$a(\xi_1,\xi_2)\,a(\xi_1\xi_2,\xi_3)={}^{j(\xi_1)}a(\xi_2,\xi_3)\,a(\xi_1,\xi_2\xi_3)\,. \tag{5.3}$$

Here we have written ${}^{j(\xi_1)}a(\xi_2,\xi_3)$ for $j(\xi_1)\,a(\xi_2,\xi_3)j(\xi_1)^{-1}$. If $\hat{\mathscr{G}}$ has a countable family of independent generators, we showed in Section IV that we could take j to be a homomorphism s and we would then get $a(\xi_1,\xi_2)\equiv 1$. However there is no reason to suppose that we can choose j to be a homomorphism in general.

Let $i:\mathscr{I}\rightarrow\mathscr{A}$ be a section of $\sigma:\mathscr{A}\rightarrow\mathscr{I}$ with $i(\iota)=I$, then from (5.3) we get

$$ia(\xi_1,\xi_2)\,ia(\xi_1\xi_2,\xi_3)=z(\xi_1,\xi_2,\xi_3)\,[j(\xi_1)\,(ia(\xi_2,\xi_3))]\,ia(\xi_1,\xi_2\xi_3) \tag{5.4}$$

where $z(\xi_1, \xi_2, \xi_3) \in \boldsymbol{T}$, the centre of $\mathscr{A}$. Routine computations show that z is actually a 3-cocycle and that changing the sections i and j changes z by a coboundary. The cohomology class of z is thus uniquely determined by the extension Γ_C of $\mathscr{I}$ by $\hat{\mathscr{G}}$.

Now suppose that $\mathscr{F}$ is a field group and let $k: \hat{\mathscr{G}} \to \mathscr{F}$ be a section of the canonical map $\mathscr{F} \to \hat{\mathscr{G}}$ with $j = \sigma \circ k$ and $k(\imath) = I$. Then

$$k(\xi_1)\, k(\xi_2) = b(\xi_1, \xi_2)\, k(\xi_1 \xi_2) \quad \text{for} \quad \xi_1, \xi_2 \in \hat{\mathscr{G}} \tag{5.5}$$

where $b(\xi_1, \xi_2) \in \mathscr{A}$ satisfies the identity

$$b(\xi_1, \xi_2)\, b(\xi_1 \xi_2, \xi_3) = [j(\xi_1)(b(\xi_2, \xi_3))]\, b(\xi_1, \xi_2 \xi_3)\,[14]. \tag{5.6}$$

As $j = \sigma \circ k$, $a(\xi_1, \xi_2) = \sigma_{b(\xi_1, \xi_2)}$ and hence

$$b(\xi_1, \xi_2) = \lambda(\xi_1, \xi_2)\, i a(\xi_1, \xi_2) \tag{5.7}$$

with $\lambda \in C^2(\hat{\mathscr{G}}, \boldsymbol{T})$. Comparing (5.4) and (5.6), we see that $z = \delta\lambda$ is a necessary condition for a field group to exist. On the other hand we shall show that this condition is also sufficient because if $z = \delta\lambda$, we may define $b(\xi_1, \xi_2)$ by (5.7) and derive (5.6) using (5.4). We now construct the field group $\mathscr{F}_\lambda$ by setting $\mathscr{F}_\lambda = \{(U, \xi) \mid U \in \mathscr{A}, \xi \in \hat{\mathscr{G}}\}$ with the multiplication law

$$(U_1, \xi_1)(U_2, \xi_2) = (U_1 j(\xi_1)(U_2)\, b(\xi_1, \xi_2), \xi_1 \xi_2)\,. \tag{5.8}$$

The associative law for $\mathscr{F}_\lambda$ follows from (5.6), (5.7), and (5.3), and we verify that $\mathscr{F}_\lambda$ is a group. $\mathscr{A}$ is identified with the subgroup $\{(U, \imath) \mid U \in \mathscr{A}\}$. Defining

$$\sigma_{(U, \xi)} = \sigma_U j(\xi) \tag{5.9}$$

we get a homomorphism σ from $\mathscr{F}_\lambda$ onto Γ_C extending $\sigma: \mathscr{A} \to \mathscr{I}$. Conditions (i) to (iv) are easily verified and we have proved

5.1. Theorem. *There exists a field group if and only if the* 3*-cocycle* z *defined by* (5.4) *is a coboundary.*

As explained in Appendix I, this Theorem is a consequence of a standard result in the theory of group extensions [5].

We showed in the previous section that field groups $\mathscr{F}$ exist if $\hat{\mathscr{G}}$ is a countable restricted product of cyclic groups and we use this result to prove the general existence theorem.

