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The Damped Self-Interaction

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Abstract. A self-interaction with damped off-diagonal coefficients is used to illustrate techniques for dealing directly with Hamiltonians in strange representations of the CCR.

Introduction

It is clear by now that the existence of many inequivalent representations of the canonical commutation relations (CCR) is both the hope and the bane of the Hamiltonian approach to Quantum Field Theory. Since even the simplest Hamiltonians $\left(\text{for example: } \sum_{k=1}^{\infty} \omega_k (p_k^2 + q_k^2) + a_k q_k + b_k p_k - \lambda_k \text{ if the sequences of real numbers } \{a_k\} \text{ and } \{b_k\} \text{ are large enough}\right)$ do not make sense in the Fock representation one can hope that everything would be all right in another representation. However, the problem of finding the "right" representation and carrying through the analysis of the Hamiltonian in it does not seem, to say the least, to be easy. The usual approach to these problems is first to cut-off the Hamiltonian and develop a well-defined theory on Fock space and then to try to remove the cut-off (using the vacuum expectation values and/or the algebraic approach of Segal) and thereby recover a limiting theory and the "right" representation.

In this note we will sketch how the theory of infinite sums of self-adjoint operators on infinite tensor product spaces developed in [5] and analytic perturbation theory can be used to analyze directly the operator

$$A_\infty = \sum_{k=1}^{\infty} (\omega_k (p_k^2 + q_k^2) - \omega_k \tau_k) + \sum_{k,l,m,n=1}^{\infty} d_{k\,l\,m\,n}\, q_k\, q_l\, q_m\, q_n \,.$$

There will be no restriction on the on-diagonal d_{kkkk} except that they be positive, the off-diagonal $d_{k\,l\,m\,n}$ must be small. We will find a representation of the CCR such that A_∞ is well-defined and self-adjoint (for an appropriate choice of the renormalizing sequence $\{\tau_k\}$). We show that A_∞ is bounded below and has point spectrum of unit multiplicity as lowest point in its spectrum. We determine sufficient conditions on test functions so that when the field and its conjugate momentum are smeared with them in this representation they are self-adjoint. Finally, we show that if one developes a cut-off theory in this representation then the physical vacuum (the eigenfunction corresponding to the lowest spectrum of A_∞) is the limit in the Hilbert space of the cut-off physical vacuums.

§ 1. Self-adjointness

We begin by writing the formal operator A_∞ in the form

$$A_\infty = \sum_{k=1}^{\infty} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right) + \sum_{k,l,m,n}^{\infty}{}' d_{k\,l\,m\,n}\, q_k\, q_l\, q_m\, q_n$$

$\left(\omega_k = \sqrt{k^2 + m^2},\ d_k = d_{k\,k\,k\,k} \geqslant 0\right.$, the $d_{k\,l\,m\,n}$ are real and the prime means the sum is only over the off-diagonal $\left.k, l, m, n\right)$.

The following two lemmas give information about the differential operator $B = -\frac{d^2}{dx^2} + x^2 + cx^4$, $c \geqq 0$. Proofs may be found in § 5.

Lemma 1. *B is essentially self-adjoint or $\mathcal{S}(R)$, and it has discrete spectrum, its eigenvalues have multiplicity one, and the corresponding eigenfunctions are in $\mathcal{S}(R)$.*

Lemma 2. *Let λ be the lowest eigenvalue of B, λ_2 the next highest eigenvalue. Then $\lambda_2 - \lambda_1 > 3/2$ (independent of c) and $\lambda_1^3 \leqq 2\,(3/2)^3\,(1 + c)$.*

We now define τ_k to be the lowest eigenvalue of $-\frac{d^2}{dx^2} + x^2 + \frac{d_k}{\omega_k} x^4$ and let $\chi_k(x)$ be the corresponding normalized eigenfunction. Set $\chi = \Pi \bigotimes_{k=1}^{\infty} \chi_k(x)$ and let $H(\chi)$ be the infinite tensor product Hilbert space generated by the C_0-vector χ (von Neumann's terminology [4]). If we let q_k be the operator which acts by multiplication by x on the k^{th} component of vectors in $H(\chi)$ and p_k the operator which acts as $\frac{1}{i}\frac{d}{dx}$ on the k^{th} component then $\{q_k, p_k\}$, $k = 1, 2, \ldots$, is a representation of the canonical commutation relations on $H(\chi)$. For a short description of the infinite tensor product spaces of von Neumann and the infinite tensor product representations of the CCR see [7] or the Appendices in [5].

We now have a specific Hilbert space $H(\chi)$, and representation of the CCR. Let D be the finite span of the set $\left\{\psi;\ \psi \in H(\chi),\ \psi = \Pi \bigotimes_{k=1}^{\infty} \psi_k(x),\right.$ $\left.\psi_k(x) \in \mathcal{S}(R),\ \psi_k(x) = \chi_k(x) \text{ for } k \geqq N, N \text{ arbitrary}\right\}$. Then D is dense in $H(\chi)$ and $\sum_{k=1}^{\infty} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$ certainly makes sense on D since for any given $\psi \in D$ the sum is actually finite.

