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An Elementary Proof of the Plancherel Theorem for the Classical Groups

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Abstract. We show how to reduce the Plancherel theorem to one lemma which is proved by elementary means.

1. Introduction

The theory of Fourier transformations on Lie groups though being rather young and still incomplete has recently attracted the attention of theoretical physicists. They are interested in particular in the basic theorem on the inversion of the Fourier transformation, the so-called Plancherel theorem. The groups, physicists are mainly concerned with, are the homogeneous Lorentz group $SL(2, C)$, which belongs to the classical simple Lie groups, and the group $SU(1, 1)$ which is non-classical. The proof of the Plancherel theorem for classical groups has first been established by GELFAND and NAIMARK [1]. In its original form it is extremely tedious. Lateron Gelfand himself found a more elegant proof which exploits the connection between the Plancherel theorem and integral transforms of the Riesz type [2]. Both proofs are repeated in [3][1]. HARISH-CHANDRA [4] dropped most of this dead weight and reduced the proof to a few lemmas. For the convenience of the physicists we undertake in this note to prove the Plancherel theorem for classical groups with a minimal number of elementary arguments, which in particular keep all constant factors under control.

Actually we perform the proof explicitly only for the complex unimodular groups $SL(n, C)$. The orthogonal and symplectic groups can be treated quite the same way. Even for the non-classical group $SU(1, 1)$ the Plancherel theorem can be proved in this fashion [5]. In the case of the homogeneous Lorentz group the correctness of our lemma can directly be inspected and the whole proof of the Plancherel theorem becomes pedestrian. Since our proof is simple even in the case of a general group $SL(n, C)$, we refrain from separately dealing with the case $SL(2, C)$ which is of greatest interest for physicists.

[1] In fact GELFAND proves two Plancherel formulae which differ by a constant factor, compare [3], Eq. 26.83 and ibid. App. III, Eq. 2.42. Our proof agrees with his second result [2].

2. Notations and a First Reduction

Let $x(a)$ be a function on $SL(n, C)$ with compact support and derivatives of all orders. e be the unit element of $SL(n, C)$. The Plancherel theorem in the simplest of several equivalent forms is (see [3])

$$x(e) = \frac{1}{(2\pi)^{(n-1)(n+2)}\, n!} \int_{-\infty}^{+\infty} d\varrho_2 \int_{-\infty}^{+\infty} d\varrho_3 \ldots \int_{-\infty}^{+\infty} d\varrho_n \sum_{m_2, m_3, \ldots, m_n = -\infty}^{+\infty} \omega(\chi)\, \mathrm{Tr} K_x^\chi .$$

By χ we denote a representation T_a^χ of the principal (non-degenerate) series, we can fix the meaning of this symbol by a $2n$-tupel of numbers

$$\chi = (m_1, m_2, \ldots, m_n; \varrho_1, \varrho_2, \ldots, \varrho_n)$$

where the m_i are integers and the ϱ_i are real. We normalize these numbers for convenience by

$$m_1 = \varrho_1 = 0 .$$

The weight function $\omega(\chi)$ is defined to be

$$\omega(\chi) = \prod_{n \geqq p > q \geqq 1} [(\varrho_p - \varrho_q)^2 + (m_p - m_q)^2] .$$

K_x^χ is the Hilbert-Schmidt operator

$$K_x^\chi = \int x(a)\, T_a^\chi\, d\mu(a)$$

(for the normalization of the invariant measure $d\mu(a)$ see below). Its trace is [3]

$$\mathrm{Tr} K_x^\chi = \int d\mu(\delta)\, \chi(\delta) \int x(z^{-1} k z)\, d\mu(z)\, d\mu'(k) ,$$

where δ, z, k are elements of $SL(n, C)$ of the typical form

$$\delta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix},$$

$$z = \begin{pmatrix} 1 & & & 0 \\ z_{21} & 1 & & \\ z_{31} & z_{32} & 1 & \\ \vdots & & & \\ z_{n1} & z_{n2} & \ldots & 1 \end{pmatrix},$$

$$k = \begin{pmatrix} \lambda_1 & \mu_{12} & \mu_{13} \ldots & \mu_{1n} \\ & \lambda_2 & \mu_{23} \ldots & \mu_{2n} \\ & & & \vdots \\ 0 & & & \lambda_n \end{pmatrix}.$$

For any complex variable w we use the notation

$$Dw = dw_1\, dw_2 , \quad w = w_1 + i w_2 , \quad -\infty < w_{1,2} < +\infty .$$