5.2. Theorem. *Every observable algebra satisfying duality in the vacuum sector has at least one associated field group.*

Proof. The set of functions from $\hat{\mathscr{G}} \times \hat{\mathscr{G}}$ into $\boldsymbol{T}$ with the topology of pointwise convergence is a compact space by Tychonoff's Theorem

[14] We may regard a and b as 2-cochains with values in $\mathscr{I}$ and $\mathscr{A}$ respectively and Eqs. (5.3) and (5.6) as saying that these 2-cochains are 2-cocycles provided we recognize that $\mathscr{I}$ and $\mathscr{A}$ are not Abelian and that $\hat{\mathscr{G}}$ acts non-trivially on $\mathscr{I}$ and $\mathscr{A}$.

[6; I § 9 N° 5]. $C^2(\hat{\mathscr{G}}, \boldsymbol{T})$ is clearly a closed subset of this compact space and is hence itself a compact space with the topology of pointwise convergence. Let G be any finitely generated subgroup of $\hat{\mathscr{G}}$ and let $F(G) = \{\lambda \in C^2(\hat{\mathscr{G}}, \boldsymbol{T}) \mid \delta\lambda = z \text{ on } G\}$. $F(G)$ must be a closed subset since the condition $\delta\lambda = z$ on G is just a simple restriction of the values of λ at specific points of $\hat{\mathscr{G}} \times \hat{\mathscr{G}}$. $F(G)$ is non-empty by the results of the previous section since every finitely generated Abelian group is a finite product of cyclic groups [7, § 20]. If $G_1 \subset G_2$, then $F(G_1) \supset F(G_2)$. Thus $\{F(G) \mid G$ a finitely generated subgroup of $\hat{\mathscr{G}}\}$ has the finite intersection property. Hence $\bigcap_G \{F(G) \mid G$ a finitely generated subgroup of $\hat{\mathscr{G}}\}$ is non-empty [6; Chapter I, § 9, N° 1]. Now $\hat{\mathscr{G}}$ is the union of its finitely generated subgroups so if λ is in this intersection, $\delta\lambda = z$ and the result follows from Theorem 5.1.

Once the existence of field groups has been established, the classification of all possible field groups is a standard exercise in the theory of group extensions [5]. We call two field groups $\mathscr{F}$ and $\mathscr{F}'$ *equivalent* if there is an isomorphism of $\mathscr{F}$ onto $\mathscr{F}'$ which reduces to the identity on $\mathscr{A}$. This is the natural notion of equivalence based on the conditions (i) to (iv) of the previous section since the homomorphism $\sigma : \mathscr{F} \to \Gamma_C$ is uniquely determined by $\mathscr{F}$. The discussion preceding Theorem 5.1 shows that every field group is equivalent to a field group of the form $\mathscr{F}_\lambda$ where λ is a 2-cochain with $\delta\lambda = z$. Any two such cochains λ, λ' differ by a 2-cocycle: $\lambda'\lambda^{-1} = f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$. If $f = \delta h$ then

$$(U, \xi) \to (U h(\xi), \xi) \tag{5.10}$$

is an isomorphism of $\mathscr{F}_\lambda$ onto $\mathscr{F}_{\lambda'}$ establishing the equivalence of $\mathscr{F}_\lambda$ and $\mathscr{F}_{\lambda'}$. Conversely, any equivalence of these field groups must be realized by an isomorphism of the form (5.10), so if $\mathscr{F}_\lambda$ and $\mathscr{F}_{\lambda'}$ are equivalent $\lambda'\lambda^{-1} = \delta h$. Hence the equivalence classes of field groups are in 1–1 correspondence with the elements of $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$.

This correspondence can be interpreted by relating it to the commutation structure of the field group. Theorem 4.1 shows that the commutator cocycle ε_λ of any field group $\mathscr{F}_\lambda$ determines the commutation relations of spacelike separated elements of $\mathscr{F}_\lambda$. Let $\lambda' = f\lambda$ with $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ then

$$\varepsilon_{\lambda'} = \hat{f}\varepsilon_\lambda \tag{5.11}$$

where $\hat{f}(\xi_1, \xi_2) = f(\xi_1, \xi_2) f(\xi_2, \xi_1)^{-1}$. As shown in Lemma A.1.2 $\hat{f}$ may be identified with the cohomology class of f. Thus a field group is determined to within equivalence by the commutation relations of its spacelike separated elements. Let $n(\xi)$ be 1 or i according as ξ is a Bose or Fermi sector, i.e. according as ε_ξ is $+1$ or -1. Then the condition that $\mathscr{F}_\lambda$ has normal commutation relations may be written in the form

$$\varepsilon_\lambda = \delta n\,. \tag{5.12}$$

5.3. Theorem. *There is to within equivalence one field group with normal commutation relations. If $\mathscr{F}$ is any field group the corresponding commutator cocycle ε has the form*

$$\varepsilon = \hat{f}\,\delta n \tag{5.13}$$

where $\hat{f} \in H^2(\hat{\mathscr{G}}, \boldsymbol{T})$ and determines $\mathscr{F}$ to within equivalence. Thus (5.13) *determines a natural* $1-1$ *correspondence between the equivalence classes of field groups and the elements of $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$.*