Theorem 1. $\sum_{k=1}^{\infty} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$ *is essentially self-adjoint on D.*

Proof. The theorem follows as a special case of a general theorem on the essential self-adjointness of diagnonal sums in infinite tensor product spaces [5]. It can also be proven by observing that D contains a dense set of analytic vectors; namely, finite combinations of c_0-vectors of the form

$$\Theta_1 \otimes \Theta_2 \cdots \otimes \Theta_n \otimes \chi_{n+1} \otimes \cdots$$

Θ_i an eigenfunction of $p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4$.

We need the following estimate:

Lemma 3. *Let* $B = \omega\left(-\frac{d^2}{dx^2} + x^2 + cx^4 - \tau\right)$ *and* $f \in \mathscr{S}(R)$. *Then* $(x^8 f, f) \leqq \frac{2}{\omega^2 c^2}(B^2 f, f) + \left\{\frac{2}{c}\left(\frac{2}{3}\tau\right)^3 + 3\left(\frac{6}{c}\right)^{4/3}\right\}\|f\|^2$.

Proof. We denote $\frac{1}{i}\frac{d}{dx}$ by p and x by q. Then

$$\begin{aligned}(B^2 f, f) &= \left((\omega(p^2 + q^2 - \tau))^2 f, f\right) + ([\omega(p^2 + q^2 - \tau), \omega c q^4]_+ f, f) \\ &\quad + \omega^2 c^2 (q^8 f, f) \geqq ([\omega(p^2 + q^2 - \tau), \omega c q^4]_+ f, f) + \omega^2 c^2 (q^8 f, f) \\ &= \omega^2 c((p^2 q^4 + q^4 p^2) f, f) + 2\omega^2 c(q^4(q^2 - \tau) f, f) + \omega^2 c^2 (q^8 f, f).\end{aligned}$$

Since $pq^4 = \frac{4}{i} q^3 + q^4 p$ and $pq^3 = \frac{3}{i} q^2 + q^3 p$ we have:

$$\begin{aligned}((p^2 q^4 + q^4 p^2) f, f) &= \left(\left(p\left(q^4 p + \frac{4}{i} q^3\right) + \left(pq^4 - \frac{4q^3}{i}\right)p\right) f, f\right) \\ &\geqq \frac{4}{i}((pq^3 - q^3 p) f, f)) = -12(q^2 f, f).\end{aligned}$$

Furthermore since $q^4(q^2 - \tau) \geqq -\frac{1}{2}\left(\frac{2}{3}\tau\right)^3$ we have

$$2\omega^2 c(q^4(q^2 - \tau) f, f) \geqq -\omega^2 c\left(\frac{2}{3}\tau\right)^3 \|f\|^2.$$

Thus

$$(q^8 f, f) \leqq \frac{1}{\omega^2 c^2}(B^2 f, f) + \frac{1}{c}\left(\frac{2}{3}\tau\right)^3 \|f\|^2 + \frac{12}{c}(q^2 f, f).$$

The estimate $\frac{12}{c} q^2 \leqq \frac{1}{2} q^8 + 3\left(\frac{6}{c}\right)^{4/3}$ now proves the lemma.

Theorem 2. *Suppose that the off-diagonal coefficients satisfy:*

1. $\sum'' |d_{klmn}|\, |d_{rstu}| \left(1 + \left(\frac{\omega_k}{d_k}\right)^{4/3} + \left(\frac{\omega_l}{d_l}\right)^{4/3} + \cdots \left(\frac{\omega_u}{d_u}\right)^{4/3}\right) < \infty.$

2. $\frac{1}{4}\sum'' |d_{klmn}|\, |d_{rstu}| \left(\frac{1}{d_k^2} + \frac{1}{d_l^2} + \cdots + \frac{1}{d_u^2}\right) < 1.$

(The double prime means that no term where $k = l = m = n$ *or* $r = s = t = u$ *appears.) Then*

$$\sum_{k=1}^{\infty} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right) + \sum_{k,l,m,n=1}^{\infty}{}' d_{klmn}\, q_k q_l q_m q_n$$

is essentially self-adjoint on D.