Then the measures $d\mu(\delta)$, $d\mu(z)$, and $d\mu'(k)$ are defined by

$$d\mu(z) = \prod_{i>j} Dz_{ij},$$
$$d\mu'(k) = \prod_{i<j} D\mu_{ij},$$
$$d\mu(\delta) = \prod_{i\geqq 2} |\lambda_i|^{-2} D\lambda_i .$$

We note that $d\mu(z)$ and $d\mu(\delta)$ are invariant on the respective subgroups. In addition

$$d\mu_l(k) = \beta(k)\, d\mu(\delta)\, d\mu'(k)$$

is left invariant on the subgroup K of matrices k if

$$\beta(k) = \prod_{i\geqq 2} |\lambda_i|^{2(i-1)} .$$

If $a = kz$ we have in this normalization

$$d\mu(a) = d\mu_l(k)\, d\mu(z) .$$

$\chi(\delta)$ is the "infinitesimal character" for the representation χ,

$$\chi(\delta) = \prod_{i\geqq 2} \lambda_i^{-\frac{1}{2}m_i + \frac{i}{2}\varrho_i}\, \bar{\lambda}_i^{+\frac{1}{2}m_i + \frac{i}{2}\varrho_i} .$$

A representation of the principal series is obtained in the following fashion. We characterize the right cosets of the subgroup K by the matrices z. We introduce transformations of these cosets z due to group elements $a \in SL(n, C)$ by

$$za = kz_a .$$

The principal series results if we set

$$T_a^{\chi} f(z) = \chi(\delta)\, \beta(k)^{-1} f(z_a)$$

where δ is the diagonal part of k, $\chi(\delta)$ and $\beta(k)$ are as defined earlier, and the functions $f(z)$ constitute an L^2-space with respect to the measure $d\mu(z)$.

Now we make one reduction of the Plancherel theorem which is standard. With the short hand $G(\delta)$ for the inner integral in $\mathrm{Tr} K_x^{\chi}$ we can write

$$\mathrm{Tr} K_x^{\chi} = \int d\mu(\delta)\, \chi(\delta)\, G(\delta) .$$

We notice that

$$\omega(\chi)\, \chi(\delta) = (-4)^{\frac{1}{2}n(n-1)} L\bar{L}\, \chi(\delta),$$

L being the differential operator

$$L = \prod_{n\geqq p>q\geqq 2} \left(\lambda_p \frac{\partial}{\partial\lambda_p} - \lambda_q \frac{\partial}{\partial\lambda_q}\right) \prod_{s\geqq 2} \lambda_s \frac{\partial}{\partial\lambda_s} .$$

Partial integrations followed by summations over m_i and integrations over ϱ_i reduce the Plancherel theorem to

$$x(e) = \frac{(-4)^{\frac{1}{2}n(n-1)}}{(2\pi)^{n(n-1)} n!} \{L\bar{L}\, G(\delta)\}_{\delta=e} \,.$$

This assertion is proved in the subsequent section.

3. Further Reductions and a Lemma

We introduce a function $\hat{x}(a)$ with compact support and derivatives of all orders in the $2n^2$ real variables of a, which coincides with $x(a)$ on the submanifold $\det a = 1$. $f(b)$ be its Fourier transform such that

$$\hat{x}(a) = \int \prod_{ij} Db_{ij}\, f(b) \exp\{i \operatorname{Re} \operatorname{Tr}(ab)\} \,.$$

Inserting this expression into $G(\delta)$ yields

$$G(\delta) = \int \prod_{ij} Db_{ij}\, f(b) \int d\mu(z) \prod_{l<m} D\mu_{lm}$$
$$\cdot \exp i \operatorname{Re}\left[\sum_{s=1}^{n} \lambda_s h_{ss} + \sum_{p<q} \mu_{pq} h_{qp}\right]$$

where we use the notation

$$h = zbz^{-1} \,.$$

The operation $L\bar{L}$ can be exerted on $G(\delta)$ as follows. First we notice that the dependent variable λ_1 enters $G(\delta)$ symmetrically with the other λ_i and that we may therefore replace L by the operator

$$\Lambda = \prod_{n \geqq p > q \geqq 1} \left(\lambda_p \frac{\partial}{\partial \lambda_p} - \lambda_q \frac{\partial}{\partial \lambda_q}\right)$$

i.e. formally include the differentiation with respect to λ_1. This operator Λ acts like

$$\Lambda \exp \sum_{i=1} \alpha_i \lambda_i = \prod_{n \geqq p > q \geqq 1} (\alpha_p \lambda_p - \alpha_q \lambda_q) \exp \sum_{i=1}^{n} \alpha_i \lambda_i \,,$$

