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Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories V*

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Abstract. We continue here the series of papers treated by LUDWIG in [1—5]. Using some results of DÄHN in [6], we point out that each irreducible solution of the axiomatic scheme set up in [5] is represented by a system of positive-semi-definite operator pairs of a finite-dimensional Hilbert-space over the real, complex or quaternionic numbers.

I. Introduction

Following MACKEY'S [7] general outline of axiomatic quantum theory, MACLAREN [11] and ZIERLER [8] or PIRON [12] and JAUCH [13] introduce two final axioms concerning the topological structure of the lattice G of questions (also called propositions or decision effects). This means strictly speaking that G and each sublattice of G is a compact set and that the set $A(G)$ of all atoms of G is connected. These axioms characterize the division ring appearing in the representation theorem for G.

In his axiomatic scheme (cited in [5]), LUDWIG starts from a pair of sets $(K, \hat{L})$ imbedded in a dual pair (B, B') of finite-dimensional real Banach-spaces. Hence the lattice G of decision effects, being the set of all extreme points of $\hat{L}$, carries a topological structure inherited from B'.

In [5] it was already shown that the first of the axioms mentioned above is a theorem in this exposition.

The purpose of this paper is to show that also the second axiom can be deduced. Furthermore, the following representation theorem for the system $(K, \hat{L})$ will be shown.

Theorems 20, 21. *If the dimension of the finite-dimensional Banach-spaces B, B' is large enough, then there holds:*

1. *Every irreducible solution of the axiomatic system $(K, \hat{L})$ is isomorphic to a system $(\mathcal{K}, \hat{\mathcal{L}})$ of linear operators of a finite-dimensional Hilbert-space H.*

2. *The division ring of H is isomorphic to either the real, the complex or the quaternionic number ring.*

3. *The set $\mathcal{K}$ consists of all positive-semidefinite operators V with $\mathbf{Tr}\, V = 1$.*

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4. $\hat{\hat{\mathscr{L}}}$ is the set of all positive-semidefinite operators $F \leqq 1$.

5. The operators $V \in \mathscr{K}$ and $F \in \hat{\hat{\mathscr{L}}}$ are put in duality by the operator trace $\boldsymbol{Tr}\, VF$.

II. Preliminaries

Let us sketch the axioms and some of the most important propositions cited in the papers [1–5] by G. Ludwig.

We will start from a dual pair (B, B') of finite-dimensional topological vector-spaces over the field $\boldsymbol{R}$ of the real numbers, where B is spanned by the closed convex hull K of the set $\underline{K}$ of all physical ensembles v and whereas B' is spanned by the closed convex hull $\hat{L}$ of the set $\underline{L}$ of all physical effects f. $(K, \hat{L})$ is a dual pair according to the following.

Axiom 1. *There exists a mapping μ of $K \times \hat{L}$ into $\boldsymbol{R}_+$ satisfying.*

α) $0 \leqq \mu(v, f) \leqq 1$ for all $(v, f) \in K \times \hat{L}$.

β) $\mu(v_1, f) = \mu(v_2, f)$ for all $f \in \hat{L}$ and $v_1, v_2 \in K$ implies $v_1 = v_2$.

γ) $\mu(v, f_1) = \mu(v, f_2)$ for all $v \in K$ and $f_1, f_2 \in \hat{L}$ implies $f_1 = f_2$.

δ) There exists $f \in \hat{L}$ (denoted by 0) such that $\mu(v, 0) = 0$ for all $v \in K$.

ε) For each $v \in K$ there exists $f \in \hat{L}$ such that $\mu(v, f) = 1$. ⌟

μ can be extended to the canonical bilinear functional over $B \times B'$. Then in B a norm $\|\cdot\|$ is defined by $\|x\| := \sup(|\mu(x, f)| : f \in \hat{L})$ for $x \in B$. Hence the finite-dimensional $\boldsymbol{R}$-vector space B is a Banach-space. With respect to the norm $\|y\| := \sup(|\mu(x, y)| : x \in B, \|x\| = 1)$ and to the partial ordering defined by $y_1 \leqq y_2$ *for* $y_1, y_2 \in B'$ *iff* $\mu(v, y_1) \leqq \mu(v, y_2)$ *for all* $v \in K$, B' becomes a partially ordered real Banach-space. For the further axioms we need the following sets. Let $i = 0, 1$; $l \subseteq \hat{L}$, $k \subseteq K$.

$K_i(l) := \{v \in K : \mu(v, f) = i \text{ for all } f \in l\}$.

$\hat{L}_i(k) := \{f \in \hat{L} : \mu(v, f) = i \text{ for all } v \in k\}$.

$\hat{\hat{L}}$ is defined to be the closure of the set $\{y \in B' : y = \lambda f, \lambda \geqq 0, f \in \hat{L}$ and $\lambda\mu(v, f) \leqq 1$ for all $v \in K\}$.