Proof. Let $\psi \in D$. For any finite N, $\sum_{k,l,m,n=1}^{N}{}' d_{klmn} q_k q_l q_m q_n$ is a well-defined symmetric operator on D. Let $C_k = \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$ and

$$S_N = \{(k, l, m, n); k, l, m, n \leqq N\}$$

$$T_N = \{(k, l, m, n, r, s, t, u); k, l, m, n, r, s, t, u \leqq N\}$$

$$S_{N,M} = S_N - S_M, T_{N,M} = T_N - T_M .$$

We observe that $S_{N,M} \times S_{N,M} \subset T_{N,M}$. We now estimate

$$\left\|\sum_{S_{M,N}}{}' d_{klmn} q_k q_l q_m q_n \psi\right\|^2$$

$$\leqq \sum_{T_{M,N}}{}'' |d_{klmn}| \, |d_{rstu}| \, |(q_k q_l q_m q_n q_r q_s q_t q_u \psi, \psi)|$$

$$\leqq \frac{1}{8} \sum_{T_{M,N}}{}'' |d_{klmn}| \, |d_{rstu}| \, |(q_k^8 + q_l^8 + \cdots + q_u^8 \psi, \psi)|$$

$$\leqq \frac{1}{4} \sum_{T_{M,N}}{}'' |d_{klmn}| \, |d_{rstu}| \left(\left(\frac{C_k^2}{d_k^2} + \frac{C_l^2}{d_l^2} + \cdots + \frac{C_u^2}{d_u^2}\right) \psi, \psi\right)$$

$$+ \left(\frac{1}{4}\right) \sum_{T_{M,N}}{}'' |d_{klmn}| \, |d_{rstu}| \left(\left(\frac{\omega_k}{d_k}\right)\left(\frac{2}{3}\tau_k\right)^3 + 20 \left(\frac{\omega_k}{d_k}\right)^{4/3} + \cdots \right.$$

$$\left. + \left(\frac{\omega_u}{d_u}\right)\left(\frac{2}{3}\tau_u\right)^3 + 20 \left(\frac{\omega_u}{d_u}\right)^{4/3}\right) \|\psi\|^2$$

$$\leqq \left(\sup_k (C_k^2 \psi, \psi)\right) \cdot \sum_{T_{M,N}}{}'' \frac{1}{4} |d_{klmn}| \, |d_{rstu}| \left(\frac{1}{d_k^2} + \frac{1}{d_l^2} + \cdots + \frac{1}{d_u^2}\right)$$

$$+ 10 \sum_{T_{M,N}}{}'' |d_{klmn}| \cdot |d_{rstu}| \left(1 + \left(\frac{\omega_k}{d_k}\right)^{4/3} + \left(\frac{\omega_l}{d_l}\right)^{4/3} + \cdots + \left(\frac{\omega_u}{d_u}\right)^{4/3}\right) \|\psi\|^2$$

(we have used Lemma 2 and Lemma 3). For large enough k, $(C_k^2 \psi, \psi) = 0$ so $\sup_k (C_k^2 \psi, \psi) < \infty$. By hypotheses 1. and 2.

$$\sum_{T_{M,N}}{}'' |d_{klmn}| \, |d_{rstu}| \left(\frac{1}{d_k^2} + \frac{1}{d_l^2} + \cdots + \frac{1}{d_u^2}\right) \xrightarrow[N, M \to \infty]{} 0$$

and

$$\sum_{T,M}{}'' |d_{klmn}| \, |d_{rstu}| \left(1 + \left(\frac{\omega_k}{d_k}\right)^{4/3} + \cdots + \left(\frac{\omega_u}{d_u}\right)^{4/3}\right) \xrightarrow[N, M \to \infty]{} 0_z .$$

Thus

$$\left\|\sum_{S_{N,M}}{}' d_{klmn} q_k q_l q_m q_n \psi\right\|^2 \xrightarrow[N, M \to \infty]{} 0 .$$

Therefore $\sum_{k,l,m,n=1}^{\infty}{}' d_{klmn} q_k q_l q_m q_n$ is a well-defined symmetric operator on D since it is the strong limit of $\sum_{k,l,m,n=1}^{N}{}' d_{klmn} q_k q_l q_m q_n$. Further, if we set

$$T_2 = 10 \sum'' |d_{klmn}|\,|d_{rstu}| \left(1 + \left(\frac{\omega_k}{d_k}\right)^{4/3} + \left(\frac{\omega_l}{d_l}\right)^{4/3} + \cdots + \left(\frac{\omega_u}{d_u}\right)^{4/3}\right)$$

then we have the estimate

$$\begin{aligned}\|\textstyle\sum' d_{klmn}\, q_k\, q_l\, q_m\, q_n\, \psi\|^2 &\leqq \frac{1}{4} \sum^{\infty}{}' |d_{klmn}|\,|d_{rstu}| \left(\left(\frac{C_k^2}{d_k^2} + \cdots + \frac{C_u^2}{d_u^2}\right)\psi, \psi\right) + T_2\,\|\psi\|^2 \\ &= \frac{1}{4} \sum_{v=1}^{\infty} \frac{1}{d_v^2} (D_{1,v} + D_{2,v} + \cdots + D_{8,v})\,(C_v^2 \psi, \psi) + T_2\,\|\psi\|^2\end{aligned}$$

where

$$D_{1,v} = \sum_{l,m,n,r,s,t,u}'' |d_{vlmn}|\,|d_{rstu}|$$

$$D_{2,v} = \sum_{k,m,n,r,s,t,u}'' |d_{kvmn}|\,|d_{rstu}|$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_{8,v} = \sum_{k,l,m,n,r,s,t}'' |d_{klmn}|\,|d_{rstv}|\,.$$