Proof. We first adapt the proof of Theorem 5.2. If G is a finitely generated subgroup of $\hat{\mathscr{G}}$ and $\delta\lambda = z$ on G, let $\mathscr{F}_{G,\lambda}$ be the corresponding extension of $\mathscr{A}$ by G, and $\varepsilon_{G,\lambda}$ be the commutator cocycle of $\mathscr{F}_{G,\lambda}$. We now define

$$F'(G) = \{\lambda \in C^2(\hat{\mathscr{G}}, \boldsymbol{T}) \mid \delta\lambda = z \text{ on } G \text{ and } \varepsilon_{G,\lambda} = \delta n \text{ on } G\}\,.$$

$F'(G)$ is again closed because the condition $\varepsilon_{G,\lambda} = \delta n$ on G may, by virtue of (5.7) and (5.8), be written:

$$U_1 j(\xi_1)(U_2)\,\lambda(\xi_1, \xi_2) = \delta n(\xi_1, \xi_2)\, U_2 j(\xi_2)(U_1)\,\lambda(\xi_2, \xi_1) \tag{5.14}$$

for all $(U_1, \xi_1), (U_2, \xi_2)$ such that $\xi_1, \xi_2 \in G$ and the automorphisms $\sigma_{U_1} j(\xi_1)$ and $\sigma_{U_2} j(\xi_2)$ have spacelike separated supports. The existence of a field group with normal commutation relations now follows as in Theorem 5.2. Eqs. (5.11) and (5.12) show that ε has the form (5.13). The rest of the Theorem follows from the previous discussion.

The cohomology groups $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$ are calculated explicitly in Appendix I for the case that $\hat{\mathscr{G}}$ is a finite product of cyclic groups. The possible values for the commutator cocycle ε may then be calculated using (5.13). In any case ε has the following properties:

$$\begin{aligned} \varepsilon(\xi_2, \xi_1) &= \varepsilon(\xi_1, \xi_2)^{-1}\,, \\ \varepsilon(\xi_1\xi_2, \xi_3) &= \varepsilon(\xi_1, \xi_3)\,\varepsilon(\xi_2, \xi_3)\,, \\ \varepsilon(\xi, \xi) &= \varepsilon_\xi = \pm 1\,, \end{aligned} \tag{5.15}$$

where the sign $(+)$ corresponds to Bose fields and the $(-)$ sign to Fermi fields, and the second relation follows from the associativity of the product in $\mathscr{F}$ (cf. Theorem 4.1 and Lemma A.1.1).

VI. Field Algebras and Twisted Duality

Let $\mathscr{F}$ be a field group and define $\tilde{\mathscr{F}} = \mathscr{C} \times \mathscr{F}/N$ and the homomorphism $\sigma : \tilde{\mathscr{F}} \to \Gamma_C$ as in Section IV. We first show that any field group is equivalent to a subgroup of $\tilde{\mathscr{F}}$. Any element of $\tilde{\mathscr{F}}$ may be written uniquely in the form (g, ψ) where $g \in C^1(\hat{\mathscr{G}}, \boldsymbol{T}) \subset \mathscr{C}$ and $\psi \in \mathscr{F}$. Given

$f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ define a map $\xi \to h_\xi$ from $\hat{\mathscr{G}}$ to $C^1(\hat{\mathscr{G}}, \boldsymbol{T})$ by setting

$$h_{\xi_2}(\xi_1) = f(\xi_1, \xi_2)^{-1}. \tag{6.1}$$

Since $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ we have

$$h_{\xi_1}(\xi)\, h_{\xi_2}(\xi\xi_1) = f(\xi_1, \xi_2)\, h_{\xi_1\xi_2}(\xi) \quad \text{for all} \quad \xi \in \hat{\mathscr{G}}. \tag{6.2}$$

Let $\psi_1, \psi_2 \in \mathscr{F}$ with $\xi_1 = \xi_{\psi_1}$ and $\xi_2 = \xi_{\psi_2}$ then by (6.2)

$$(h_{\xi_1}, \psi_1)(h_{\xi_2}, \psi_2) = (h_{\xi_1\xi_2}, f(\xi_1, \xi_2)\psi_1\psi_2). \tag{6.3}$$