Axiom 2a. *For each pair $f_1, f_2 \in \hat{L}$ there exists $f_3 \in \hat{L}$ so that $f_3 \geqq f_1, f_2$ and $K_0(f_3) \supseteq K_0(f_1) \cap K_0(f_2)$.* ⌟

Let lg be the greatest subset of $\hat{L}$ such that $K_0(l) = K_0(lg)$. According to axiom 2a, lg is an ascending directed set possessing a greatest element e_{lg} called *decision effect*. It is defined by

$$\mu(v, e_{lg}) := \sup\{\mu(v, f) : f \in lg\}$$

for $v \in K$ and satisfies $\|e_{lg}\| = 1$. Let G be the set of all decision effects e of $\hat{L}$.

In [1] and [5] it was shown that G and the set $\hat{W} := \{K_1(l) : l \subseteq \hat{L}\}$ are complete, orthocomplemented and orthoisomorphic lattices. The zero elements in G, $\hat{W}$ are 0, $\emptyset$, respectively; the unit element in G is 1 given by $\mu(v, 1) = 1$ for all $v \in K$, whereas the unit element in $\hat{W}$ is K. The orthocomplementation in G, $\hat{W}$ is given by $e \to 1 - e$ and $K_1(l) \to K_0(l)$, respectively. Because of $\dim B = \dim B' < \infty$, G and $\hat{W}$ are atomic lattices.

Axiom 2b. *For $f \in \hat{\hat{L}}$ and $e \in G$, $K_0(f) \supseteqq K_0(e)$ implies $f \leqq e$.* ⌟

$A\,(\neq \emptyset) \subseteqq K$ is called an *extremal set* iff;

α) A is convex and closed;

β) Every open line segment $S \subseteqq K$ with $S \cap A \neq \emptyset$ is contained in A.

Let $C(v)$ denote the *smallest extremal set of K containing v*. $v \in A \subseteqq K$ is an *extreme point* iff there is no open line segment in A containing v.

Axiom 3. *$\hat{L}_0(v_1) = \hat{L}_0(v_2)$ implies $C(v_1) = C(v_2)$.* ⌟

The following theorems are proved in [5].

Theorem 1. $\hat{L} = \hat{\hat{L}} = (y \in B' : 0 \leqq \mu(v, y) \leqq 1$ *for all* $v \in K)$.

Theorem 2. $K = (x \in B : 0 \leqq \mu(x, f)$ *for all* $f \in \hat{L}$ *and* $\|x\| = \mu(x, 1) = 1)$.

Theorem 3. *G is the set of all extreme points of $\hat{L}$.*

Theorem 4. $\sum_{i=1}^{m} e_i \leqq 1$, $e_i \in G$ *implies* $\sum e_i = \vee e_i$ *and* $e_i \perp e_k$ *for* $i \neq k$.

Theorem 5. *Every $f \in \hat{L}$ allows an unique decomposition $f = \sum_{i=1}^{m} \lambda_i e_i$, with $e_i \in G$ pairwise orthogonal and $1 \geqq \lambda_1 > \cdots > \lambda_m > 0$.*

According to the Theorems 1 and 2, K and $\hat{L}$ are bounded sets. Since B and B' are topologically isomorphic to $\boldsymbol{R}^n$, K and $\hat{L}$ are compact sets.

Let $A \subseteqq K$ be convex and $M(A)$ denote the linear manifold generated by A. $x \in A$ is called an *internal point of A relative to $M(A)$* iff for every line $g \subseteqq M(A)$ through x there exists an open segment $S \subseteqq g \subseteqq A$ with $x \in S$. The set of all internal points of A is denoted by A^i. A point of an extremal set $A \subseteqq K$ not being an internal point is called a *bounding point*. Let $\mathrm{Bd}\,A$ denote the set of all bounding points of A.

G. Dähn proves then the following theorems:

Theorem 6. *$v_1 \in C(v)$, iff there is $\lambda \in\,]0, 1[\, \subseteqq \boldsymbol{R}$ and $v_2 \in K$ with $v = \lambda v_1 + (1 - \lambda)\, v_2$.*

Theorem 7. *$C(v) = C(\bar{v})$ iff $\bar{v} \in C(v)^i$.*

Corollar. *$C(v_1) \subset C(v_2)$, $v_1, v_2 \in K$ implies $C(v_1) \subseteqq \mathrm{Bd}\,C(v_2)$.*

Theorem 8. *There exists $e \in G$ with $C(v) = K_1(e)$, hence:*

$$\hat{W} = (C(v) : v \in K)\,.$$

Theorem 9. *The extreme points of K are the atoms of the lattice $\hat{W}$.*

Theorem 10. *Each extreme point of $C(v) \subseteqq K$ is also an extreme point of K.*

Theorem 11. *Each extremal set of K contains at least one extreme point.*

Together with the isomorphism between G and $\hat{W}$ the Theorems 9 and 11 imply:

Theorem 12. *The set $E(K)$ of all extreme points of K is bijectively mapped onto the set $A(G)$ of all atoms of G.*

Axiom 4. *For* $v_1, v_2, v_3 \in K$:
$C(v_1) \cap C(v_2) = \emptyset$, $\emptyset \neq C(v_3) \subseteq C(\frac{1}{2}v_1 + \frac{1}{2}v_2)$ *and* $C(v_1) \perp C(v_3)$ *implies* $C(\frac{1}{2}v_1 + \frac{1}{2}v_3) \cap C(v_2) \neq \emptyset$. ⌟

This axiom is equivalent to

Axiom 4'. *The orthoisomorphic lattices* $\hat{W}$ *and* G *are modular.*

In [5] the implication is proved: Axiom 4 ⇒ Axiom 4'. The converse implication may be seen as follows.