Now since by hypotheses 2)

$$\frac{1}{4} \sum_{v=1}^{\infty} \frac{1}{d_v^2} (D_{1,v} + D_{2,v} + \cdots + D_{8,v}) = T_1$$

we must have $\frac{1}{4}\left(\frac{1}{d_v^2}\right)(D_{1,v} + D_{2,v} + \cdots + D_{8,v}) \leqq T_1$. Thus

$$\begin{aligned}\|\textstyle\sum' d_{klmn}\, q_k\, q_l\, q_m\, q_n\, \psi\|^2 &\leqq T_1 \sum_{v=1}^{\infty} (C_v^2 \psi, \psi) + T_2\,\|\psi\|^2 \\ &\leqq T_1 \sum_{v=1}^{\infty} (C_v^2 \psi, \psi) + T_1 \sum_{v \neq \omega}^{\infty} (C_v C_\omega \psi, \psi) \\ &\qquad + T_2\,\|\psi\|^2 \\ &\leqq T_1 \left(\left(\sum_{v=1}^{\infty} C_v\right)^2 \psi, \psi\right) + T_2\,\|\psi\|^2 \\ &= T_1 \left\|\sum_{v=1}^{\infty} C_v \psi\right\|^2 + T_2\,\|\psi\|^2\,.\end{aligned}$$

We have used the fact that $(C_v C_\omega \psi, \psi) > 0$ since the C_k are positive and they commute. Therefore we have

$$\|\textstyle\sum' d_{klmn}\, q_k\, q_l\, q_m\, q_n\, \psi\| \leqq T_1^{1/2} \left\|\sum_{v=1}^{\infty} C_v \psi\right\|^2 + T_2^{1/2}\,\|\psi\|$$

which proves that

$$\sum_{k=1}^{\infty} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right) + \sum_{k,l,m,n=1}^{\infty}{}' d_{klmn}\, q_k\, q_l\, q_m\, q_n$$

is essentially self-adjoint on D since $T_1^{1/2} < 1$ (see KATO [9]).

§ 2. The Physical Vacuum

Since $-\frac{d^2}{dx^2} + x^2 + cx^4$, $c \geqq 0$, has pure point spectrum and the gap between the lowest eigenvalue and the next is at least 3/2 (Lemma 2) the operator $\omega_k\left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$ must have a gap of at least $3m/2$ between its zero eigenvalue and the next highest $\left(\text{where } m < \inf_k \omega_k \text{ is just the mass}\right)$. Since the zero eigenvalue of $\omega_k\left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$ has multiplicity one, $\sum_{k=1}^{\infty} \omega_k\left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$ on $H(\chi)$ has a zero eigenvalue of multiplicity one and the next point in its spectrum is $> \frac{3m}{2}$.

Theorem 3. *If the off-diagonal coefficients satisfy*

$$\sum{}'' |d_{klmn}|\, |d_{rstu}| \cdot \left\{2\left(\frac{1}{d_k^2} + \cdots \frac{1}{d_u^2}\right) + 20\left(\frac{4}{3m}\right)^2\left(1 + \left(\frac{\omega_k}{d_k}\right)^{4/3} + \cdots \left(\frac{\omega_u}{d_u}\right)^{4/3}\right)\right\} < 1$$

then A_∞ is essentially self-adjoint on $D \subset H(\chi)$ and has an isolated point spectrum $< \frac{3m}{4}$ of unit multiplicity as lowest point in its spectrum.

Proof. We use a standard theorem of analytic perturbation theory (Kato [3], p. 214) which states. Suppose a symmetric operator V is dominated by a self-adjoint operator H_0 in the sense of Kato ($\|Vu\| \leqq a\|u\| + b\|H_0 u\|$, $0 \leqq b < 1$). Let Γ be a closed curve in the resolvent set of H_0 which surrounds a finite number, n, of point spectra of H_0. If $\sup_{\zeta \in \Gamma} (a\|R(\zeta, H_0)\| + b\|H_0 R(\zeta, H_0)\|) < 1$ then Γ is in the resolvent set of $H_0 + V$ and surrounds n point spectra of $H_0 + V$. In our case $H_0 = \sum_{k=1}^{\infty} \omega_k\left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k\right)$, $V = \sum{}' d_{klmn}\, q_k q_l q_m q_n$ and we will take Γ to be the circle with center the origin of radius $\frac{3m}{4}$. Let $\sigma(H_0)$ denote the spectrum of H_0. Then

$$\sup_{\zeta \in \Gamma}\left(a \sup_{\lambda \in \sigma(H_0)} \frac{1}{|\lambda - \zeta|} + b \sup_{\lambda \in \sigma(H_0)} \left|\frac{\lambda}{\lambda - \zeta}\right|\right) \leqq \frac{4a}{3m} + 2b$$

since H_0 has point spectrum at the origin and the rest of its spectrum is $> \frac{3m}{2}$. From the proof of Theorem 2 we have $\|V\psi\| \leqq \sqrt{T_1}\,\|H_0\psi\| + \sqrt{T_2}\,\|\psi\|$ so the condition to be fulfilled is $\frac{4}{3m}\sqrt{T_2} + 2\sqrt{T_1} < 1$. A condition which implies this is $2T_2\left(\frac{4}{3m}\right)^2 + 8T_1 < 1$ which is just the hypothesis of this theorem. (T_1 and T_2 are defined in the proof of Theorem 2).