Thus $\mathscr{F}^h = \{(h_\xi, \psi) \mid \psi \in \mathscr{F}, \xi = \xi_\psi\}$ is a field group and a subgroup of $\tilde{\mathscr{F}}$. The multiplication law in $\mathscr{F}^h$ differs from that in $\mathscr{F}$ by the presence of the phase factors $f(\xi_1, \xi_2)$. Since $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ was arbitrary we have shown that every field group is equivalent to a subgroup of $\tilde{\mathscr{F}}$. Every field group which is a subgroup of $\tilde{\mathscr{F}}$ must have the form $\mathscr{F}^h$ where h satisfies (6.1) for some $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$. The map $\psi \in \mathscr{F} \to (h_{\xi_\psi}, \psi) \in \mathscr{F}^h$ is a generalization of the Klein transformation (4.7) so we may sum up the results of this paragraph by saying that the field groups are related to each other by generalized Klein transformations.

Now if $\psi_1 \in \mathscr{F}(\mathcal{O}_1)$ and $\psi_2 \in \mathscr{F}(\mathcal{O}_2)$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated double cones then $(1, \psi_1)$ and (h_{ξ_2}, ψ_2) commute in $\tilde{\mathscr{F}}$ if and only if

$$f(\xi\xi_1, \xi_2)^{-1} f(\xi, \xi_2) = \varepsilon(\xi_1, \xi_2) \tag{6.4}$$

where ε is the commutator cocycle of $\mathscr{F}$. However ε satisfies (5.15) so that

$$f = \varepsilon^{-1} \tag{6.5}$$

is the unique solution of (6.4) for all $\xi_1, \xi_2 \in \hat{\mathscr{G}}$. The subgroup of $\tilde{\mathscr{F}}$ defined by taking $f = \varepsilon^{-1}$ in (6.1) will be called the *twisted field group* corresponding to $\mathscr{F}$ and will be denoted by $\mathscr{F}^t$. It is characterized by the condition that $\mathscr{F}(\mathcal{O}_1)$ and $\mathscr{F}^t(\mathcal{O}_2)$ commute whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated double cones. $\mathscr{F}$ and $\mathscr{F}^t$ will be equivalent if and only if $\hat{\varepsilon} \equiv 1$ and from (5.13) this condition may be written

$$\varepsilon^2 \equiv 1 \tag{6.6}$$

and is always fulfilled if $\mathscr{F}$ has normal commutation relations.

The construction of a field algebra proceeds as in Section IV. The action of the gauge group on $\tilde{\mathscr{F}}$ is defined by (4.8) and the action of the Poincaré group on $\mathscr{F}$ is defined analogously to (4.9) by

$$_L\psi = (\alpha_L(U_\psi)\, U_L(j(\xi_\psi)), \xi_\psi). \tag{6.7}$$

The representation π of $\mathfrak{A}$ is defined by

$$\pi = \bigoplus_{\xi \in \hat{\mathscr{G}}} \pi_\xi, \quad \text{with} \quad \pi_\xi = \pi \circ j(\xi) \tag{6.8}$$

and acts on the Hilbert space $\mathscr{H}$ of (4.10′). $\mathscr{U}_f$ is defined by (4.11) but instead of (4.12) we now get

$$(\mathscr{V}_\xi \Psi)(\xi') = \pi_0(b(\xi', \xi))\,\Psi(\xi'\xi), \qquad \Psi \in \mathscr{H},\ \xi, \xi' \in \hat{\mathscr{G}}\,, \tag{6.9}$$

where $\mathscr{F}$ is defined by (5.8) and b satisfies (5.6). We have

$$\mathscr{V}_\xi \mathscr{V}_{\xi'} = \pi(b(\xi', \xi))\,\mathscr{V}_{\xi\xi'} \tag{6.10}$$

and

$$\mathscr{V}_\xi \pi(A) \mathscr{V}_\xi^{-1} = \pi(j(\xi)(A))\,. \tag{6.11}$$

Eqs. (5.8), (6.10), and (6.11) show that we can define a representation of $\tilde{\mathscr{F}}$ on $\mathscr{H}$ by (4.14). $\mathfrak{F}(\mathcal{O})$ is defined as in Section IV and Proposition 4.5 is valid without modification.

As we have a representation of the group $\tilde{\mathscr{F}}$ at our disposal, we may also define in a natural way a field algebra $\mathfrak{F}^h$ corresponding to the field group $\mathscr{F}^h$. In particular for $\mathfrak{F}^t$ we have immediately

6.1. Proposition. *$\mathfrak{F}$ satisfies twisted locality (cf. [1], Definition 4.2), i.e.*

$$\mathfrak{F}(\mathcal{O}')^- \subset \mathfrak{F}^t(\mathcal{O})' \tag{6.12}$$

for every double cone $\mathcal{O}$.