Let $C(v_1) = a$, $C(v_2) = b$, $C(v_3) = c$. In [6] there is shown

$$C(\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2) = C(v_1) \vee C(v_2)\,.$$

Hence with $a \wedge b = 0$, $0 \neq c \leqq a \vee b$, $a \perp c$ and the assumption $(a \vee c) \wedge b = 0$, we find by using the modularity:

$$\begin{aligned} c &= c \wedge (a \vee b) = c \wedge (c \vee a) \wedge (a \vee b) \\ &= c \wedge \big(a \vee (b \wedge (a \vee c))\big) = c \wedge a\,, \quad \text{i.e.} \quad \emptyset \neq c \leqq a \end{aligned}$$

contrary to $c \perp a$. ⌟

Let $\mathbf{dim}\, G$ ($\mathbf{dim}(e)$ for $e \in G$) denote the greatest number of pairwise orthogonal atoms $p_i \in A(G)$ with $\sum p_i = 1$ ($p_i \leqq e$ with $\sum p_i = e$) respectively. Then in [5] there is shown.

Theorem 13. G *is closed and* $e_i \to e$ *implies* $\mathbf{dim}(e_i) \to \mathbf{dim}(e)$. *Hence also* $A(G)$ *is closed.*

The lattice G is a direct sum of irreducible sublattices $G(0, e_i)$, $\sum_{i=1}^{k} e_i = 1$ of the same structure as G.

Hence each $f \in \hat{L}$ has the form $f = \sum_{i=1}^{k} f_i$ with $0 \leqq f_i \leqq e_i$ $(i = 1 \ldots k)$.

Each $v \in K$ may be written $v = \sum_{i=1}^{k} \omega_i v_i$ defined by $\mu(v, f) = \sum_{i=1}^{k} \omega_i \mu(v_i, f_i)$ for all $f \in \hat{L}$, $\omega_i \in \boldsymbol{R}_+$, $\sum \omega_i = 1$ and $\mu(v_i, e_i) = 1$.

Thus, without restriction of generality, we may postulate.

P: *The lattices* $\hat{W}$ *and* G *are irreducible.*

III. Some Further Consequences of the Axiomatic Scheme

Theorem 14. *Let* C *be an extremal set. The set* $\mathbf{Bd}\, C$ *of all bounding points of* C *equals the boundary* ∂C *of* C *relative to* $M(C)$.

Proof. Since $\mathbf{Bd}\, C \subseteq \partial C$, let us take $v \in \partial C \subseteq C^- = C$ with $v \notin \mathbf{Bd}\, C$. Obviously $v \in C^i$.

The n-dimensional Banach-space B is homeomorphic to $\boldsymbol{R}^n$ under its euclidean norm $|x| := \sqrt{\sum \alpha_i^2}$ with $x = \sum \alpha_i x_i$, x_i $(i = 1 \ldots n)$ being a cartesian base. Let us assume $\mathbf{dim}\, M(C) = m \leqq n$. By $\mathfrak{U}_\delta(x)$ we denote the spherical open neighbourhood $\{y \in B : |y - x| < \delta\}$ of x. Since v is an internal point of C, we find $m + 1$ independent points $v_i \in C$ with

$v = \sum \lambda_i v_i$, $\sum \lambda_i = 1$ and $0 < \lambda_i < 1$ $(i = 1 \ldots m+1)$. Since all λ_i are positive, there exist open neighbourhoods $\mathfrak{V}_i(\lambda_i)$ in $\boldsymbol{R}_+$.

Now every $x \in M(C)$ may be written in the general form $x = \sum_{i=1}^{m+1} \alpha_i v_i$, $\alpha_i \in \boldsymbol{R}$ and $\sum_{i=1}^{m+1} \alpha_i = 1$. Because all λ_i depend continuously on the cartesian coordinates of $v \in \boldsymbol{R}^n$, we find $\delta > 0$ such that for all $x = \sum \alpha_i v_i$, $\sum \alpha_i = 1$, contained in $\mathfrak{U}_\delta(v) \cap M(C)$, $\alpha_i \in \mathfrak{V}_i(\lambda_i) \subsetneqq \boldsymbol{R}_+$ consequently holds, i.e., $\alpha_i > 0$ $(i = 1 \ldots m+1)$. Therefore, every $x \in \mathfrak{U}_\delta(v) \cap M(C)$ is a convex combination of $v_1 \ldots v_{m+1}$; i.e. $\mathfrak{U}_\delta(v) \cap M(C)$ is an open neighbourhood of v relative to $M(C)$ and totally contained in C. Thus, we have $v \notin \partial C$ contrary to our assumption. $\lrcorner$

Because of Theorem 14 we need not distinguish between boundary points and bounding points of an extremal set C. C is said to be *strictly convex* iff one of the of the following equivalent conditions is satisfied:

1. The boundary of C includes no line segment.
2. Each boundary point of C is an extreme point.

Theorem 15. *If p, q are orthogonal atoms of the irreducible lattice G, then the extremal set $C(v) := K_1\left(p \overset{\perp}{\vee} q\right)$ is strictly convex.*

Proof. Maeda [14] shows that in an irreducible atomic and modular lattice an atom $r \neq p, q$ exists which is covered by $p \overset{\perp}{\vee} q$. Since $\hat{W}$ is isomorphic to G, $K_1(r) =: v_r \in C(v)$ is an atom contained in the boundary of $C(v)$ by Theorem 7 and corollary. Supposing the boundary of $C(v)$ contains a line segment $[v_1, v_2]$ with distinct end-points, it also contains $\bar{v} := \frac{1}{2} v_1 + \frac{1}{2} v_2$. If $C(v)$ is not included in $\mathbf{Bd}\, C(v)$, then $\bar{v} \in C(v)^i$, would follow from Theorem 7 contrary to the choice of $\bar{v}$. Thus, observing $v \in C(v)^i$ we have the inclusion $C(\bar{v}) \subsetneqq \mathbf{Bd}\, C(v) \subset C(v)$. On account of $C(\bar{v}) \neq \emptyset$, $\emptyset \subset C(\bar{v}) \subset C(v)$ is a chain of length two. Yet all chains between $\emptyset$ and $C(v) := K_1\left(p \overset{\perp}{\vee} q\right)$ of length two are covering chains in the modular lattice $\hat{W}$. Thus, contrary to containing the segment $[v_1, v_2]$, $C(\bar{v})$ is an atom, i.e., an extreme point. This completes the proof. $\lrcorner$

Theorem 16. *The bijective mapping $E(k) \leftrightarrow A(G)$ is a homeomorphism.*

Proof. Since $\hat{L}$ is compact, the closed set $A(G)$ is also compact. Let (p_α) be a convergent sequence in $A(G)$ with $\lim_\alpha p_\alpha = p$. On account of the compactness of K, we can find a convergent subsequence $(p_\alpha)'$ so that $(v_\alpha := K_1(p_\alpha))'$ is a convergent sequence in $E(K)$ with $v_\alpha \to v \in K$. If we choose $\varepsilon > 0$, almost all v_α satisfy the inequality $\|v_\alpha - v\| = \sup_{f \in \hat{L}} |\mu(v_\alpha, f) - \mu(v, f)| < \varepsilon$; in particular we have $|\mu(v_\alpha, p_\alpha) - \mu(v, p_\alpha)| < \varepsilon$; i.e., $|1 - \mu(v, p_\alpha)| < \varepsilon$ for almost all p_α converging to p. Thus $\mu(v, p) = \lim_{p_\alpha \to p} \mu(v, p_\alpha) = 1$; i.e., $v \in K_1(p) =: v_p \in E(K)$. Hence the mapping

$A(G) \to E(K)$ is continuous. Since $A(G)$ is compact, the continuous bijection $A(G) \to E(K)$ is by a topological theorem even bicontinuous. ⌟

This implies immediately

Corollary. *The set $E(K)$ of all extreme points of K is compact.*

Theorem 17. *The set of all atoms of every lattice segment $G(0, e) = \{x \in G : 0 \leqq x \leqq e\}$, $e \in G$ is connected.*

Proof. Let $p, q (\neq p)$ be atoms of G. We must find a continuous mapping f of a closed segment $[\alpha, \beta] \subseteqq \boldsymbol{R}$, $\alpha \neq \beta$, into the set $A(p \vee q) := \{r \in A(G) : r < p \vee q\}$ with $f(\alpha) = p$, $f(\beta) = q$. This will be shown in several steps.

1. G is irreducible; hence there is a third atom $r < p \vee q$. Therefore, the corresponding three distinct extreme points $v_p, v_q, v_r \in K_1(p \vee q)$ are independent; i.e., they span a plane $\mathscr{E} = \mathscr{E}(v_p, v_q, v_r)$ in B. Since $\mathscr{E}$ is closed, $\mathscr{C} := K_1(p \vee q) \cap \mathscr{E}$ is an extremal set and strictly convex. Now let us construct the above function f in four steps.

2. For each $x \in \mathscr{E} \backslash \mathscr{C}^i$, there exists one and only one euclidean nearest point on the boundary of $\mathscr{C}$.

To show this, let $|\cdot|$ be the euclidean norm and assume $x \in \mathscr{E}$ but $x \notin \mathscr{C}^i$. Being a closed subset of K, $\mathscr{C} - x$ is compact. Since $|\cdot|$ is a continuous function on $\mathscr{C} - x$, $\inf |\mathscr{C} - x| := \inf(|y - x|; y \in \mathscr{C})$ exists; i.e., there is $\bar{v} \in \mathscr{C}$ so that $|\bar{v} - x| = \inf |\mathscr{C} - x|$. $\bar{v}$ is said to be an *euclidean nearest point of $\mathscr{C}$ relative to x.*

$\bar{v} \in \mathscr{C}$ is a boundary point of $\mathscr{C}$, for otherwise, the line through $\bar{v}$ and x would intersect the boundary of $\mathscr{C}$ in a point $v' (\neq \bar{v}) \in \mathscr{C}$ between $\bar{v}$ and x. This would imply $|v' - x| < |\bar{v} - x|$ contrary to $|\bar{v} - x| = \inf |\mathscr{C} - x|$.