§ 3. Test Functions for the Field and Conjugate Momentum

In this setup where we have expanded the field into modes test functions are test sequences of real numbers $a = \{a_k\}_{k=1}^{\infty}$. The smeared field and conjugate momentum operators are $\varphi(a) = \sum_{k=1}^{\infty} a_k q_k$ and $\pi(a) = \sum_{k=1}^{\infty} a_k p_k$ respectively.

Theorem 4. *Suppose* $\{a_k\}_{k=1}^{\infty}$ *satisfies* $\sum_{k=1}^{\infty} |a_k|^2 \left(1 + \left(\frac{d_k}{\omega_k}\right)^{1/3}\right) < \infty$. *Then* $\varphi(a) = \sum_{k=1}^{\infty} a_k q_k$ *and* $\pi(a) = \sum_{k=1}^{\infty} a_k p_k$ *are essentially self-adjoint on* $D \subset H(\chi)$.

Proof. To show that $\varphi(a)$ is essentially self-adjoint on D it is sufficient to show

$$\sum_{k=1}^{\infty} |a_k| \, |(q_k \chi_k, \chi_k)| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} a_k^2 (q_k^2 \chi_k, \chi_k) < \infty$$

(Streit [7] shows that these conditions are sufficient for $\varphi(a)$ to be self-adjoint on some domain in $H(\chi)$; for a proof that this domain can be taken to be D see § 1.2 of (5)).

From Lemma 1 we know that $\chi_k(x) \in \mathscr{S}(r)$ so the expressions $(q_k \chi_k, \chi_k)$ and $(q_k^2 \chi_k, \chi_k)$ make sense. $\chi_k(x)$ satisfies

$$-\chi_k''(x) + \left(x^2 + \frac{d_k}{\omega_k} x^4\right) \chi_k(x) = \tau_k \chi_k(x) \tag{3.1}$$

but if $\chi_k(x)$ is a solution so is $\chi_k(-x)$. Since each eigenvalue has multiplicity one we must have $\chi_k(x) = z \chi_k(-x)$, $|z| = 1$. Since every solution of (3.1) is a constant times an everywhere real solution, $\chi_k(x) = \pm \chi_k(-x)$. Therefore $(q_k \chi_k(x), \chi_k(x)) = 0$. Now

$$\begin{aligned}
(q_k^2 \chi_k(x), \chi_k(x)) &= \frac{1}{\omega_k} (\omega_k q_k^2 \chi_k(x), \chi_k(x)) \\
&\leqq \frac{1}{\omega_k} ((\omega_k p_k^2 + \omega_k q_k^2 + d_k q_k^4) \chi_k(x), \chi_k(x)) \\
&= \tau_k (\chi_k(x), \chi_k(x)) \\
&\leqq (2^{1/3}) \left(\frac{3}{2}\right) \left(1 + \frac{d_k}{\omega_k}\right)^{1/3} \quad \text{by Lemma 2}
\end{aligned}$$

which by the hypotheses of the theorem proves that

$$\sum_{k=1}^{\infty} a_k^2 (q_k^2 \chi_k, \chi_k) < \infty .$$

The proof for $\pi(a) = \sum_{k=1}^{\infty} a_k p_k$ is similar.

§ 4. The Convergence of the Cut-off Physical Vacuums

Let A_N be the formal operator

$$A_N = \sum_{k=1}^{N} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k \right) + \sum_{k,l,m,n}^{N}{}' d_{klmn} q_k q_l q_m q_n \,.$$

If the off-diagonal coefficients are suitably small, then by the same proofs as in § 1 and § 2 A_N can be shown to be essentially self-adjoint on $\mathcal{S}(R^N) \subset L^2(R^N)$ and to have point spectrum of unit multiplicity as lowest point in its spectrum. Let ψ_N be the corresponding eigenfunction. (See JAFFE [2] for a detailed analysis of the cut-off case where the off-diagonal d_{klmn} are not required to be small.)