as can be inspected by induction with respect to n, if we write Λ as a Vandermonde's determinant

$$\Lambda = \begin{vmatrix} \left(\lambda_n \frac{\partial}{\partial \lambda_n}\right)^{n-1}, & \left(\lambda_n \frac{\partial}{\partial \lambda_n}\right)^{n-2}, \ldots, & 1 \\ \left(\lambda_{n-1} \frac{\partial}{\partial \lambda_{n-1}}\right)^{n-1}, & \left(\lambda_{n-1} \frac{\partial}{\partial \lambda_{n-1}}\right)^{n-2}, \ldots, & 1 \\ \left(\lambda_1 \frac{\partial}{\partial \lambda_1}\right)^{n-1}, & \left(\lambda_1 \frac{\partial}{\partial \lambda_1}\right)^{n-2}, \ldots, & 1 \end{vmatrix} .$$

This gives

$$\{L\bar{L}\,G(\delta)\}_{\delta=e}$$
$$= (2\pi)^{n(n-1)}\,(-4)^{-\frac{1}{2}n(n-1)} \int \prod_{ij} Db_{ij}\, f(b)\exp i\,\mathrm{Re}(\mathrm{Tr}\,b)$$
$$\cdot \left\{\int d\mu(z) \prod_{n\geqq p>q\geqq 1} |h_{pp}-h_{qq}|^2 \prod_{n\geqq r>s\geqq 1} \delta(h_{rs})\right\}$$

where $\delta(\ldots)$ are two-dimensional delta functions. Thus our proof will be completed if we have shown that

$$\int d\mu(z) \prod_{n\geqq p>q\geqq 1} |h_{pp}-h_{qq}|^2 \prod_{n\geqq r>s\geqq 1} \delta(h_{rs}) = n!\,.$$

This we call our lemma. The condition for a coset z to be stationary under the action of a group element a is

$$(zaz^{-1})_{ij} = 0 \quad \text{for} \quad i>j\,.$$

The integral in the lemma defines therefore an average over all the stationary cosets for any given element $a = b$.

The lemma is proved in two steps. We write

$$h = z' + k' - e$$

and show first that the Jacobian between z' and z is

$$\left.\frac{\partial(z'_{lm})}{\partial(z_{ij})}\right|_{z'=e} = \prod_{n\geqq p>q\geqq 1} (h_{qq}-h_{pp})\,.$$

Second we prove that for any given b (apart from a set of b-measure zero) the equation

$$k' = zbz^{-1}$$

has exactly $n!$ different solutions z. Both issues are easily settled.

By straightforward computation we derive the relation

$$\frac{\partial z'_{pq}}{\partial z_{lm}} = \delta_{pl}(bz^{-1})_{mq} - h_{pl}(z^{-1})_{mq}$$

which if multiplied with z_{sm}, $s<l$, and summed over m yields

$$\sum_{m=1}^{s} z_{sm}\frac{\partial z'_{pq}}{\partial z_{lm}} = \delta_{pl}\,h_{sq} - \delta_{sq}\,h_{pl}\,.$$

At $z' = e$ we obtain this way

$$\left.\frac{\partial z'_{pq}}{\partial z_{lm}}\right|_{z'=e} = 0\,, \qquad p>l\,,$$

which implies

$$\left.\frac{\partial(z'_{lm})}{\partial(z_{ij})}\right|_{z'=e} = \prod_{n\geqq p\geqq 2}\ \det_{q,m}\left.\left(\frac{\partial z'_{pq}}{\partial z_{pm}}\right)\right|_{z'=e}.$$

In addition we have the result

$$\det_{q,m}\left(\frac{\partial z'_{pq}}{\partial z_{pm}}\right)\Big|_{z'=e} = \prod_{q=1}^{p-1}(h_{qq} - h_{pp}) ,$$

from which our assertion follows immediately.

The second part of the proof of our lemma makes use of the fact that a matrix b with distinct eigenvalues (almost all b are of such type) may be decomposed into the product

$$b = s^{-1}\,\delta s$$

where for a given order of the eigenvalues, s is unique up to a left factor δ_1. Moreover we may split up s uniquely as

$$s = k_1 z$$

again for almost all b, the ambiguous factor δ_1 being absorbed in k_1. This yields

$$k' = z b z^{-1} = k_1^{-1}\,\delta k_1 .$$

Given the order of the eigenvalues of b, z is unique. Since there are $n!$ different orders of the eigenvalues, for almost all b there exist $n!$ different solutions z as asserted.

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