Now let us show that there is only one euclidean nearest point. On account of the Minkowski-inequality: $|x + y| < |x| + |y|$ iff $x \neq \lambda y$, $\lambda \neq 0$, we find for $x, y (\neq \lambda x) \in \mathscr{S} := (z \in \boldsymbol{R}^n : |z| = 1)$ the inequality $|\frac{1}{2}(x + y)| < 1$; i.e. the boundary of $\mathscr{S}$ containing no line segment is strictly convex.

Suppose v_1 and $v_2 (\neq v_1)$ are two nearest points of $\mathscr{C}$ relative to x. Being convex, $\mathscr{C}$ contains $v_0 := \frac{1}{2}(v_1 + v_2)$. Then $|v_1 - x| = |v_2 - x| =: d$ yields the contradiction $|v_0 - x| = |\frac{1}{2}[(v_1 - x) + (v_2 - x)]| < d := \inf |\mathscr{C} - x|$. Hence $v_1 = v_2$.

3. The mapping $\varphi : x \to v$ attaching to every $x \in \mathscr{E} \backslash \mathscr{C}^i$, its nearest point $v \in \mathbf{Bd}\, \mathscr{C}$ is continuous.

Let $(x_n) \subseteqq \mathscr{E} \backslash \mathscr{C}^i$ be a convergent sequence $x_n \to x_0$. The compactness of $\mathbf{Bd}\, \mathscr{C}$ implies the existence of a convergent subsequence $v_m \to \bar{v}$, of nearest points $v_m = \varphi(x_m) \in \mathbf{Bd}\, \mathscr{C}$. We need only show $\bar{v} = v_0 := \varphi(x_0)$. Now let $\varepsilon > 0$ and $|x_n - x_0| < \varepsilon$. Then $|v_n - x_0| \leqq |v_n - x_n| + |x_n - v_0| \leqq \inf |\mathscr{C} - x_n| + \varepsilon \leqq \inf |\mathscr{C} - x_0| + 2\varepsilon = |v_0 - x_0| + 2\varepsilon$, but $|v_0 - x_0| = \inf |\mathscr{C} - x_0| \leqq |v_n - x_0|$. Hence

α): $x_n \to x_0$ implies $|v_n - x_0| \to |v_0 - x_0|$. On the other hand, noticing $0 \leqq ||v_m - x_0| - |\bar{v} - x_0|| \leqq |v_m - \bar{v}|$ we find;

β): $|v_m - \bar{v}| \to 0$ implies $|v_m - x_0| \to |\bar{v} - x_0|$. α) and β) together yield $|v_0 - x_0| = |\bar{v} - x_0|$. Hence $v_0 = \bar{v}$, because v_0 is the unique nearest point of $\mathscr{C}$ relative to x_0.

4. Let us consider the intersections of the supporting hyperplanes Hp, Hq with the plane $\mathscr{E}$. These are lines l_p, l_q. Assume l_p, l_q not to be parallel and let $\mathscr{M}$ be the set union of l_p, l_q; i.e., $\mathscr{M} := (x \in \mathscr{E} : \mu(x, p) = 1) \cup (y \in \mathscr{E} : \mu(y, q) = 1)$ obviously $\mathscr{M} \cap \mathscr{C} = \{v_p, v_q\} \subseteqq \mathscr{C}^i$. Let $\bar{v}$ be the nearest point of $\mathscr{C}$ relative to $\bar{x} = l_p \cap l_q$.

Then

$$x_{pq}(t) := \begin{cases} v_p + t(\bar{x} - v_p) \text{ for } t \in [0, 1] \subseteqq \boldsymbol{R} \\ \bar{x} + (t - 1)(v_q - \bar{x}) \text{ for } t \in [1, 2] \subseteqq \boldsymbol{R} \end{cases}$$

is a continuous mapping of $[0, 2] \subseteqq \boldsymbol{R}$ onto the union $\mathscr{U} \subseteqq \mathscr{M}$ of the line segments $[v_p, \bar{x}]$, $[\bar{x}, v_q]$. Let $v(t) = \varphi(x(t))$. Hence $v(t)$ is a continuous mapping of the connected intervall $[0, 2]$ onto the arc $\mathscr{A}$ joining v_p with $\bar{v}$ and $\bar{v}$ with v_q on the boundary of $\mathscr{C}$. Since $\mathscr{C}$ is strictly convex, $\mathscr{A}$ consists only of extreme points and is a connected set. Using the homeomorphism $E(K) \leftrightarrow A(G)$ we find, in $A(p \vee q)$, an arc joining p and q. $\lrcorner$

5. If l_p and l_q are parallel, we take $l_r := (x \in \mathscr{E} : \mu(x, r) = 1)$ as auxiliary line being parallel neither with l_p nor with l_q. Then the foregoing scheme applied twice yields also an connected arc in $A(p \vee q)$. $\lrcorner$

Using Theorem 13 and Theorem 17 and the notion of states instead of ensembles ZIERLER shows in [8]:

Theorem 18. *G is a topological lattice; i.e. orthocomplementation, lattice union and intersection are continuous operations.*

Collecting the results of ZIERLER [8, 9] and former results [10], MACLAREN has in [11] given the following representation theorem for G.