A_N is a well-defined operator on $H(\chi)$ also (it only operates on the first N components) and is essentially self-adjoint on D. Although the lowest point in its spectrum is point spectrum, it has infinite multiplicity since any vector of the form $\psi_N \otimes \Pi \bigotimes_{k=N+1}^{\infty} f_k$ where $\sum_{N+1}^{\infty} |1 - (f_k, \chi_k)| < \infty$ is an eigenvector in $H(\chi)$ corresponding to the lowest eigenvalue. Let us choose for the cut-off physical vacuum the vector $\theta_N = \psi_N \otimes \Pi \bigotimes_{k=N+1}^{\infty} \chi_k$. We then can state

Theorem 5. *Suppose that the off-diagonal coefficients satisfy*

$$\sum{}'' |d_{klmn}| \, |d_{rstu}| \left\{ 2 \left(\frac{1}{d_k^2} + \cdots \frac{1}{d_u^2} \right) + 20 \left(\left(\frac{4}{3m} \right)^2 + \frac{1}{2} \right) \left(1 + \left(\frac{\omega_k}{d_k} \right)^{4/3} + \cdots \left(\frac{\omega_u}{d_u} \right)^{4/3} \right) \right\} < 1$$

and let θ be the unique physical vacuum given by Theorem 3. Then $\theta_N \xrightarrow{H(\chi)} \theta$.

Proof. Let

$$B_N = \sum_{k=1}^{\infty} \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k \right) + \sum^{N}{}' d_{klmn} q_k q_l q_m q_n \,.$$

The operators A_∞ and B_N are all essentially self-adjoint *on* D and the lowest point in the spectrum of B_N is a point spectrum with unit multiplicity and corresponding eigenvector θ_N. For $\psi \in D$ we have the estimate (from the proof of Theorem 2)

$$\left\| \sum_{N}^{\infty}{}' d_{klmn} q_k q_l q_m q_n \psi \right\| \leqq \sqrt{T_1^N} \left\| \sum_{k=1}^{\infty} C_k \psi \right\| + \sqrt{T_2^N} \, \|\psi\|$$

where

$$C_k = \omega_k \left(p_k^2 + q_k^2 + \frac{d_k}{\omega_k} q_k^4 - \tau_k \right)$$

$$T_1^N = \frac{1}{4} \sum_{N}^{\infty}{}'' |d_{klmn}| \, |d_{rstu}| \left(\frac{1}{d_k^2} + \cdots \frac{1}{d_u^2} \right)$$

$$T_2^N = 10 \sum_{N}^{\infty}{}'' |d_{klmn}| \, |d_{rstu}| \left(1 + \left(\frac{\omega_k}{d_k} \right)^{4/3} + \cdots + \left(\frac{\omega_u}{d_u} \right)^{4/3} \right) .$$

Further, by the hypothesis on the off-diagonal coefficients $T_1^N < 1$, $T_2^N < 1$ for all N and $T_1^N \to 0$, $T_2^N \to 0$ as $N \to \infty$. Let

$$V = \sum{}' d_{k\,l\,m\,n}\, q_k\, q_l\, q_m\, q_n\,,$$

then

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} C_k \psi\right\| &\leqq \left\|\left(\sum_{k=1}^{\infty} C_k + V\right)\psi\right\| + \|V\psi\| \\ &\leqq \left\|\left(\sum_{k=1}^{\infty} C_k + V\right)\psi\right\| + \sqrt{T_1^1}\left\|\sum_{k=1}^{\infty} C_k \psi\right\| + \sqrt{T_2^1}\,\|\psi\|\end{aligned}$$

so,

$$\left\|\left(\sum_{k=1}^{\infty} C_k + V\right)\psi\right\| + \|\psi\| \geqq \left(1 - \sqrt{T_1^1}\right)\left\|\sum_{k=1}^{\infty} C_k \psi\right\| + \left(1 - \sqrt{T_2^1}\right)\|\psi\|\,.$$

Therefore

$$\begin{aligned}\sup_{\psi\in D} \frac{\|(A_\infty - B_N)\,\psi\|}{\|\psi\| + \|A_\infty \psi\|} &= \sup_{\psi\in D} \frac{\left\|\sum_N{}' d_{k\,l\,m\,n}\, q_k\, q_l\, q_m\, q_n\, \psi\right\|}{\|\psi\| + \left\|\left(\sum_{k=1}^{\infty} C + V\right)\psi\right\|} \\ &\leq \sup_{\psi\in D} \frac{\sqrt{T_1^N}\left\|\sum_{k=1}^{\infty} C_k\, \psi\right\| + \sqrt{T_2^N}\,\|\psi\|}{\left(1 - \sqrt{T_1^1}\right)\left\|\sum_{k=1}^{\infty} C_k\, \psi\right\| + \left(1 - \sqrt{T_2^1}\right)\|\psi\|} \to 0\end{aligned}$$

as $N \to \infty$. Thus B_N converges strongly relatively uniformly to A_∞ in the terminology of Sz. NAGY ([6], p. 370). From [6] we therefore get

$$\left\|E^N\left(-\infty, \frac{3m}{4}\right) - E\left(-\infty, \frac{3m}{4}\right)\right\| \to 0 \quad \text{as} \quad N \to \infty.$$

where $E^N(s, t)$ and $E(s, t)$ are the spectral projectors of B_N and A_∞ respectively. Now, $E\left(-\infty, \frac{3m}{4}\right)$ has a one-dimensional range, namely θ, and as soon as $\left\|E^N\left(-\infty, \frac{3}{4}m\right) - E\left(-\infty, \frac{3}{4}m\right)\right\| < 1$, $E^N\left(-\infty, \frac{3}{4}m\right)$ will also have a one-dimensional range which must be θ_N. Since

$$\left\|E^N\left(-\infty, \frac{3}{4}m\right) - E\left(-\infty, \frac{3}{4}m\right)\right\| \to 0 \quad \text{as} \quad N \to \infty$$

we have $\theta_N \xrightarrow{H(\chi)} \theta$ as $N \to \infty$.