We are prevented from proving twisted duality (cf. [1], Definition 4.3) for all double cones because it is not clear whether there are states of each possible charge strictly localized in an arbitrarily small double cone. However we have

6.2. Proposition. *If $\mathcal{O}$ is a double cone such that $\mathfrak{F}(\mathcal{O})$ contains fields of all charges, i.e. if $\Gamma(\mathcal{O})$ maps onto $\hat{\mathscr{G}}$, then $\mathfrak{F}$ satisfies twisted duality for $\mathcal{O}$, i.e.*

$$\mathfrak{F}(\mathcal{O}')^- = \mathfrak{F}^t(\mathcal{O})'\,. \tag{6.13}$$

Proof. Let $A \in m(\mathfrak{F}(\mathcal{O}')')$ then by duality in each sector for the observable algebra, $A = \bigoplus_{\xi \in \hat{\mathscr{G}}} \pi_\xi(A_\xi)$ with $A_\xi \in \mathfrak{A}(\mathcal{O})$. Without loss of generality we may suppose the section $j : \hat{\mathscr{G}} \to \Gamma_C$ has been chosen so that $j(\xi)$ has its support spacelike to $\mathcal{O}$. Then $\mathscr{V}_\xi \in \mathfrak{F}(\mathcal{O}')$ and commutes with A. Hence $A_\iota = A_\xi$. But this must hold for all $\xi \in \hat{\mathscr{G}}$ so $A = \pi(A_\iota)$ and

$$m(\mathfrak{F}(\mathcal{O}')') = \mathfrak{A}(\mathcal{O}) = m(\mathfrak{F}^t(\mathcal{O}))\,. \tag{6.14}$$

Let $F \in \mathfrak{F}(\mathcal{O}')'$ have tensor character ξ and pick a unitary $U \in \mathfrak{F}^t(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O}')'$ of this same character, then $FU^{-1} \in m(\mathfrak{F}(\mathcal{O}')') = m(\mathfrak{F}^t(\mathcal{O})) \subset \mathfrak{F}^t(\mathcal{O})$. Hence $F \in \mathfrak{F}^t(\mathcal{O})$. But $\mathfrak{F}(\mathcal{O}')'$ is a gauge invariant von Neumann algebra and is therefore generated by its elements of definite tensor character under $\mathscr{G}$. Thus $\mathfrak{F}(\mathcal{O}')' \subset \mathfrak{F}^t(\mathcal{O})$ and taking commutants and using (6.12) we have (6.13) as required.

Appendix I

For the reader's convenience we collect in this Appendix a few definitions and results in the cohomology theory of groups and refer him to Eilenberg and MacLane [5, 8] for further details. Let G be a group then an *n-cochain* with values in $\boldsymbol{T}$ is a function $f(g_1, g_2, \ldots, g_n)$ of n group elements, $g_1, g_2, \ldots g_n \in G$, with values in $\boldsymbol{T}$ such that $f(g_1, g_2, \ldots, g_n) = 1$ if any g_i is the identity. The set of n-cochains forms a group under pointwise multiplication which is denoted by $C^n(G, \boldsymbol{T})$. We define the coboundary homomorphism $\delta : C^n(G, \boldsymbol{T}) \to C^{n+1}(G, \boldsymbol{T})$ by

$$\begin{aligned}(\delta f)&(g_1, g_2, \ldots, g_{n+1})\\ &= f(g_2, g_3, \ldots, g_{n+1}) \prod_{i=1}^{n} f(g_1, g_2, \ldots, g_i g_{i+1}, \ldots, g_{n+1})^{(-1)^i}\\ &\quad \cdot f(g_1, \ldots, g_n)^{(-1)^{n+1}}.\end{aligned} \tag{A.1.1}$$

An n-cochain is called an *n-cocycle* if $\delta f \equiv 1$ and an *n-coboundary* if it is of the form $f = \delta f'$ with $f' \in C^{n-1}(G, \boldsymbol{T})$. A routine calculation shows that $\delta\delta f \equiv 1$ so that every coboundary is a cocycle. The subgroups of n-cocycles and n-coboundaries are denoted by $Z^n(G, \boldsymbol{T})$ and $B^n(G, \boldsymbol{T})$ respectively. The quotient group of these two groups is called the n-th cohomology group $H^n(G, \boldsymbol{T}) \equiv Z^n(G, \boldsymbol{T})/B^n(G, \boldsymbol{T})$.