Theorem 19. *If G contains at least four orthogonal atoms, then G is isomorphic to the lattice of all subspaces of a finite-dimensional Hilbert-space H over the real, complex or quaternionic numbers.*

IV. The Representation Theorem for the Dual Pair $(K, \hat{L})$

Theorem 20. *Let $\mathscr{P}$ be the cone of all positive semidefinite operators of the finite-dimensional Hilbert-space H. By $\mathscr{K}$ and $\hat{\mathscr{L}}$ we denote the subsets $(V \in \mathscr{P} : \boldsymbol{Tr}\, V = 1)$ and $(F \in \mathscr{P} : F \leqq 1)$, respectively.*

Then there exists a pair of topological isomorphisms $(\psi, \chi) : (K, \hat{L}) \to (\mathscr{K}, \hat{\mathscr{L}})$ such that:

1. *ψ preserves extremality in both directions.*
2. *χ preserves partial ordering in both directions.*

3. *The mapping* μ *and the trace of operators are related by* $\mu(v, f) = \boldsymbol{Tr}(\psi(v) \cdot \chi(f))$.

Proof. The proof is divided into five steps.

1. The lattice $\mathfrak{P}$ of projections of the finite-dimensional *Hilbert*-space H is orthoisomorphic to the subspace-lattice $L(H)$ of H which is orthoisomorphic to G by Theorem 19.

Let $\mathscr{H}$ be the real linear space spanned by the cone $\mathscr{P}$ of all positive semidefinite linear operators of H and let $\bar{\chi}$ be the orthoisomorphic mapping $G \to \mathfrak{P}$. By $\chi(\sum \lambda_i e_i) = \sum \lambda_i \bar{\chi}(e_i)$, $\lambda_i \in \boldsymbol{R}$, $e_i \in G$, we have extended $\bar{\chi}$ to a linear mapping χ of B' into $\mathscr{H}$. Since each real operator of H has an unique decomposition $\sum \lambda_i E_i$, with $E_i \in \mathfrak{P}$ pairwise orthogonal and $\lambda_i \in \boldsymbol{R}$, the mapping χ is also an isomorphism of B' onto $\mathscr{H}$.

2. By a fundamental theorem of Gleason [15], to each orthomeasure m of $\mathfrak{P}$ with $\dim \mathfrak{P} \geqq 3$, there exists a positive-semidefinite operator V of H defined by $m_v(E) = \boldsymbol{Tr}(VE)$ for all $E \in \mathfrak{P}$. Hence to each $v \in K$ there corresponds an operator $V = \bar{\psi}(v) \in \mathscr{P}$ and only one. For if there is another $V' \in \mathscr{P}$ satisfying $m_v(E) = \boldsymbol{Tr}\, V'E$, we should have $\boldsymbol{Tr}(V' - V)E = 0$ for all $E \in \mathfrak{P}$ and particularly for all atoms $P_x \in \mathfrak{P}$. That would mean $\boldsymbol{Tr}(V' - V)\, P_x = \langle x, (V - V')\, x\rangle = 0$ for all $x \in H$, contrary to $V \neq V'$. For all $v \in K$, we have $\boldsymbol{Tr}\, \bar{\psi}(v) = m_v(1) = \mu(v, 1) = 1$; i.e., $\bar{\psi}$ is a mapping of K into $\mathscr{K}$. By $\psi(\sum \lambda_i v_i) := \sum \lambda_i \bar{\psi}(v_i)$, with $v_i \in K$, $\lambda_i \in \boldsymbol{R}$, we have extended $\bar{\psi}$ to a linear mapping ψ of B into the real linear space $\mathscr{H}$.

3. The linear mapping ψ is injective and bicontinuous. To prove the first property it suffices to show that $\bar{\psi}$ is an injective mapping. Let $v_1, v_2 \in K$ with $\bar{\psi}(v_1) = \bar{\psi}(v_2)$. Hence $\mu(v_1, f) = \boldsymbol{Tr}(\bar{\psi}(v_1) \cdot \chi(f)) = \boldsymbol{Tr}(\bar{\psi}(v_2) \cdot \chi(f)) = \mu(v_2, f)$ for all $f \in \hat{L}$; by axiom 1 β, we find then $v_1 = v_2$.

According to $\|v\| = 1 = \mu(v, 1) = \boldsymbol{Tr}\, \psi(v) \geqq |\psi(v)|$, with the operator-norm $|\cdot|$, the injective mapping ψ is continuous. Yet with the compactness of K this implies the bicontinuity of ψ.

4. The mapping $\psi : K \to \mathscr{K}$ being linear, injective and bicontinuous obviously preserves extremality in both directions.

Because of the inequality $1 = \boldsymbol{Tr}\, V \geqq \sup(\boldsymbol{Tr}\, V P_x : P_x \in A(\mathfrak{P})) = \sup(\langle x, Vx\rangle : x \in H \text{ and } \|x\| = 1) =: |V|$, the set $\mathscr{K} \supseteqq \psi(K)$ is a subset of the unit ball $\mathscr{B}$ of the linear space $\mathscr{H}$.