§ 5. The Anharmonic Oscillator

In this section we prove the two lemmas about the operator $B = -\frac{d^2}{dx^2} + x^2 + cx^4$ which were stated in § 1. We always assume $c \geqq 0$.

Lemma 1. *B is essentially self-adjoint on $\mathscr{S}(R)$, it has discrete spectrum, its eigenvalues have multiplicity one and the corresponding eigenfunctions are in $\mathscr{S}(R)$.*

Proof. B is essentially self-adjoint on any dense domain contained in $\mathscr{S}(R)$ (JAFFE [2]) and since $x^2 + cx^4 \to \infty$ as $x \to \pm\infty$, B has discrete spectrum (TITCHMASSH [8]). JAFFE [2] has also shown that the eigenfunctions are in $\mathscr{S}(R)$. The eigenvalues of B have unit multiplicity essentially because B has the limit point case at both $\pm\infty$. Suppose an eigenvalue had multiplicity two; that is, both solutions of $B\varphi = \mu\varphi$ are in $L^2(R)$. Then by a well-known theorem of Weyl both solutions of $B\varphi = \lambda\varphi$ for any complex λ are in $L^2(R)$. In particular, there are two $L^2(R)$ functions such that $B\varphi = i\varphi$. These functions are in the domain of $(B|_{C_0^\infty(R)})^*$ which implies that the range of $B + i$ on $C_0^\infty(R)$ is not dense which contradicts the essential self-adjointness of B. Thus the lemma is proven.

Let $\{\lambda_n(c)\}_{n=1}^\infty$ and $\{\mu_n(c)\}_{n=1}^\infty$ denote the eigenvalues of $-\frac{d^2}{dx^2} + x^2 + cx^4$ and $\frac{d^2}{dx^2} + cx^4$ respectively. Each sequence is listed in increasing order. It is convenient to prove Lemma 2 by a sequence of lemmas.

Lemma 3. $\lambda_1(c) \leqq \frac{13}{12} + \frac{c}{3}$, $\lambda_2(c) \geqq 3$.

Proof. Let $h(a, x) = \left(\frac{\pi}{a}\right)^{-1/4} e^{-ax^2/2}$, then $\|h(a,x)\|_{L^2} = 1$ and $\lambda_1(c) \leqq (Bh(a,x), h(a,x)) = \frac{1}{2}a + \frac{1}{2a} + \frac{3c}{4a^2}$. If we choose $a = \frac{3}{2}$ we find $\lambda_1(c) \leqq \frac{13}{12} + \frac{c}{3}$. Since $x^2 + cx^4 \geqq x^2$, $\lambda_2(c)$ is larger than the second eigenvalue of $-\frac{d^2}{dx^2} + x^2$, thus $\lambda_2(c) \geqq 3$.

Lemma 4. $c^{1/3}\mu_n(1) \leqq \lambda_n(c) \leqq \left(c^{1/3} + \frac{1}{12\,\delta c^{2/3}}\right)\mu_n(1) + \delta$ *for all* $\delta > 0$.

Proof. Suppose $\varphi(x)$ satisfies $-\varphi''(x) + cx^4\varphi(x) = \tau\varphi(x)$.

Let $x = c^{-1/6}y$ and define $\psi(y) = \varphi(c^{-1/6}y)$. Then $-\psi''(y) + y^4\psi(y) = \frac{\tau}{c^{1/3}}\psi(y)$ so $\mu_n(c) = c^{1/3}\mu_n(1)\ \forall_n$. Since $x^2 + cx^4 \geqq cx^4$, $\lambda_n(c) \geqq \mu_n(c) = c^{1/3}\mu_n(1)$. Furthermore, $x^2 + cx^4 \leqq \left(c + \frac{1}{4\delta}\right)x^4 + \delta$ for all $\delta > 0$ so

$$\lambda_n(c) \leqq \mu_n\left(c + \frac{1}{4\delta}\right) + \delta = \left(c + \frac{1}{4\delta}\right)^{1/3}\mu_n(1) + \delta$$

$$\leqq \left(c^{1/3} + \frac{1}{12\delta c^{2/3}}\right)\mu_n(1) + \delta\,.$$

Lemma 5. $\mu_1(1) \leqq \frac{13}{12}$, $\mu_2(1) \geqq 3\sqrt{2} - 1$.