As pointed out in Section V, we may regard a field group $\mathscr{F}$ as an *extension* of $\mathscr{A}$ by $\hat{\mathscr{G}}$, that is $\mathscr{F}$ contains $\mathscr{A}$ as a normal subgroup and $\hat{\mathscr{G}} \cong \mathscr{F}/\mathscr{A}$. Let $A(\mathscr{A})$ denote the group of automorphisms of $\mathscr{A}$ and $I(\mathscr{A})$ the normal subgroup of inner automorphisms. Any extension $\mathscr{F}$ of $\mathscr{A}$ by $\hat{\mathscr{G}}$ is associated with a homomorphism $\theta : \hat{\mathscr{G}} \to A(\mathscr{A})/I(\mathscr{A})$. θ is constructed as follows: let $k : \hat{\mathscr{G}} \to \mathscr{F}$ be any section of the canonical map $\mathscr{F} \to \hat{\mathscr{G}}$, then $U \to k(\xi)\, U k(\xi)^{-1}$ is an automorphism of $\mathscr{A}$ and determines a mapping of $\hat{\mathscr{G}}$ into $A(\mathscr{A})$. The automorphisms $U \to k(\xi_1 \xi_2)\, U k(\xi_1 \xi_2)^{-1}$ and $U \to k(\xi_1)\, k(\xi_2)\, U k(\xi_2)^{-1} k(\xi_1)^{-1}$ differ by an inner automorphism. Hence the composed map $\hat{\mathscr{G}} \to A(\mathscr{A}) \to A(\mathscr{A})/I(\mathscr{A})$ defines a homomorphism θ. θ is independent of the choice of section.

Now if $\mathscr{F}$ is actually a field group then $j = \sigma \circ k$ is a section for $\Gamma_C \to \hat{\mathscr{G}}$ and condition (iii) of Section IV gives

$$k(\xi)\, U k(\xi)^{-1} = j(\xi)(U). \tag{A.1.2}$$

Hence θ is determined by the extension Γ_C of $\mathscr{I}$ by $\hat{\mathscr{G}}$ since the image of $U \to j(\xi)(U)$ in $A(\mathscr{A})/I(\mathscr{A})$ is again independent of the choice of section j. Now Γ_C may be considered as a subgroup of $A(\mathscr{A})$ since a $\gamma \in \Gamma_C$ extends uniquely from $\mathscr{A}$ to give an automorphism of $\mathfrak{A}$. Thus we see that θ determines the extension Γ_C of $\mathscr{I}$ by $\hat{\mathscr{G}}$ since Γ_C considered as a subgroup of $A(\mathscr{A})$ is just the inverse image of $\theta(\hat{\mathscr{G}})$ under the canonical map $A(\mathscr{A}) \to A(\mathscr{A})/I(\mathscr{A})$. The problem of constructing field groups is thus equivalent to the problem of constructing extensions of $\mathscr{A}$ by $\hat{\mathscr{G}}$ corresponding to the homomorphism θ. The discussion in Section V leading to Theorem 5.1 corresponds to the discussion given by Eilenberg and MacLane [5]. As they showed, whenever θ allows extensions, the equivalence classes of such extensions are in $1-1$ correspondence with the elements of $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$. Note that because $j(\xi)$ being linear leaves the centre $\boldsymbol{T}$ of $\mathscr{A}$ pointwise fixed, $\hat{\mathscr{G}}$ does not act on $\boldsymbol{T}$ and the cohomology groups are as defined above.

The problem of determining all possible field groups is now reduced to that of calculating the cohomology groups $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$. Given $f \in C^2(\hat{\mathscr{G}}, \boldsymbol{T})$ define

$$\hat{f}(\xi_1, \xi_2) = f(\xi_1, \xi_2) f(\xi_2, \xi_1)^{-1} \tag{A.1.3}$$

A.1.1. Lemma. *$f \to \hat{f}$ is a homomorphism of $C^2(\hat{\mathscr{G}}, \boldsymbol{T})$ into $C^2(\hat{\mathscr{G}}, \boldsymbol{T})$ such that*

a) *If $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ then $\hat{f} \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$.*
b) *If $f \in B^2(\hat{\mathscr{G}}, \boldsymbol{T})$ then $\hat{f} \equiv 1$.*
c) $\hat{\hat{f}} = \hat{f}^2$.
d) $\hat{f}(\xi_2, \xi_1) = \hat{f}(\xi_1, \xi_2)^{-1}$.
e) $\hat{f}(\xi, \xi) = 1$.
f) *If $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ then $\hat{f}(\xi_1\xi_2, \xi_3) = \hat{f}(\xi_1, \xi_3)\hat{f}(\xi_2, \xi_3)$.*

The proof of this Lemma involves straightforward computations which we omit. Since $f \to \hat{f}$ is a homomorphism, b) implies that $\hat{f}$ is constant on $B^2(\hat{\mathscr{G}}, \boldsymbol{T})$-cosets. Hence for $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$, $\hat{f}$ depends only on the cohomology class of f.