Now, according to a theorem of Kadison [16] the set of all extreme points of $\mathscr{P} \cap \mathscr{B}$ equals the set $\mathfrak{P}$.

Since $\psi(K) \subseteqq \mathscr{K} \subseteqq \mathscr{P} \cap \mathscr{B}$ and $\boldsymbol{Tr}\, V = 1$ for all $V \in \psi(K)$, the set $E(\psi(K))$ of all extreme points of $\psi(K)$ must be even a subset of $A(\mathfrak{P})$. So, to every $v_p = K_1(p) \subseteqq K$ there corresponds only one atom $P \in A(\mathfrak{P}) \cap \psi(K)$. But being isomorphic to $A(G)$, $E(K)$ is also isomorphic to $A(\mathfrak{P})$. Hence we have the equation $A(\mathfrak{P}) = E(\mathscr{K}) = E(\psi(K))$. Now $\psi : K \to \mathscr{K}$

is a surjection. This may be seen as follows: since $\boldsymbol{Tr}\, V = 1$, each $V \in \mathscr{K} \subseteq \mathscr{H}$ has a convex decomposition $V = \sum \lambda_i P_i$, with $\lambda_i \in \boldsymbol{R}_+$ and $\sum \lambda_i = 1$. Because of $E(\psi(K)) = A(\mathfrak{P})$, each $\psi^{-1}(P_i)$ is defined and is an extreme point $v_{pi} \in K$. Being a convex set, K contains the convex decomposition $v = \psi^{-1}(V) = \sum \lambda_i v_{pi}$.

5. Now we will prove that the mapping $\chi : \hat{L} \to \hat{\mathscr{L}}$ is an isomorphism and preserves order in both directions.

First we show that $\boldsymbol{Tr}\, VF_1 \leqq \boldsymbol{Tr}\, VF_2$ holds for all $V \in \mathscr{K}$ iff $F_1 \leqq F_2$, $\leqq$ being the ordering in $\mathscr{P}$. This may be seen as follows: $\boldsymbol{Tr}\, VF_1 \leqq \boldsymbol{Tr}\, VF_2$ for all $V \in \mathscr{K}$ implies $\boldsymbol{Tr}\, P_x F_1 \leqq \boldsymbol{Tr}\, P_x F_2$ for all $P_x \in A(\mathfrak{P}) = E(\mathscr{K})$; i.e., $\langle x, F_1 x\rangle \leqq \langle x, F_2 x\rangle$ for all $x \in H$. Thus $F_1 \leqq F_2$. Therefore, because of the convex decomposition $V = \sum \lambda_i P_i$ for each $V \in \mathscr{K}$ and the linearity of the trace, $F_1 \leqq F_2$ being equivalent to $\boldsymbol{Tr}\, P_i F_1 \leqq \boldsymbol{Tr}\, P_i F_2$ for all $P_i \in A(\mathfrak{P})$ implies $\boldsymbol{Tr}\, VF_1 \leqq \boldsymbol{Tr}\, VF_2$ for all $V \in \mathscr{K}$.

Since for $f_1, f_2 \in \hat{L}$, $f_1 \leqq f_2$ means $\mu(v, f_1) \leqq \mu(v, f_2)$ for all $v \in K$ and because $\psi : K \to \mathscr{K}$ is an isomorphism, χ is obviously order preserving in both directions and hence maps $\hat{L}$ onto $\hat{\mathscr{L}} := (F \in \mathscr{P} : F \leqq 1)$.

The inequality $\|f\| = \sup(|\mu(x, f)| : x \in B, \|x\| = 1) \geqq \sup(\boldsymbol{Tr}\, PF : P \in A(\mathfrak{P})) = |F|$ and the compactness of $\hat{L}$ imply the bicontinuity of χ. ⊣

Now it remains to show that the system $(\mathscr{K}, \hat{\mathscr{L}})$ is a solution of the axiomatic scheme $(K, \hat{L})$. For that we must know the annihilator sets in $\mathscr{K}$ and $\hat{\mathscr{L}}$ as well as the $C(V)$-sets in $\mathscr{K}$.

The set $\mathfrak{M} := (x \in H : Fx = 0, F \in \hat{\mathscr{L}})$ is a subspace of H. Let E be the projection onto the subspace complementary to $\mathfrak{M}$. Then we have $F(1 - E)x = 0$ for all $x \in H$; i.e., $F = FE = EFE$.

Since $0 \leqq F \leqq 1$ and $\langle x, Fx\rangle = \langle x, EFEx\rangle = \langle Ex, FEx\rangle \leqq \langle Ex, Ex\rangle = \langle x, Ex\rangle$ for all $x \in H$, we find $F \leqq E$.

Now this being the case, we find $E_i \leqq E$ for the pairwise orthogonal projections E_i in the unique decomposition $F = \sum \lambda_i E_i$; hence $\sum E_i \leqq E$ and finally $FE = \sum \lambda_i E_i E = \sum \lambda_i E_i = F$. However, being the smallest projection with $FE = F$, E must be equal to $\sum E_i$ and is said to be the *carrier of* F. Hence, denoting by E_A the carrier of $A \in \mathscr{P}$ and noticing $\boldsymbol{Tr}\, VF = 0$ iff $VF = 0$ for all $V, F \in \mathscr{P}$ we find.