Proof. For all $a > 0$, $\mu_1(1) \leqq \left(\left(-\frac{d^2}{dx^2} + x^4\right)h(a,x), h(a,x)\right) = \frac{a}{2} + \frac{3}{4}a^{-2}$. Choosing $a = \frac{3}{2}$ we get $\mu_1(1) \leqq \frac{13}{12}$. Now $x^4 \geqq \sqrt{4\delta}\,x^2 - \delta$

for all $\delta > 0$. Thus $\mu_n(1)$ is larger than the n^{th} eigenvalue of

$$-\frac{d^2}{dx^2} + \sqrt{4\delta}\, x^2 - \delta$$

which is $(4\delta)^{1/4}(2n-1) - \delta$. In particular, $\mu_2(1) \geqq 3(4\delta)^{1/4} - \delta$, $\delta > 0$. If we choose $\delta = 1$, we get $\mu_2(1) \geqq 3\sqrt{2} - 1$.

Finally we have

Lemma 2. $\lambda_2(c) - \lambda_1(c) > 3/2$ *(independent of c) and*

$$\lambda_1^3 < 2\left(\frac{3}{2}\right)^3 (1+c)\,.$$

Proof. For $c \leqq 1$ the fact that $\lambda_2(c) - \lambda_1(c) > 3/2$ follows immediately from Lemma 3. For $c > 1$ we observe that $\lambda_1(c) \leqq \left(c^{1/3} + \frac{1}{12\delta c^{2/3}}\right)\mu_n(1) + \delta$, $\forall \delta > 0$ (Lemma 4). Taking the minimum over all $\delta > 0$ we find $\lambda_1(c) \leqq c^{1/3}\mu_1(1) + 2\left(\frac{\mu_1(1)}{12c^{2/3}}\right)^{1/2} \leqq c^{1/3}\mu_1(1) + \frac{\sqrt{13}}{6}$ (using Lemma 5). Since $\lambda_2(c) \geqq c^{1/3}\mu_2(1)$, $\lambda_2(c) - \lambda_1(c) \geqq c^{1/3}(\mu_2(1) - \mu_1(1)) - \frac{\sqrt{13}}{6} > \frac{3}{2}$.

If $c \leqq 1$, then $\lambda_1(c) \leqq 1\frac{1}{12}$, so $\left(\frac{2}{3}\lambda_1\right)^3 < 1$. If $c > 1$, then from the proof of Lemma 4 we have for all $\delta \leqq 0$

$$\frac{2}{3}\lambda_1(c) \leqq \frac{2}{3}\left(c + \frac{1}{4\delta}\right)^{1/3}\mu_1(1) + \frac{2}{3}\delta \leqq \left(c + \frac{1}{4\delta}\right)^{1/3} + \frac{2}{3}\delta\,.$$

Choosing $\delta = \frac{1}{4}$ and expanding $\left(\frac{2}{3}\lambda_1(c)\right)^3$ we find $\left(\frac{2}{3}\lambda_1(c)\right)^3 \leqq 2c + 2$.

§ 6. Remarks

The point of this example is the following: For diagonal Hamiltonians it is easy to pick an infinite tensor product representation of the CCR so that the Hamiltonian is self-adjoint (a theorem in [5] gives an explicit domain of self-adjointness) and questions about the spectrum reduce to questions about the spectrum of the component operators in the sum $\left(\text{in this case } -\frac{d^2}{dx^2} + x^2 + cx^4\right)$. For a "diagonally dominated" Hamiltonian the diagonal part may be used to determine the "right" representation and then the whole operator proved self-adjoint by the method of KATO. Further analysis of the Hamiltonian can then be carried out treating the whole operator as a perturbation of the diagonal part. Of course "real" Hamiltonians are not diagonally dominated (the off-diagonal d_{klmn} are too big). Nevertheless, damping the off-diagonal coefficients for large k, l, m, n gives in some sense a better approximation to the "real" Hamiltonian than cutting them off completely. It would be interesting to carry through an analysis similar to JAFFE [2] or CANNON [1] and prove the existence of the vacuum expectation values.

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References

1. CANNON, J.: Field theoretic properties of Nelson's model. Princeton University Thesis, 1968.
2. JAFFE, A.: Dynamics of a cut-off $\lambda\phi^4$ field theory. Princeton University Thesis, 1965.
3. KATO, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966.
4. NEUMANN, J. VON: On infinite direct products. Compos. Math. **6**, 1—77 (1939).
5. REED, M.: On the self-adjointness of quantum fields and Hamiltonians. Stanford University Thesis, 1968.
6. RIESZ, F., and B. SZ.-NAGY: Functional analysis. New York: Frederick Ungar 1955.
7. STREIT, L.: Test function spaces for direct product representations. Commun. Math. Phys. **4**, 22—31 (1967).
8. TITCHMARSH, E.: Eigenfunction expansions I. London: Oxford University Press 1962.

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