A.1.2. Lemma. *Let $[f]$ denotes the cohomology class of f, then $[f] \to \hat{f}$ defines an isomorphism of $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$ onto a subgroup of $Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$.*

Proof. In view of the above remarks $[f] \to \hat{f}$ is a homomorphism and it thus suffices to show that $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$, $\hat{f} \equiv 1$ implies $f \in B^2(\hat{\mathscr{G}}, \boldsymbol{T})$. Using the compactness argument as in Theorem 5.2, it suffices to take $\hat{\mathscr{G}}$ to be a finitely generated Abelian group. Suppose that $\hat{\mathscr{G}} = G_1 \times G_2$ and that we have proved the result for G_1 and G_2. Given $f \in Z^2(\hat{\mathscr{G}}, \boldsymbol{T})$ with $\hat{f} \equiv 1$, let f_i denote the restriction of f to G_i, $i = 1, 2$. Then $f_i = \delta g_i$, $i = 1, 2$. Given $\xi \in \hat{\mathscr{G}}$, $\xi = \xi_1 \xi_2$ with $\xi_i \in G_i$, $i = 1, 2$. Define

$$g(\xi) = f(\xi_1, \xi_2)^{-1} g_1(\xi_1) g_2(\xi_2) \tag{A.1.4}$$

Then

$$\begin{aligned} f(\xi_1\xi_2, \xi_1'\xi_2') &= f(\xi_1, \xi_2)^{-1} f(\xi_1, \xi_2\xi_1'\xi_2') f(\xi_2, \xi_1'\xi_2') \\ &= f(\xi_1, \xi_2)^{-1} f(\xi_1', \xi_2\xi_2')^{-1} f(\xi_1\xi_1', \xi_2\xi_2') f(\xi_1, \xi_1') f(\xi_2', \xi_1')^{-1} \\ &\quad \cdot f(\xi_2\xi_2', \xi_1') f(\xi_2, \xi_2') . \end{aligned}$$

Substituting on the right hand side for f in terms of g, g_1 and g_2 using (A.1.3), $f_i = \delta g_i$, $i = 1, 2$ and $\hat{f} \equiv 1$, we get:

$$f(\xi_1 \xi_2, \xi_1' \xi_2') = g(\xi_1 \xi_2) g(\xi_1' \xi_2') g(\xi_1 \xi_1' \xi_2 \xi_2')^{-1}.$$

Hence $f = \delta g$ and $f \in B^2(\hat{\mathscr{G}}, \boldsymbol{T})$ as required. Thus by induction it suffices to consider the case where $\hat{\mathscr{G}}$ is a cyclic group. However in this case $Z^2(\hat{\mathscr{G}}, \boldsymbol{T}) = B^2(\hat{\mathscr{G}}, \boldsymbol{T})$ and the result is trivial.

The fact that the homomorphism $[f] \to \hat{f}$ is $1-1$ has another interpretation, namely that there are no non-trivial Abelian extensions of $\boldsymbol{T}$ by $\hat{\mathscr{G}}$.

If $\hat{\mathscr{G}} = \prod_{i=1}^{n} \boldsymbol{Z}_{v_i}$, we may calculate $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$ explicitly. By Lemma A.1.1 f) it suffices to give the values of $\hat{f}$ on the generators $\xi_1, \xi_2, \ldots, \xi_n$ say of $\hat{\mathscr{G}}$. By Lemma A.1.1 we must have

$$\begin{aligned} \hat{f}(\xi_i, \xi_j) &= \hat{f}(\xi_j, \xi_i)^{-1}, \\ \hat{f}(\xi_i, \xi_i) &= 1, \qquad i, j = 1, 2 \ldots n, \\ \hat{f}(\xi_i, \xi_j)^{v_i} &= \hat{f}(\xi_i, \xi_j)^{v_j} = 1. \end{aligned} \tag{A.1.5}$$

Constructing 2-cocycles explicitly it is easy to show that (A.1.5) is the only restriction on $\hat{f}(\xi_i, \xi_j)$.

Appendix II. The Field Algebra as a Covariance Algebra

In Section VI we defined the field algebra $\mathfrak{F}$ through a special covariant representation (π, V) of $\{\mathfrak{A}, \sigma\}$. We could also have obtained that representation by considering the covariance algebra $\mathfrak{A}^{\mathscr{F}}$ [9] of $\mathfrak{A}$ with respect to the discrete group $\mathscr{F}$ and defining on it an invariant pure state $\widehat{\omega_0}$ by the relation

$$\widehat{\omega_0}(X) = \sum_{\psi \in \mathscr{A}} \omega_0(X(\psi)), \qquad X \in \mathfrak{A}^{\mathscr{F}}; \tag{A.2.1}$$

in fact (π, V) is easily identified with the representation $\pi_{\widehat{\omega_0}}$ of $\mathfrak{A}^{\mathscr{F}}$. This algebra appears then as an abstract field algebra; it has however the feature of being too large in the sense that some of its representations map $\mathscr{A} \subset \mathscr{F}$ and $\mathscr{A} \subset \mathfrak{A}$ into different sets.