Lemma 1. $\mathscr{K}_0(F) = (V \in \mathscr{K} : VE_F = 0)$ *and* $\hat{\mathscr{L}}_0(V) = (F \in \hat{\mathscr{L}} : FE_V = 0)$.

Next, we show.

Lemma 2. $C(V) = \mathscr{K}_0(1 - E_V)$.

Proof. Obviously $\mathscr{K}_0(1 - E_V)$ is closed and convex. Let $]V_1, V_2[$ be an open line segment in $\mathscr{K}$ containing $\tilde{V} \in \mathscr{K}_0(1 - E_V)$. Then there holds $\tilde{V} = \lambda V_1 + (1 - \lambda) V_2$, $\lambda \in\,]0, 1[$; This implies $0 = \tilde{V}(1 - E_V) = \lambda V_1(1 - E_V) + (1 - \lambda) V_2(1 - E_V)$. Since λ and $1 - \lambda$ are positive numbers, we find $V_1(1 - E_V) = V_2(1 - E_V) = 0$. On account of the convexity of $\mathcal{K}_0(1 - E_V)$, this implies $[V_1, V_2] \subseteqq \mathcal{K}_0(1 - E_V)$. Hence, $\mathcal{K}_0(1 - E_V)$ is an extremal set. Obviously $V \in \mathcal{K}_0(1 - E_V)$; hence $C(V) \subseteqq \mathcal{K}_0(1 - E_V)$. To show the converse inclusion, let $\overline{V}(\neq V)$ be another point of $\mathcal{K}_0(1 - E_V)$. We decompose E_V into a sum of m pairwise orthogonal atoms P_i. Being internal point of the simplex $(P_1 \ldots P_m)$ which spans $M(\mathcal{K}_0(1 - E_V))$, V is also internal point of $\mathcal{K}_0(1 - E_V)$. Hence a $V' \in \mathcal{K}$ exists such that $V \in]\overline{V}, V'[$. Then, by Theorem 6, there follows $\overline{V} \in C(V)$. Thus we have shown $C(V) \supseteqq \mathcal{K}_0(1 - E_V)$, too. Now we are able to verify the axioms. By the remark that $\boldsymbol{Tr} A P = 0$ for all $P \in A(\mathfrak{P})$ iff $A = 0$, we see that Axiom 1 holds.

Axiom 2a: Let E_1, E_2 be the carrier of F_1, F_2, respectively. For all $V \in \overline{\mathcal{K}}_0 := (\overline{V} \in \mathcal{K} : \overline{V} E_1 = \overline{V} E_2 = 0)$ we find $V(1 - E_1) = V(1 - E_2) = V$. Hence for $F := (1 - E_1)(1 - E_2)$ there holds $0 \leqq F = F(1 - E_i)$, i.e., $0 \leqq F \leqq 1 - E_i \leqq 1$ or $1 \geqq 1 - F \geqq E_i$ $(i = 1, 2)$. On the other hand, $V(1 - F) = 0$. Thus $F_3 := 1 - F \in \hat{\mathcal{L}}$ satisfies the conditions $F_3 \geqq E_i \geqq F_i$ and $\mathcal{K}_0(F_3) = \overline{\mathcal{K}}_0 = \mathcal{K}_0(F_1) \cap \mathcal{K}_0(F_2)$ of Axiom 2a.

Axiom 2b: Obviously we have $\hat{\mathcal{L}} = \hat{\hat{\mathcal{L}}}$. Since $E_F \geqq F$ is the smallest projection E satisfying $\mathcal{K}_0(F) \supseteqq \mathcal{K}_0(E)$ Axiom 2b holds.

Axiom 3: $\hat{\mathcal{L}}_0(V_1) = \hat{\mathcal{L}}_0(V_2)$ means $(F \in \hat{\mathcal{L}} : F E_1 = 0) = (F \in \hat{\mathcal{L}} : F E_2 = 0)$, with E_1, E_2 the carriers of V_1, V_2, respectively; hence $F \leqq 1 - E_1$ iff $F \leqq 1 - E_2$ for all $F \in \hat{\mathcal{L}}$. This implies $1 - E_1 = 1 - E_2$ and $C(V_1) = \mathcal{K}_0(1 - E_1) = \mathcal{K}_0(1 - E_2) = C(V_2)$.

Axiom 4: The lattice $\mathfrak{P}$ of projections is modular. Thus Axiom 4′ holds equivalently.

Summarizing Theorem 20 with the above results, we have shown.

Theorem 21. *The system* $(\mathcal{K}, \hat{\mathcal{L}}) := ((V \in \mathcal{P} : \boldsymbol{Tr}\, V = 1), (F \in \mathcal{P} : F \leqq 1))$ *of positive-semidefinite linear operators of the finite-dimensional Hilbert-space H, given by Theorem 19, is a categorical solution of the axiomatic scheme* $(K, \hat{L})$.

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