We generalize here the notion of covariance algebra of $\mathfrak{A}$ relative to $\hat{\mathscr{G}}$ for our case where $\mathfrak{A}$ is acted upon (not by $\hat{\mathscr{G}}$ itself but) by the extension $\mathscr{F}$ of $\hat{\mathscr{G}}$ characterized up to an isomorphism by the element b of $H^2(\hat{\mathscr{G}}, \boldsymbol{T})$.

Let $j: \hat{\mathscr{G}} \to \mathscr{F}$ be a section, and define the collection $\mathfrak{A}^{\hat{\mathscr{G}}, b}$ of all those functions X from $\mathscr{F}$ to $\mathfrak{A}$ satisfying

(i) $\quad X(U^{-1}\psi) = X(\psi) U, \qquad U \in \mathscr{A},\ \psi \in \mathscr{F};$

(ii) $\quad \sum_{\psi' \in j(\hat{\mathscr{G}})} \|X(\psi')\| \equiv \|X\|_1 < \infty.$

If $X, Y \in \mathfrak{A}^{\hat{\mathscr{G}},b}$, and λ is a complex number, define the linear combination, adjoint and convolution by the relations (cf. [9], Eq. (5, 8, 10)):

$$\begin{aligned}(X+\lambda Y)(\psi) &= X(\psi)+\lambda Y(\psi), \qquad \psi \in \mathscr{F},\\ X^*(\psi) &= \sigma_\psi(X(\psi^{-1}))^*,\\ (X*Y)(\psi) &= \sum_{\psi' \in j(\hat{\mathscr{G}})} X(\psi')\,\sigma_{\psi'}(Y(\psi'^{-1}\psi)).\end{aligned} \tag{A.2.2}$$

If $j' : \hat{\mathscr{G}} \to \mathscr{F}$ is another section, $j'(\xi) = V_\xi^{-1} j(\xi)$ with $V_\xi \in \mathscr{A}$ for all $\xi \in \hat{\mathscr{G}}$. One verifies then easily that $\|X\|_1$ and $X * Y$ are independent of the section. Using this fact it is possible to verify by simple computations that, under the operations (A.2.2) and the norm $\|\,.\,\|_1$, $\mathfrak{A}^{\hat{\mathscr{G}},b}$ is a Banach $*$-algebra.

Defining (cf. [9], Eq. (18)):

$$\begin{aligned}\psi \in \mathscr{F} \to V(\psi) &: (V(\psi)X)(\psi') = \sigma_\psi(X(\psi^{-1}\psi')),\\ A \in \mathfrak{A} \to \varrho(A) &: (\varrho(A)X)(\psi') = AX(\psi');\end{aligned}$$

we can see that $\mathfrak{A}^{\hat{\mathscr{G}},b}$ is stable under the operations $V(\psi)$ and $\varrho(A)$ and that the following property holds

$$V(U) = \varrho(U) \quad \text{if} \quad U \in \mathscr{A}.$$

If $\hat{\pi}$ is an essential representation of $\mathfrak{A}^{\hat{\mathscr{G}},b}$, we obtain a covariant representation (π, V) of $\{\mathfrak{A}, \sigma\}$ setting

$$\begin{aligned}\hat{\pi} \to (\pi, V) : \pi(A)\,\hat{\pi}(X) &= \hat{\pi}(\varrho(A)X)\\ V(\psi)\,\hat{\pi}(X) &= \hat{\pi}(V(\psi)X).\end{aligned} \qquad A \in \mathfrak{A},\ \psi \in \mathscr{F},\ X \in \mathfrak{A}^{\hat{\mathscr{G}},b} \tag{A.2.3}$$

The map (A.2.3) sets up a one-to-one correspondence between the essential representations of $\mathfrak{A}^{\hat{\mathscr{G}},b}$ and the covariant representations (π, V) of $\{\mathfrak{A}, \sigma\}$ such that

$$\pi \mid \mathscr{A} = V \mid \mathscr{A}.$$

The results analogous to those of [9] are valid without further modifications.

Acknowledgements. R. Haag and J. E. Roberts would like to acknowledge the hospitality of the Faculté des Sciences de Marseille-Luminy in the final stage of this research, under contract D.G.R.S.T. 6700976 and D.R.M.E. 46/68.

J. E. Roberts wishes to thank the Royal Society for a research followship under their European Programme.

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